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INHOMOGENEOUS PRODUCTS OF CYCLIC IRREDUCIBLE NONNEGATIVE MATRICES

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Abstract

The now well-known Coale–Lopez theorem, and variants of it, state that, under certain conditions, the product of the first r members of an infinite inhomogeneous sequence of nonnegative matrices approaches, as $r \to \infty$, the class of matrices which, apart from scalar multiples, have only one distinct column. The aim of the present paper is to lay down conditions under which such products approach the class of matrices which, apart from scalar multiples, have no more than d distinct columns. A stronger result is then obtained by considering stochastic matrices instead of just nonnegative ones.

1. Terminology

Before stating and attempting to accomplish our aims, it will be apposite to review some of the standard terminology surrounding nonnegative matrices. In doing this, we shall follow the terminology described in the recent elegant exposition by Seneta (1973).

DEFINITIONS. A matrix $A = (a_{ij})$ is said to be *nonnegative* if $a_{ij} \geqq 0$ for each (i, j). Write $A \geqq 0$. The matrix is said to be *positive* if $a_{ij} > 0$ for each (i, j). Write $A > 0$. The *period of an index* i of a nonnegative matrix A is the gcd of those positive integers k for which the (i, i)-element of A^k is > 0. A nonnegative matrix A is *irreducible* if, for any given pair i, j of its index set, there exists a positive integer k for which the (i, j)-element of A^k is > 0.

It can be shown (see e.g. Seneta, 1973, 14) that, in an irreducible nonnegative matrix, all members of the index set have the same period. This enables the next

DEFINITION. An irreducible nonnegative matrix is said to be *cyclic with period* d if one of (and therefore each of) its indexes has period d (> 1). If

$d = 1$, the matrix is called *acyclic*. An acyclic irreducible nonnegative matrix is sometimes called a *primitive* matrix.

In what follows the *norm* $\|A\|$ of an $n \times n$ matrix $A = (a_{ij})$ will have the meaning:

$$\|A\| = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.$$

2. Some standard results

One standard result to which we shall have recourse is the fact that any irreducible nonnegative matrix has a canonical form in which primitive matrices play an important role. For the present purposes, the important result is as stated in the following. It summarizes the relevant portions of Seneta (1973, 15–7).

THEOREM 0.1. *If A is a cyclic matrix with period d, then by a permutation of its index set A can be put in a canonical form A_c such that A_c^d has the following partitioned form*:

$$A_c^d = \begin{bmatrix} Z^{(1)} & & & \\ & Z^{(2)} & & \\ & & \ddots & \\ & & & Z^{(d)} \end{bmatrix}$$

where each of the blocks $Z^{(1)}, \cdots, Z^{(d)}$ is a primitive matrix.

Let us now consider a sequence $A^{(1)}, A^{(2)}$, etc. of $n \times n$ matrices. We can form products from this sequence:

$$B^{(p,r)} = A^{(p+r)} A^{(p+r-1)} \cdots A^{(p+1)}.$$

If the $A^{(k)}$ are not all equal we refer to these products as *inhomogeneous*.

Now a considerable amount of work has been done on the properties of these inhomogeneous products. The general form of the results which have been obtained is contained in Theorem 0.2 below. The theorem is essentially due to Coale and Lopez (Lopez, 1961) in a demographic context, but is stated here in the somewhat generalized form given by Seneta (1973, 69).

THEOREM 0.2 (*Coale–Lopez*). *Let $A^{(k)}$ be a sequence of $n \times n$ nonnegative matrices satisfying the two conditions*:

(i) $B^{(p,r)} = A^{(p+r)} \cdots A^{(p+1)} > 0$ *for $p \geqq 0$ and $r \geqq r_0$, where $r_0 \geqq 1$ is some fixed integer independent of p*;

(ii) $\min^+_{i,j} a_{ij}^{(k)} \geqq \lambda > 0$ *and* $\max_{i,j} a_{ij}^{(k)} \leqq \gamma < \infty$ *uniformly for* $k = 1, 2, \cdots$, *where* $\min^+$ *refers to the minimum among all positive entries.*

Then as $r \to \infty$, *for all* i, j, p, s:

$$\frac{b_{is}^{(p,r)}}{b_{js}^{(p,r)}} \to V_{ij}^{(p)} > 0,$$

where the limit $V_{ij}^{(p)}$ *is independent of* s.

