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SOBOLEV INEQUALITIES FOR RIESZ POTENTIALS OF FUNCTIONS IN $L^{p(\cdot)}$ OVER NONDOUBLING MEASURE SPACES

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Dedicated to Professor Noriaki Suzuki on the occasion of his sixtieth birthday

Abstract

Our aim in this paper is to deal with Sobolev inequalities for Riesz potentials of functions in Lebesgue spaces of variable exponents near Sobolev's exponent over nondoubling metric measure spaces.

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1. Introduction

In the present paper, we are concerned with Sobolev inequalities for Riesz potentials of functions in Lebesgue spaces of variable exponents near Sobolev's exponent in the nondoubling setting.

Sobolev space is a useful tool for studying the existence and regularity of solutions of partial differential equations. For $0 < \alpha < n$, we define the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^n$ by

$$U_\alpha f(x) = \int_{\mathbf{R}^n} |x - y|^{\alpha - n} f(y)\, dy.$$

Here it is natural to assume that

$$\int_{\mathbf{R}^n} (1 + |y|)^{\alpha - n} |f(y)|\, dy < \infty$$

(see [9, Theorem 1.1, Ch. 2]). The famous Sobolev inequality says that the Riesz potential $U_\alpha f$ of order α with $f \in L^p(\mathbf{R}^n)$ belongs to $L^{p^*}(\mathbf{R}^n)$ when $1 < p < \infty$ and $1/p^* = 1/p - \alpha/n > 0$.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with nonstandard growth condition. For a survey, see [2, 4].

Let G be a bounded Borel set in $\mathbf{R}^n$. Let $p(\cdot) : G \to (1, \infty)$ be a variable exponent satisfying the log-Hölder conditions on G. Denote by $L^{p(\cdot)}(G)$ the family of all measurable functions f on G such that

$$\|f\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G |f(x)/\lambda|^{p(x)}\, dx \le 1\right\} < \infty.$$

If $p(x) < n/\alpha$, then we set

$$1/p^*(x) = 1/p(x) - \alpha/n.$$

We take the following result from [6]. There are related results in [5, 10]. The case $\sup_{x\in G} p(x) < n/\alpha$ was shown in [3].

Theorem 1.1 [6]. *Let $p(\cdot)$ be a variable exponent on G satisfying log-Hölder conditions on G such that*

$$1 < p^- := \inf_{x\in G} p(x) \le p(x) < n/\alpha$$

for $x \in G$. Then there exists a constant $c > 0$ such that

$$\|\gamma(\cdot)^{-1} U_\alpha f\|_{L^{p^*(\cdot)}(G)} \le c\|f\|_{L^{p(\cdot)}(G)}$$

for all $f \in L^{p(\cdot)}(G)$, where

$$\gamma(x) = p^*(x)^{(p(x)-1)/p(x)}.$$

We denote by (X, d, μ) a metric measure space, where X is a bounded set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centred at x with radius r and we set $d_X = \sup\{d(x, y) : x, y \in X\}$. We assume for simplicity that $0 < d_X < \infty$, $\mu(\{x\}) = 0$ for $x \in X$ and $\mu(B(x, r)) > 0$ for $x \in X$ and $r > 0$.

In the present paper, we do not postulate on μ the so-called 'doubling condition'. Recall that a Radon measure μ is said to be doubling if there exists a constant $c_0 > 0$ such that $\mu(B(x, 2r)) \le c_0\mu(B(x, r))$ for all $x \in \mathrm{supp}(\mu)\,(= X)$ and $r > 0$. Otherwise μ is said to be nondoubling. We say that a measure μ is lower Ahlfors Q-regular if there exists a constant $c_1 > 0$ such that

$$\mu(B(x, r)) \ge c_1 r^Q \tag{1.1}$$

for all $x \in X$ and $0 < r < d_X$. We assume here that μ is lower Ahlfors Q-regular. Note that if μ is a doubling measure and $d_X < \infty$, then μ is lower Ahlfors $\log_2 c_0$-regular since

$$\frac{\mu(B(x, r))}{\mu(B(x, d_X))} \ge c_0^{-2}\left(\frac{r}{d_X}\right)^{\log_2 c_0}$$

