

# ENGEL CONGRUENCES IN GROUPS OF PRIME-POWER EXPONENT

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**1. Introduction.** In this paper a simplified proof of a theorem of Sanov (4) is given. No mention is required of Lie elements or of the Baker–Hausdorff formula, both of which played central roles in Sanov’s proof.

Let  $G$  be a group. Define, for all  $x, y \in G$ ,

$$(y, x; 0) = y, \quad (y, x; 1) = (y, x) = y^{-1}x^{-1}yx,$$

$$(y, x; n + 1) = ((y, x; n), x).$$

Define, as usual,

$$G_1 = G, \quad G_{n+1} = (G_n, G) = \{(c, x) \mid c \in G_n, x \in G\}.$$

Sanov (4) proved that if  $G$  is a group of prime-power exponent,  $p^e$ , then

$$(1.1) \quad (y_m, x; k(p^k - p^{k-1}))^{p^{e-k}} \in G_q, \quad k = 1, 2, \dots, e,$$

for all  $y_m \in G_m$  and all  $x \in G$ , where

$$q = \min(pm + 1, m + k(p^k - p^{k-1}) + 1).$$

This is a non-trivial result only if

$$(1.2) \quad m \geq kp^{k-1}.$$

In particular, if

$$y_m = (y, x; kp^{k-1} - 1),$$

then (1.1) becomes

$$(1.3) \quad (y, x; kp^k - 1)^{p^{e-k}} \in G_{kp^{k+1}}.$$

In this paper (1.1) is proved with

$$(1.4) \quad q = \min(pm + p - 1, m + k(p^k - p^{k-1}) + 1),$$

but even with this larger  $q$ , (1.1) is still non-trivial only if (1.2) holds.

In §2 we list some elementary results about commutators and binomial coefficients. In §3 we study the ideal in  $Z[X]$  associated with the ring of endomorphisms of an abelian normal subgroup of a group of prime-power exponent. Properties of this ideal yield a commutator relation (3.29) between

the subgroup and the group, which is strengthened in §4 and is shown in §5 to yield (1.1).

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**2. LEMMA 2.1.** *Let  $G$  be a group. Then for all  $x, y, z \in G$ ,*

$$(2.1) \quad (xy, z) = (x, z)((x, z), y)(y, z),$$

$$(2.2) \quad (x, yz) = (x, z)(x, y)((x, y), z).$$

*Proof.* See, for example **(2, p. 150)**.

Let

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

be the binomial coefficient, i.e., the coefficient of  $x^r$  in  $(1+x)^n$ .

**LEMMA 2.2.** *Let  $A$  be an abelian normal subgroup of a group  $G$ ;  $r, n, m$  be positive integers;  $a, a_i \in A$ ;  $x \in G$ . Then*

$$(2.3) \quad \left(\prod_i a_i, x; r\right) = \prod_i (a_i, x; r),$$

$$(2.4) \quad (xa)^n = x^n a^n \prod_{i=1}^{n-1} (a, x; i) \binom{n}{i+1},$$

$$(2.5) \quad (a, x^m; r) = \prod_{w=r}^{m-r} (a, x; w)^{E(r, m, w)},$$

where

$$E(r, m, w) = \sum \binom{m}{s_1} \binom{m}{s_2} \dots \binom{m}{s_r}$$

and the sum is taken over all distinct (ordered)  $r$ -tuples of positive integers  $(s_1, s_2, \dots, s_r)$  such that  $s_1 + s_2 + \dots + s_r = w$  and  $s_i \leq m$ .

*Proof.* To establish (2.3) use induction and (2.1). To establish (2.4) use induction and the following argument:

$$(xa)^{n+1} = (xa)^n xa = x^n a^n \prod_{i=1}^{n-1} (a, x; i) \binom{n}{i+1} xa.$$

Now move the right-most  $x$  to the far left by repeated use of the identity

$$ux = xu(u, x),$$

to obtain

$$\begin{aligned} (xa)^{n+1} &= x^{n+1} a^{n+1} \prod_{i=1}^{n-1} (a, x; i)^{\binom{n}{i+1}} \prod_{i=0}^{n-1} (a, x; i+1)^{\binom{n}{i+1}} \\ &= x^{n+1} a^{n+1} \prod_{i=1}^n (a, x; i)^{\binom{n}{i+1} + \binom{n}{i}}. \end{aligned}$$

Finally observe that

$$\binom{n}{i+1} + \binom{n}{i} = \binom{n+1}{i+1}.$$

One also proves (2.5) by induction; see (3, Lemma 4.1).