The general idea of this theorem is that, for any fixed p, the sequence $B^{(p,r)}$ converges in some sense to the class of matrices of rank one. A precise statement of the result in this form is given by Parlett (1970):

THEOREM 0.3 (*Alternative statement of Theorem* 0.2). *If the hypotheses of Theorem* 0.2 *hold, then, for each fixed* p, *there exist sequences* $\{G^{(p,r)}\}$ *and* $\{H^{(p,r)}\}$ *of matrices with the following three properties*:

(a) $B^{(p,r)} = G^{(p,r)} + H^{(p,r)}$;

(b) $G^{(p,r)}$ *is positive with rank* 1;

(c) $\|H^{(p,r)}\| / \|G^{(p,r)}\| \to 0$ *as* $r \to \infty$.

Theorems such as 0.2 and 0.3 are called *weak ergodic theorems.* Wolfowitz (1963) proved such a theorem for the case where the $A^{(k)}$ were chosen from a finite set of stochastic matrices. Hajnal (1956, 1958) also dealt with stochastic matrices but under more general conditions. The whole subject of weak ergodicity is thoroughly surveyed by Seneta (1973, chapters 3 and 4).

As a motivation of the next section, it is instructive to examine a simple corollary of Theorem 0.2. If condition (i) is replaced by the stronger condition that the $A^{(k)}$ be all primitive with identical graphs, then the conclusion of the theorem still holds. Now primitive matrices have period 1, and this theorem says that inhomogeneous products of them tend to the class of matrices of rank 1. Is there any relation between period and rank? If we were to replace the primitive matrices with cyclic matrices of period d, would the inhomogeneous products of them tend to the class of matrices of rank d?

3. A weak ergodic theorem for cyclic irreducible nonnegative matrices

In this section, we answer the questions raised at the end of the last section.

THEOREM 1. *Let* $\{A^{(k)}\}$ *be a sequence of* $n \times n$ *nonnegative matrices satisfying the two conditions*:

(i) *for each fixed* p, *there exist positive integers* d, r_0 (*independent of* p) *and a matrix* M *of period* d *such that, for* $r \geqq r_0$ *and* $0 \leqq s < d$, $B^{(p,rd+s)} = A^{(p+rd+s)} \cdots A^{(p+1)}$ *has the same graph as* M^{rd+s};

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(ii) $\min^{+}_{i,j} a^{(k)}_{ij} \geqq \lambda > 0$ *and* $\max_{i,j} a^{(k)}_{ij} \leqq \gamma < \infty$ *uniformly for* $k = 1, 2, \ldots$.

Then, for each fixed p, *there exist sequences* $\{G^{(p,r)}\}$ *and* $\{H^{(p,r)}\}$ *of matrices with the following three properties*:

(a) $B^{(p,r)} = G^{(p,r)} + H^{(p,r)}$;

(b) $G^{(p,r)}$ *is nonnegative and, apart from scalar multiples, has no more than* d *distinct columns*;

(c) $\|H^{(p,r)}\| / \|G^{(p,r)}\| \to 0$ *as* $r \to \infty$.

Proof. Without loss of generality, we may assume M to be in the canonical form described in Theorem 0.1, whereupon we can write, for some $q \geqq r_0$,

$$M^{qd} = \begin{bmatrix} Z^{(1)} & & & & \\ & Z^{(2)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & Z^{(d)} \end{bmatrix}$$

where $Z^{(1)}, \cdots, Z^{(d)}$ are primitive. By condition (i), it then follows that, for any given p, we have

$$B^{(p,qd)} = \begin{bmatrix} Z^{(1,p,qd)} & & & & \\ & Z^{(2,p,qd)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & Z^{(d,p,qd)} \end{bmatrix}, \tag{1}$$

with $Z^{(1,p,qd)}, \cdots, Z^{(d,p,qd)}$ primitive. We note also that the graph of $Z^{(l,p,qd)}$ is independent of p.

It follows from (1) that

$$B^{(p,tqd)} = B^{(p+(t-1)qd,qd)} B^{(p+(t-2)qd,qd)} \cdots B^{(p,qd)}$$

$$= \begin{bmatrix} Z^{(1,p,tqd)} & & & & \\ & Z^{(2,p,tqd)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & Z^{(d,p,tqd)} \end{bmatrix} \tag{2}$$

where

$$Z^{(l,p,tqd)} = Z^{(l,p+(t-1)qd,qd)} Z^{(l,p+(t-2)qd,qd)} \cdots Z^{(l,p,qd)}. \tag{3}$$

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Since all of the matrices on the right side of (3) are primitive with identical graphs we have immediately, for each p,

(4) $$Z^{(l,p,tqd)} > 0 \text{ for all } t \geqq \text{some } t_0.$$

It is easy to see that

(5) $$\min_{i,j}{}^{+} z_{ij}^{(l,p,qd)} \geqq \lambda^{qd} > 0,$$

and

(6) $$\max_{i,j} z_{ij}^{(l,p,qd)} \leqq (n\gamma)^{qd} < \infty$$

uniformly over l and p for fixed q.