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for all $x \in X$ and $0 < r < d_X$ (see, for example, [1, Lemma 3.3]). However, there exist lower Ahlfors measures which are nondoubling. For example, let

$$X_1 = \{x = (x_1, 0) \in \mathbf{R}^2 : 0 \le x_1 < 1\}, \quad X_2 = \{x = (x_1, x_2) \in \mathbf{R}^2 : |x| < 1, x_1 < 0\}$$

and define $(X, d, \mu) = (X_1, d_2, m_1) \cup (X_2, d_2, m_2)$, where d_2 denotes the two-dimensional Euclidean distance and m_i denotes the i-dimensional Lebesgue measure. It is easy to show that μ is nondoubling and lower Ahlfors 2-regular. For other examples of nondoubling metric measure spaces, see [13].

Our aim is to give a general version of Sobolev's inequality for Riesz potentials $I_{\alpha,\tau} f$ of functions in $L^{p(\cdot)}(X)$ on nondoubling metric measure spaces X (Theorem 3.1) as an extension of Theorem 1.1 (see Section 2 for the definitions of $I_{\alpha,\tau} f$ and $L^{p(\cdot)}(X)$). To this end, we apply Hedberg's trick (see Hedberg [8]) by the use of the Hardy–Littlewood maximal operator M_λ adapted to our setting (see Theorem 2.4).

For variable exponents attaining the value 1, we refer to [10, 14].

2. Boundedness of the maximal operator

Throughout this paper, let C denote various positive constants independent of the variables in question and $C(a, b, \ldots)$ be a constant that depends on $a, b, \ldots$.

We consider a variable exponent $p(\cdot)$ such that

$$|p(x) - p(y)| \le \frac{C_p}{\log(e + 1/d(x, y))} \quad \text{for all } x, y \in X \tag{2.1}$$

and

$$1 < p^- := \inf_{x \in X} p(x) \le \sup_{x \in X} p(x) =: p^+ < \infty,$$

where $C_p > 0$.

For $\alpha > 0$ and $\tau > 0$, we define the Riesz potential of order α for a locally integrable function f on X by

$$I_{\alpha,\tau} f(x) = \int_X \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau\, d(x, y)))}\, d\mu(y)$$

(see, for example, [7, 11]).

Denote by $L^{p(\cdot)}(X)$ the family of all measurable functions f on X such that

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\left\{\lambda > 0 : \int_X |f(x)/\lambda|^{p(x)}\, d\mu(x) \le 1\right\} < \infty.$$

If $p(x) < Q/\alpha$, then we set

$$1/p^*(x) = 1/p(x) - \alpha/Q.$$

For a locally integrable function f on X and $\lambda \ge 1$, the Hardy–Littlewood maximal function $M_\lambda f$ is defined by

$$M_\lambda f(x) = \sup_{r>0} \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)\cap X} |f(y)|\, d\mu(y).$$

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For $\lambda \geq 1$, we say that X satisfies $(M\lambda)$ if there exists a constant $C > 0$ such that

$$\mu(\{x \in X : M_\lambda f(x) > k\}) \leq \frac{C}{k} \int_X |f(y)|\, d\mu(y)$$

for all measurable functions $f \in L^1(X)$ and $k > 0$.

Remark 2.1. In [12], Sawano showed that X satisfies $(M\lambda)$ for $\lambda \geq 2$ if X is a separable metric space (see also [15]).

Lemma 2.2. *Let $1 < p_0 < \infty$ and $\lambda \geq 1$. Suppose that X satisfies $(M\lambda)$. Then there exists a constant $C > 0$ such that*

$$\int_X \{M_\lambda f(x)\}^{p_0}\, d\mu(x) \leq C$$

for all measurable functions f with $\|f\|_{L^{p_0}(X)} \leq 1$.