*Note.* If  $m = p$  in (2.5), then

$$p \mid \binom{p}{s_i} \quad \text{for all } s_i < p$$

and hence  $p \mid E(r, p, w)$  for all  $w < rp$ .

The following lemma is well known.

LEMMA 2.3. *If  $p$  is a prime and  $e, r, k$  are positive integers such that  $e \geq r$ ,  $kp^r \leq p^e$  and  $(k, p) = 1$ , then*

$$(2.6) \quad p^{e-r} \mid \binom{p^e}{kp^r} \quad \text{but} \quad p^{e-r+1} \nmid \binom{p^e}{kp^r}.$$

3. Let  $G$  be a group of exponent  $p^e$ , let  $A$  be an abelian normal subgroup, let  $R$  denote the ring of endomorphisms of  $A$ , and let  $x$  be a fixed but arbitrary element of  $G$ .

Using (2.3) one verifies that the mapping from  $A$  to  $A$  defined by

$$(3.1) \quad a \rightarrow \prod_{i=0}^k (a, x; i)^{r_i} \quad (\text{all } a \in A),$$

where  $k$  and the  $r_i$  are fixed integers,  $k$  non-negative, is an endomorphism of  $A$ . One can also verify that the mapping from  $Z[X]$  to  $R$ , which sends

$$(3.2) \quad f(X) = \sum_{i=0}^k r_i X^i$$

into the endomorphism, (3.1), is a homomorphism. Denote by  $J_x$  the kernel of this homomorphism; denote by  $J$  the intersection of all the  $J_x$  ( $x \in G$ ).

If we denote by  $a^x$  the image of  $a$  under the endomorphism assigned to  $X$ , namely  $a^x = (a, x)$ ; and if we denote by  $a^r$  the image of  $a$  under the endomorphism assigned to  $r$ , namely  $a^r = a^r$ ; then it follows that if  $f(X)$  is given by (3.2),

$$(3.3) \quad a^{f(X)} = \prod_{i=0}^k (a, x; i)^{r_i}.$$

Hence  $J$  can be described as

$$J = \{f(X) \in Z[X] \mid a^{f(X)} = 1 \text{ for all } a \in A \text{ and all } x \in G\}.$$

For example, note that  $p^e \in J$  and that by (2.4)

$$\sum_{i=1}^{p^e-1} \binom{p^e}{i+1} X^i \in J.$$

**THEOREM 3.1.** *Let*

(3.4)  $G$  be a group of exponent  $p^e$ ,

(3.5)  $A$  be an abelian normal subgroup of  $G$ ,

(3.6)  $J = \{f(X) \in Z[X] \mid a^{f(X)} = 1 \text{ for all } a \in A \text{ and all } x \in G\}$ .

Then, for each  $i = 1, 2, \dots, e$ ,  $J$  contains a polynomial  $f_i(X)$  given by

$$(3.7) \quad f_i(X) = \sum_{j=1}^e p^{e-j} X^{p^i+j-1-p^{i-1}} a_{ij}(X),$$

where the  $a_{ij}(X)$  have constant coefficients prime to  $p$ , and  $a_{ie}(X) = 1$ .

*Proof.* Put  $n = p^e$  in (2.4) to obtain

$$(3.8) \quad f_1(X) = \sum_{i=1}^{p^e-1} \binom{p^e}{i+1} X^i \in J.$$

Use Lemma 2.3 to put (3.8) into the form of (3.7) with  $i = 1$ . Assume now that  $f_i(X) \in J$ ; i.e.,

$$(3.9) \quad 1 = \prod_r (a, x; r)^{s_r}$$

where

$$f_i(X) = \sum_r s_r X^r.$$

In (3.9) replace  $x$  by  $x^p$  and use (2.5) with  $m = p$ . Then

$$1 = \prod_r (a, x^p; r)^{s_r} = \prod_w (a, x; w)^{t_w}.$$

We wish to show that

$$(3.10) \quad \sum_w t_w X^w \equiv f_{i+1}(X) \pmod{J}.$$

By (2.5), and the note preceding Lemma 2.3,

$$(3.11) \quad (a, x^p; r) = (a, x; pr) \prod_{w=r}^{pr-1} (a, x; w)^{n_w},$$

where the  $n_w$  are positive integers. Hence  $\sum t_w X^w$  is  $f_i(X)$  with  $X^r$  replaced by