Considering now that sequence of matrices $Z^{(l,p,qd)}, Z^{(l,p,2qd)}, Z^{(l,p,3qd)}, \cdots$, we see, after comparing (4), (5) and (6) with conditions (i) and (ii) of Theorem 0.3, that that theorem can be applied in the present context to give, for fixed l, p and q, sequences $\{X^{(l,p,tqd)}\}$ and $\{Y^{(l,p,tqd)}\}$ of matrices with the following three properties:

(7) $$Z^{(l,p,tqd)} = X^{(l,p,tqd)} + Y^{(l,p,tqd)};$$

(8) $$X^{(l,p,tqd)} \text{ is nonnegative with rank } 1;$$

(9) $$Y^{(l,p,tqd)}/\|X^{(l,p,tqd)}\| \to 0 \text{ as } t \to \infty.$$

We can now define:

(10) $$G^{(p,tqd)} = \begin{bmatrix} X^{(1,p,tqd)} & & & \\ & X^{(2,p,tqd)} & & \\ & & \ddots & \\ & & & X^{(d,p,tqd)} \end{bmatrix}$$

and

(11) $$H^{(p,tqd)} = \begin{bmatrix} Y^{(1,p,tqd)} & & & \\ & Y^{(2,p,tqd)} & & \\ & & \ddots & \\ & & & Y^{(d,p,tqd)} \end{bmatrix}$$

By (2), (10) and (11),

(12) $$B^{(p,tqd)} = G^{(p,tqd)} + H^{(p,tqd)}.$$

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By (8) and (10),

(13) $G^{(p,tqd)}$ is nonnegative and, apart from scalar multiples, has only d distinct columns.

By (10) and (11),

$$\|H^{(p,tqd)}\|/\|G^{(p,tqd)}\| = \sum_{l=1}^{d} \|Y^{(l,p,tqd)}\|/\sum_{l=1}^{d} \|X^{(l,p,tqd)}\|,$$

whence it follows from (9) that

$$\|H^{(p,tqd)}\|/\|G^{(p,tqd)}\| \to 0 \text{ as } t \to \infty. \tag{14}$$

In (12), (13) and (14), we have proved the theorem in respect of those $B^{(p,r)}$ for which r is a multiple of qd. For $r = tqd + s$, $0 < s < qd$, define

$$G^{(p,r)} = B^{(p+tqd,s)}G^{(p,tqd)}, H^{(p,r)} = B^{(p+tqd,s)}H^{(p,tqd)}. \tag{15}$$

Since $B^{(p+tqd,s)}$ is a product of at most qd matrices, it is bounded above and below by a matrix of form ρU where ρ is a finite positive number independent of t and U is $n \times n$ and consists of one's (e.g. $B^{(p-tqd,s)} > \min_{i,j} b_{ij}^{(p+tqd,s)} U > \lambda^s U > \lambda^{qd} U$ if $\lambda < 1$). However $\|UA\| = n\|A\|$ if A is non-negative and (c) is immediate.

4. A weak ergodic theorem for cyclic irreducible stochastic matrices

In this section we restrict the set of matrices considered in Section 3 further by requiring that they be stochastic, i.e. that they have all row sums equal to unity. This enables us to obtain results which are, in some respects, stronger than Theorem 1. This is not surprising in view of the fact that the Coale–Lopez theorem can be strengthened when the matrices under consideration are stochastic instead of just nonnegative. The main result here is due to Hajnal (1958).

For a stochastic matrix A, define

$$\delta(A) = \max_{\beta} \max_{\alpha,\alpha'} |a_{\alpha\beta} - a_{\alpha'\beta}|, \tag{21}$$

and

$$\lambda(A) = \min_{\alpha,\alpha'} \sum_{\beta} \min(a_{\alpha\beta}, a_{\alpha'\beta}). \tag{22}$$

The value $\lambda(A)$ is called the *scrambling power* of A, whilst $\delta(A)$ is, in informal terms, a measure of amount by which A fails to have rank one.

THEOREM 0.4 (*Hajnal*). *If P, G are $n \times n$ stochastic matrices and $F = PG$ then $\delta(F) \leqq (1 - \lambda(P))\delta(G)$.*

THEOREM 0.5 (*Hajnal*). *Let* $\{A^{(k)}\}$ *be a sequence of* $n \times n$ *stochastic matrices. We have*

$$\delta(A^{(1)}\cdots A^{(k)}) \to 0 \text{ as } k \to \infty,$$

if

(i) $\prod_{k=1}^{\infty} (1-\lambda(A^{(k)}))$ *diverges to zero;*

or equivalently

(ii) $\sum_{k=1}^{\infty} \lambda(A^{(k)})$ *diverges.*

For our present purposes we find it advantageous to extend the definitions of $\delta(A)$ and $\lambda(A)$ to non-square A. If A is *any nonnegative matrix with all row sums equal to unity*, we define $\delta^*(A)$ and $\lambda^*(A)$ to be the right sides of (21) and (22) respectively.