Lemma 2.3. *Let $\lambda \geq 1$ and let f be a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(X)} \leq 1$ such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in X$. Set*

$$I = I(x, r, f) = \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)\cap X} f(y)\, d\mu(y)$$

and

$$J = J(x, r, f) = \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)\cap X} f(y)^{p(y)}\, d\mu(y).$$

Then there exists a constant $C > 0$ such that

$$I \leq C J^{1/p(x)}$$

for all $x \in X$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(X)} \leq 1$ such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in X$. First consider the case when $J \geq 1$. Set $k = J^{1/p(x)}$. Then

$$I \leq k + \frac{1}{\mu(B(x, \lambda r))} \int_{B(x,r)\cap X} f(y) \left(\frac{f(y)}{k}\right)^{p(y)-1} d\mu(y).$$

Since $\|f\|_{L^{p(\cdot)}(X)} \leq 1$, we find by (1.1) that

$$J \leq \frac{1}{\mu(B(x, \lambda r))} \int_X f(y)^{p(y)}\, d\mu(y) \leq \frac{1}{\mu(B(x, \lambda r))} \leq c_1^{-1} \lambda^{-Q} r^{-Q}.$$

Hence, for $y \in B(x, r)$,

$$k^{-p(y)} \leq \{J^{1/p(x)}\}^{-p(x)+C_p/\log(e+1/r)} \leq \{J^{1/p(x)}\}^{-p(x)+C_p/\log(e+1/(CJ^{-1/Q}))} \leq CJ^{-1}.$$

It follows that

$$I \leq C J^{1/p(x)}.$$

In the case $J \leq 1$, we find that

$$I \leq J \leq J^{1/p(x)}.$$

Consequently, the result follows. □

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Now we are ready to show the boundedness of the maximal operator M_λ.

Theorem 2.4. *Let $\lambda \geq 1$. Suppose that X satisfies (Mλ). Then there exists a constant $c_M > 0$ such that*

$$\int_X \{M_\lambda f(x)\}^{p(x)}\, d\mu(x) \leq c_M$$

for all measurable functions f with $\|f\|_{L^{p(\cdot)}(X)} \leq 1$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(X)} \leq 1$. Write

$$f = f\chi_{\{y:f(y)\geq 1\}} + f\chi_{\{y:f(y)<1\}} = f_1 + f_2,$$

where χ_E denotes the characteristic function of E. Since $Mf_2 \leq 1$ on X, we see from Lemma 2.3 that

$$\{M_\lambda f(x)\}^{p(x)} \leq C\{1 + M_\lambda g(x)\},$$

where $g(y) = f(y)^{p(y)}$. Now take p_1 such that $1 < p_1 < p^-$. Applying the above inequality with $p(x)$ replaced by $p(x)/p_1$,

$$\{M_\lambda f(x)\}^{p(x)} \leq C\{1 + \{M_\lambda g_1(x)\}^{p_1}\},$$

where $g_1(y) = f(y)^{p(y)/p_1}$. By Lemma 2.2,

$$\int_X \{M_\lambda f(x)\}^{p(x)}\, d\mu(x) \leq c_M,$$

as required. □

3. Sobolev's inequality

As an application of the maximal operator, we establish Sobolev's inequality for Riesz potentials of functions in Lebesgue spaces of variable exponents near Sobolev's exponent in the nondoubling setting.