$$(3.12) \quad X^{pr} + pX^r g_r(X), \quad r \geq 1;$$

here

$$g_r(X) = \sum_{w=r}^{pr-1} n_w X^{w-r} \in Z[X].$$

Making this replacement in each term of (3.7), we obtain

$$(3.13) \quad \sum t_w X^w = \sum_{j=1}^e p^{e-j} [X^{p^{i+j-p^i}} + pX^{p^{i+j-1-p^{i-1}}} h_j(X)] a'_{ij}(X),$$

where  $h_j(X) = g_{p^{i+j-1-p^{i-1}}}(X)$ , and  $a'_{ij}(X)$  has the same constant coefficient as  $a_{ij}(X)$ . We now use the fact that  $p^e \in J$  and rearrange terms:

$$(3.14) \quad \sum t_w X^w \equiv \sum_{j=1}^e p^{e-j} X^{p^{i+j-p^i}} a_{i+1,j}(X) = f_{i+1}(X) \pmod{J},$$

where

$$(3.15) \quad \begin{aligned} & p^{e-j} X^{p^{i+j-p^i}} a_{i+1,j}(X) \\ &= p^{e-j} [X^{p^{i+j-p^i}} a'_{ij}(X) + X^{p^{i+j-p^{i-1}}} h_{j+1}(X) a'_{i,j+1}(X)] \quad (1 \leq j \leq e-1), \end{aligned}$$

and

$$a_{i+1,e}(X) = a'_{ie}(X) = a_{ie}(X) = 1.$$

Since  $a_{ij}(X)$  has constant coefficient prime to  $p$  for all  $1 \leq j \leq e$ , and since

$$(3.16) \quad p^{i+j} - p^{i-1} > p^{i+j} - p^i,$$

we conclude that  $a_{i+1,j}(X)$  has constant coefficient prime to  $p$  for all

$$1 \leq j \leq e.$$

We now prove an elimination theorem, which yields the important formula (3.29).

**THEOREM 3.2.** *Let  $I$  be an ideal of  $Z[X]$  with the property that if  $a[X] \in Z[X]$  is invertible  $\pmod{I}$ , then so is  $a(X) + Xb(X)$  for all  $b(X) \in Z[X]$ . Suppose that  $I$  contains the polynomials*

$$(3.17) \quad \begin{aligned} f_1(X) &= A_1 a_{11} X^{m_{11}} + A_2 a_{12} X^{m_{12}} + \dots + A_e a_{1e} X^{m_{1e}}, \\ f_2(X) &= A_1 a_{21} X^{m_{21}} + A_2 a_{22} X^{m_{22}} + \dots + A_e a_{2e} X^{m_{2e}}, \\ &\dots \\ f_e(X) &= A_1 a_{e1} X^{m_{e1}} + A_2 a_{e2} X^{m_{e2}} + \dots + A_e a_{ee} X^{m_{ee}}, \end{aligned}$$

where the  $A_j$  and  $a_{ij}$  are in  $Z[X]$ , the  $a_{ij}$  are invertible  $\pmod{I}$ , and the  $m_{ij}$  are non-negative integers such that

$$(3.18) \quad m_{i1} < m_{i2} < \dots < m_{ie}, \quad 1 \leq i \leq e,$$

$$(3.19) \quad m_{11} < m_{21} < \dots < m_{e1},$$

$$(3.20) \quad (m_{ij} + m_{i+1,j+t}) - (m_{i,j+t} + m_{i+1,j}) > 0, \quad \begin{aligned} & 1 \leq t \leq e-j, \\ & 1 \leq i, j, i+1 \leq e. \end{aligned}$$

Then there exist  $c_k(X) \in Z[X]$  such that

$$(3.21) \quad A_k X^{m_k} \equiv X^{m_{k+1}} c_k(X) \pmod{I}, \quad 1 \leq k \leq e,$$

where

$$(3.22) \quad m_k = \sum_{r+s=k+1} m_{rs} - \sum_{r+s=k} m_{rs}$$

(with the understanding that  $m_1 = m_{11}$ ).