With this terminology, we can establish

THEOREM 2. *Let* $\{A^{(k)}\}$ *be a sequence of* $n \times n$ *stochastic matrices, and suppose that the index set of each can be subdivided into d mutually exclusive and exhaustive subsets* $\mathscr{C}_1, \cdots, \mathscr{C}_d$ *(independent of k) such that, for any given k and i,* $A^{(k)}$ *sends* $\mathscr{C}_i$ *into some* $\mathscr{C}_j$ *(j possibly varying with k). Let* $A^{(k)}_{(ij)}$ *denote the submatrix of* $A^{(k)}$ *relating to the transition from* $\mathscr{C}_i$ *to* $\mathscr{C}_j$, *and suppose that*

$$\sum_k \min\{\lambda^*(A^{(k)}_{(ij)}) \mid A^{(k)}_{(ij)} \neq 0\} \text{ diverges.} \tag{23}$$

Define $C^{(p,r)} = A^{(p+1)} \cdots A^{(p+r)}$.

Then, for each fixed p, there exist sequences $G^{(p,r)}$ *and* $H^{(p,r)}$ *of matrices with following three properties:*

(a) $C^{(p,r)} = G^{(p,r)} + H^{(p,r)}$;

(b) $G^{(p,r)}$ *is a stochastic matrix with no more than d distinct rows;*

(c) $\|H^{(p,r)}\| \to 0$ *as* $r \to \infty$.

PROOF. We begin by fixing p arbitrarily. It is clear from the hypotheses of the theorem that, for any given r and i, $C^{(p,r)}$ sends $\mathscr{C}_i$ to some $\mathscr{C}_{j_r}$ and moreover that, if $C^{(p,r)}_{(ij_r)}$ is the submatrix relating to the transition from $\mathscr{C}_i$ to $\mathscr{C}_{j_r}$, then

$$C^{(p,r)}_{(ij_r)} = A^{(p+1)}_{ij_1} A^{(p+2)}_{j_1 j_2} \cdots A^{(p+r)}_{j_{r-1} j_r}.$$

We now wish to show that $\delta^*(C^{(p,r)}_{(ij_r)}) \to 0$ as $r \to \infty$. This is quite simple upon a review of Hajnal's (1958) work, an examination of which (in particular, Lemma 3, pp. 237–8) shows that Theorem 0.4 holds with δ, λ replaced by δ^*, λ^* and P, G any nonnegative matrices with all rows sums equal to unity. This immediately yields the following result, analogous to Theorem 0.5:

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$$\delta^*(C^{(p,r)}_{(ij_r)}) \leqq \prod_{s=1}^{r-1} [1 - \lambda^*(A^{(p+s)}_{j_{s-1}j_s})]\delta^*(A^{(p+r)}_{j_{r-1}j_r})$$

$$\leqq \prod_{s=1}^{r-1} [1 - \lambda^*(A^{(p+s)}_{j_{s-1}j_s})],$$

where $j_0 = i$.
Thus,

$$\delta^*(C^{(p,r)}_{(ij_r)}) \leqq \prod_{s=1}^{r-1} [1 - \min\{\lambda^*(A^{(p+s)}_{(uv)}) \mid A^{(p+s)}_{(uv)} \neq 0\}],$$

which diverges to zero by the hypothesis (23). We have now proved that

(24) $$\delta^*(C^{(p,r)}_{(ij_r)}) \to 0 \text{ as } r \to \infty.$$

We now construct $G^{(p,r)}$ as follows. Within $C^{(p,r)}$ replace $C^{(p,r)}_{(ij_r)}$ by the matrix whose rows are all equal to the first row of $C^{(p,r)}_{(ij_r)}$. Do this for each $i\ (= 1, 2, \cdots, d)$, and call the resulting matrix $G^{(p,r)}$. It is immediately clear that condition (b) of the theorem is satisfied.

Now define $H^{(p,r)}$ so that condition (a) is satisfied and, from (24), we have

$$\|H^{(p,r)}\| = \|C^{(p,r)} - G^{(p,r)}\|$$

$$\leqq n^2 \max_i \delta^*(C^{(p,r)}_{ij_r}).$$

Hence $\|H^{(p,r)}\| \to 0$, by (24) and so condition (c) holds.

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