Theorem 3.1. *Let $\tau > \lambda \geq 1$. Suppose that $p(x) < Q/\alpha$ for $x \in X$ and X satisfies (Mλ). Then there exists a constant $c > 0$ such that*

$$\|\gamma(\cdot)^{-1} I_{\alpha,\tau} f\|_{L^{p^*(\cdot)}(X)} \leq c\|f\|_{L^{p(\cdot)}(X)}$$

for all $f \in L^{p(\cdot)}(X)$, where

$$\gamma(x) = p^*(x)^{(p(x)-1)/p(x)}.$$

Remark 3.2 ([10, Remark 3.2]). Let G be an open bounded set in $\mathbf{R}^n$. For $0 < \delta < 1$, we can find $f \in L^{p(\cdot)}(G)$ such that

$$\int_G \{\gamma(\cdot)^{-\delta} U_\alpha f(x)\}^{p^*(x)}\, dx = \infty,$$

so that the weight $\gamma(\cdot)^{-1}$ in Theorem 3.1 is needed.

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Lemma 3.3. *Let $\tau > 1$ and let f be a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(X)} \le 1$. Suppose that $p(x) < Q/\alpha$ for $x \in X$. Then there exists a constant $C > 0$ such that*

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau\, d(x,y)))}\, d\mu(y) \le C\gamma(x)\delta^{\alpha - Q/p(x)}$$

for all $x \in X$ and $0 < \delta < 1/2$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(X)} \le 1$.

First we consider the case $p^*(x)^{-1/Q} \le \delta$. By (1.1),

$$\begin{aligned}\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau\, d(x,y)))}\, d\mu(y) &\le \tau^{-Q} c_1^{-1} \int_{X\setminus B(x,\delta)} d(x,y)^{\alpha-Q} f(y)\, d\mu(y)\\ &\le \tau^{-Q} c_1^{-1} \delta^{\alpha-Q} \int_X f(y)\, d\mu(y)\\ &\le \tau^{-Q} c_1^{-1} \delta^{\alpha-Q} \int_X \{1 + f(y)^{p(y)}\}\, d\mu(y)\\ &\le C\gamma(x)\delta^{\alpha-Q/p(x)}.\end{aligned}$$

Next we consider the case $p^*(x)^{-1/Q} > \delta$. Note that

$$\int_{X\setminus B(x,p^*(x)^{-1/Q})} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau\, d(x,y)))}\, d\mu(y) \le Cp^*(x)^{1-\alpha/Q} \le C\gamma(x)\delta^{\alpha-Q/p(x)}.$$

On setting $\eta(x) = p^*(x)^{-1/p(x)}$ and $N(x,y) = d(x,y)^{-Q/p(x)}$,

$$\begin{aligned}&\int_{B(x,p^*(x)^{-1/Q})\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau\, d(x,y)))}\, d\mu(y)\\ &\quad\le \int_{B(x,p^*(x)^{-1/Q})\setminus B(x,\delta)} \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))}\{\eta(x)N(x,y)\}\, d\mu(y)\\ &\qquad + \int_{B(x,p^*(x)^{-1/Q})\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\Big(\frac{f(y)}{\eta(x)N(x,y)}\Big)^{p(y)-1}\, d\mu(y)\\ &\quad= J_1 + J_2.\end{aligned}$$

Let j_0 be the smallest positive integer such that $\tau^{j_0}\delta \ge p^*(x)^{-1/Q}$. We obtain

$$\begin{aligned}J_1 &\le \sum_{j=1}^{j_0} \int_{X\cap(B(x,\tau^j\delta)\setminus B(x,\tau^{j-1}\delta))} \frac{d(x,y)^\alpha}{\mu(B(x,\tau\, d(x,y)))}\{\eta(x)N(x,y)\}\, d\mu(y)\\ &\le \eta(x) \sum_{j=1}^{j_0} \int_{X\cap(B(x,\tau^j\delta)\setminus B(x,\tau^{j-1}\delta))} \frac{(\tau^{j-1}\delta)^{\alpha-Q/p(x)}}{\mu(B(x,\tau^j\delta))}\, d\mu(y)\\ &\le C\eta(x) \sum_{j=1}^{j_0} (\tau^j\delta)^{\alpha-Q/p(x)}.\end{aligned}$$