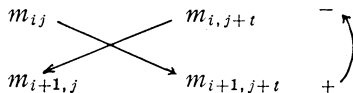
*Comments.* For our purposes, we shall be using Theorem 3.2 with  $I = J$  as described in (3.6) and the  $f_i(X)$  being the  $f_i(X)$  of (3.7); i.e.,

$$A_j = p^{e-j}, \quad a_{ij} = a_{ij}(X), \quad m_{ij} = p^{i+j-1} - p^{i-1}.$$

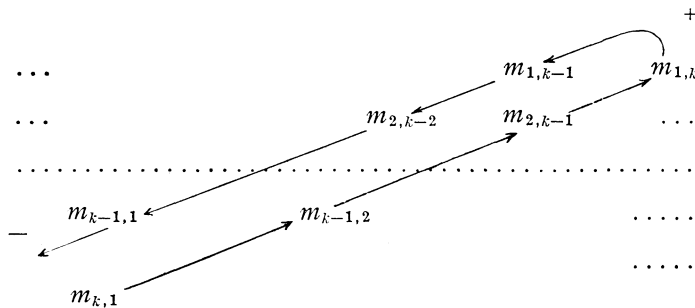
For this value of  $m_{ij}$ , one readily verifies that

$$(3.23) \quad m_k = k(p^k - p^{k-1}).$$

Condition (3.20) can be considered as a requirement that all  $2 \times 2$  “determinants”



be positive. The right-hand side of formula (3.22) can be thought of as the “sum” of the “ $k$ th diagonal” minus the “sum” of the “ $(k - 1)$ st diagonal” of the matrix of the  $m_{ij}$ :



*Proof of Theorem 3.2.* We use induction on  $e$ . If  $e = 1$ , then (3.17) is just

$$f_1(X) = A_1 a_{11} X^{m_{11}} \in I.$$

But  $a_{11}$  invertible (mod  $I$ ) implies that

$$A_1 X^{m_{11}} \equiv 0 \pmod{I};$$

so simply let  $c_1(X) = 0$  and (3.21) will be satisfied.

Now suppose the theorem is true for  $e - 1$  and consider (3.17). We rewrite  $f_1(X) \in I$  as

$$(3.24) \quad A_1 a_{11} X^{m_{11}} \equiv - \sum_{i=2}^e A_i a_{1i} X^{m_{1i}} \pmod{I}.$$

Using (3.18) with  $i = 1$  and the invertibility of  $a_{11}$ , we obtain (3.21) with  $k = 1$  and

$$c_1(X) = - \sum_{i=2}^e A_i a_{11}^{-1} a_{1i} X^{m_{1i} - m_{11} - 1}.$$

We now essentially strike out the first row and the first column from the right-hand side of (3.17). Since the  $a_{ij}(X)$  are invertible (mod  $I$ ) and the product of two invertible polynomials is invertible, we may assume that  $a_{i1} \equiv 1 \pmod{I}$ ,  $1 \leq i \leq e$ . (This amounts to multiplying each  $f_i(X)$  by  $a_{i1}^{-1}(X)$ .) Let

$$(3.25) \quad f_i^{(2)}(X) = X^{m_{i1} - m_{i-1,1}} f_{i-1}(X) - f_i(X) = \sum_{j=2}^e A_j b_{ij} X^{m_{ij}^{(2)}}, \quad 2 \leq i \leq e,$$

where

$$(3.26) \quad \begin{aligned} b_{ij} X^{m_{ij}^{(2)}} &= a_{i-1,j} X^{m_{i1} - m_{i-1,1} + m_{i-1,j}} - a_{ij} X^{m_{ij}}, \\ m_{ij}^{(2)} &= m_{i1} - m_{i-1,1} + m_{i-1,j}, \end{aligned}$$

and hence

$$(3.27) \quad b_{ij} = a_{i-1,j} - a_{ij} X^{m_{i-1,1} + m_{ij} - m_{i1} - m_{i-1,j}}.$$

Observe that (3.20) guarantees that the exponent of  $X$  in (3.27) is positive.

We consider now the following polynomials in  $I$ .

$$(3.28) \quad \begin{aligned} f_2^{(2)}(X) &= A_2 b_{22} X^{m_{22}^{(2)}} + A_3 b_{23} X^{m_{23}^{(2)}} + \dots + A_e b_{2e} X^{m_{2e}^{(2)}}, \\ f_3^{(2)}(X) &= A_2 b_{32} X^{m_{32}^{(2)}} + A_3 b_{33} X^{m_{33}^{(2)}} + \dots + A_e b_{3e} X^{m_{3e}^{(2)}}, \\ &\dots \\ f_e^{(2)}(X) &= A_2 b_{e2} X^{m_{e2}^{(2)}} + A_3 b_{e3} X^{m_{e3}^{(2)}} + \dots + A_e b_{ee} X^{m_{ee}^{(2)}}. \end{aligned}$$

It is easy to verify that the  $b_{ij}$  and the  $m_{ij}^{(2)}$ ,  $2 \leq i \leq e$ , satisfy all of the hypotheses of the theorem and hence, by induction, there exist  $c_k(X) \in Z[X]$  such that

$$A_k X^{m_k^{(2)}} \equiv X^{m_k^{(2)}+1} c_k(X) \pmod{I}$$

for all  $2 \leq k \leq e$ , where

$$m_k^{(2)} = \sum_{r+s=k+2} m_{rs}^{(2)} - \sum_{r+s=k+1} m_{rs}^{(2)} \quad (2 \leq r, s \leq e).$$

By (3.26), the right-hand side of the above reduces to  $m_k$ , and the proof is complete.

**THEOREM 3.3.** *Let  $A$  be an abelian normal subgroup of a group  $G$  of exponent  $p^e$ . Then*

$$(3.29) \quad (a, x; k(p^k - p^{k-1}))^{p^e - k} \in G_m \quad \text{for all } a \in A, x \in G,$$

where

$$m = k(p^k - p^{k-1}) + 2.$$

*Proof.* As indicated in the comment above equation (3.23), we wish to apply Theorem 3.2 with  $I = J$  as described in (3.6) and the  $f_i(X)$  of (3.17) being the  $f_i(X)$  of (3.7), i.e.,

$$A_j = p^{e-j}, \quad a_{ij} = a_{ij}(X), \quad m_{ij} = p^{i+j-1} - p^{i-1}.$$

The only hypotheses of Theorem 3.2 that are not trivial to verify are that the  $a_{ij}(X)$  are invertible, and that the invertibility of  $a(X)$  implies that of  $a(X) + Xb(X)$ . Both of these facts will follow if we can show that if  $a(X)$  has constant coefficient prime to  $p$ , then  $a(X)$  is invertible (mod  $J$ ).

By (3.7) with  $i = 1$  (recall that  $a_{ie}(X) = 1$ ),

$$(3.30) \quad X^{p^e-1} \equiv ph(X) \pmod{J}, \quad h(X) \in Z[X].$$

Raising each side to the  $e$ th power and observing (see (3.4) and (3.6)) that  $p^e \in J$ , we obtain

$$(3.31) \quad X^s \in J,$$

for  $s = e(p^e - 1)$ . (In general, this is not the minimal  $s$  for which (3.31) is true.) Hence if

$$a(X) = a_0 + a_1 X + \dots + a_j X^j \in Z[X], \quad (a_0, p) = 1,$$

then one can find  $a'(X)$  such that

$$a(X)a'(X) \equiv 1 \pmod{J}$$

by solving a finite collection of congruences of the form

$$a_0 z \equiv w \pmod{p^e} \quad (w \in Z)$$

(which is possible since  $(a_0, p^e) = 1$ ); see **(1)** for a similar argument.

**4.** Let  $G$  be a group of exponent  $p^e$ , let  $K(y, x)$  denote the subgroup of  $G$  generated by  $y$  and  $x$ , and let  $K_{m,n}(y, x)$  denote the subgroup of  $K(y, x)$  generated by all (complex) commutators built up from at least  $m$   $y$ 's and at least  $n$   $x$ 's (and nothing else) (**2**, p. 138). It is not difficult to show that  $K_{m,n}(y, x)$  is normal in  $K(y, x)$ .

**LEMMA 4.1.** *Let  $G$  be a group of exponent  $p^e$  and let  $d$  be a positive integer divisible by  $p - 1$ . Let  $t$  be an arbitrary positive integer. If*

$$(4.1) \quad (y, x; d)^t \in K_{1,d+1}(y, x)K_{2,1}(y, x) \quad \text{for all } y, x \in G,$$

then

$$(4.2) \quad (y, x; d)^t \in K_{1,d+1}(y, x)K_{p,p-1}(y, x)K_{2,d}(y, x) \quad \text{for all } y, x \in G.$$

*Proof.* Assume inductively that



$$(y, x; d)^t \in K_{1,d+1}(y, x)K_{n,1}(y, x)K_{2,d}(y, x) \quad (2 \leq n \leq p - 1).$$

Then

$$(4.3) \quad (y, x; d)^t \equiv \prod k_i \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)},$$

where the  $k_i$  are complex commutators in  $x$  and  $y$  containing exactly  $n$   $y$ 's. Let  $m$  be a primitive root  $(\text{mod } p)$ ; i.e.,

$$(4.4) \quad m^{p-1} \equiv 1 \pmod{p}, \quad m^j \not\equiv 1 \pmod{p} \text{ if } j < p - 1.$$

Replace  $y$  by  $y^m$  in (4.3). Observe that repeated use of (2.1) and (2.2) yields

$$K_{r,s}(y^m, x) \leq K_{r,s}(y, x).$$

Further use of (2.1) and (2.2) yields

$$(4.5) \quad [(y, x; d)^t]^m \equiv (\prod k_i)^{m^n} \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)}.$$

Next raise both sides of (4.3) to the  $m^n$ th power to get

$$(4.6) \quad [(y, x; d)^t]^{m^n} \equiv (\prod k_i)^{m^{n^2}} \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)}.$$

A comparison of (4.5) and (4.6) yields

$$[(y, x; d)^t]^{m^n - m} \equiv 1 \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)}.$$

But, by (4.4),  $m^n - m$  is prime to  $p$  and hence to  $p^e$ . This means that

$$(y, x; d)^t \equiv 1 \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)}.$$

By induction

$$(4.7) \quad (y, x; d)^t \equiv 1 \pmod{K_{1,d+1}(y, x)K_{p,1}(y, x)K_{2,d}(y, x)}.$$

Next we proceed from (4.7) to the inductive assumption

$$(y, x; d)^t \equiv 1 \pmod{K_{1,d+1}(y, x)K_{p,n}(y, x)K_{2,d}(y, x)} \quad (1 \leq n \leq p - 2).$$

Then

$$(4.8) \quad (y, x; d)^t \equiv \prod k_i \pmod{K_{1,d+1}(y, x)K_{p,n+1}(y, x)K_{2,d}(y, x)},$$

where the  $k_i$  are complex commutators in  $x$  and  $y$  containing exactly  $n$   $x$ 's. Choose  $m$  as in (4.4) and replace  $x$  by  $x^m$  in (4.8). Repeat the previous argument to obtain

$$[(y, x; d)^t]^{m^d - m^n} \equiv 1 \pmod{K_{1,d+1}(y, x)K_{p,n+1}(y, x)K_{2,d}(y, x)}$$

and hence, using the divisibility of  $d$  by  $p - 1$ ,

$$(y, x; d)^t \equiv 1 \pmod{K_{1,d+1}(y, x)K_{p,n+1}(y, x)K_{2,d}(y, x)}.$$

Induction now yields (4.2).

**5.** Given the group  $G$  of exponent  $p^e$ , we are to prove (1.1). We begin by taking arbitrary elements  $y, x \in G$  and forming the subgroup  $K = K(y, x)$

generated by them. We define  $K_{m,n}$  as in §4 and verify that  $K_{1,0}K_{2,1}$  is an abelian normal subgroup of  $K/K_{2,1}$ . Hence, by (3.29),

$$(y, x; k(p^k - p^{k-1}))^{p^{e-k}} \in K_{k(p^k - p^{k-1})+2} K_{2,1}.$$

Since  $K_r$  is generated by complex commutators in  $x$  and  $y$  of weight  $r$  and greater,

$$K_{n+2} K_{2,1} = K_{1,n+1} K_{2,1} = K_{1,n+1} K_{2,1} K_{2,n}$$

for all positive integers  $n$ . Hence

$$(y, x; k(p^k - p^{k-1}))^{p^{e-k}} \in K_{1,k(p^k - p^{k-1})+1} K_{2,1}.$$

Since  $k(p^k - p^{k-1})$  is divisible by  $p - 1$ , we can apply Lemma 4.1 to obtain

$$(y, x; k(p^k - p^{k-1}))^{p^{e-k}} \in K_{1,k(p^k - p^{k-1})+1} K_{p,p-1} K_{2,k(p^k - p^{k-1})}.$$

In particular, if  $y = y_m \in G_m$ , then

$$(y_m, x; k(p^k - p^{k-1}))^{p^{e-k}} \in K_q \leq G_q,$$

where  $q$  is given in (1.4).

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