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Since $p(x) < Q/\alpha$,

$$J_1 \le \frac{C}{\log\tau}\eta(x)\int_\delta^{2d_X} t^{\alpha-Q/p(x)}\frac{dt}{t} \le \frac{C}{\log\tau}\eta(x)(Q/p(x)-\alpha)^{-1}\delta^{\alpha-Q/p(x)}$$
$$\le C\gamma(x)\delta^{\alpha-Q/p(x)}.$$

Next we estimate J_2. From (2.1), for $y \in B(x, p^*(x)^{-1/Q})$,

$$\{\eta(x)N(x,y)\}^{-p(y)} \le C\eta(x)^{-p(x)}d(x,y)^Q.$$

Therefore, by (1.1),

$$\begin{aligned} J_2 &\le \tau^{-Q}c_1^{-1}\int_{B(x,p^*(x)^{-1/Q})\setminus B(x,\delta)} d(x,y)^{\alpha-Q}\Big(\frac{1}{\eta(x)N(x,y)}\Big)^{p(y)-1} f(y)^{p(y)}\,d\mu(y) \\ &\le C\eta(x)^{1-p(x)}\int_{B(x,p^*(x)^{-1/Q})\setminus B(x,\delta)} d(x,y)^{\alpha-Q/p(x)}f(y)^{p(y)}\,d\mu(y) \\ &\le C\gamma(x)\delta^{\alpha-Q/p(x)}\int_{B(x,p^*(x)^{-1/Q})\setminus B(x,\delta)} f(y)^{p(y)}\,d\mu(y) \\ &\le C\gamma(x)\delta^{\alpha-Q/p(x)}, \end{aligned}$$

which proves the lemma. □

PROOF OF THEOREM 3.1. Suppose that f is a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(X)} \le 1$. By Lemma 3.3,

$$\begin{aligned} I_{\alpha,\tau}f(x) &= \int_{B(x,\delta)}\frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) + \int_{X\setminus B(x,\delta)}\frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau\, d(x,y)))}\,d\mu(y) \\ &\le C\{\delta^\alpha M_\lambda f(x) + \gamma(x)\delta^{\alpha-Q/p(x)}\} \end{aligned}$$

for $0 < \delta < 1/2$, since

$$\begin{aligned} &\int_{B(x,\delta)}\frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau\, d(x,y)))}\,d\mu(y) \\ &\quad\le \sum_{j=1}^\infty \int_{X\cap(B(x,(\tau/\lambda)^{-j+1}\delta)\setminus B(x,(\tau/\lambda)^{-j}\delta))}\frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau\, d(x,y)))}\,d\mu(y) \\ &\quad\le \sum_{j=1}^\infty \int_{X\cap(B(x,(\tau/\lambda)^{-j+1}\delta)\setminus B(x,(\tau/\lambda)^{-j}\delta))}\frac{((\tau/\lambda)^{-j+1}\delta)^\alpha f(y)}{\mu(B(x,\lambda(\tau/\lambda)^{-j+1}\delta))}\,d\mu(y) \\ &\quad\le \frac{\delta^\alpha}{1-(\tau/\lambda)^{-\alpha}}M_\lambda f(x). \end{aligned}$$

Here note that there exists a constant $m > 0$ such that $t^{-p(x)/Q} < m/2$ for all $x \in X$ and $t \ge 1$. Considering

$$\delta = \frac{1}{m}(\gamma(x)^{-1}M_\lambda f(x))^{-p(x)/Q}$$

when $\gamma(x)^{-1}M_\lambda f(x) \geq 1$, we find that

$$I_{\alpha,\tau}f(x) \leq C\gamma(x)^{\alpha p(x)/Q}\{M_\lambda f(x)\}^{1-\alpha p(x)/Q}.$$

If $\gamma(x)^{-1}M_\lambda f(x) < 1$, then

$$I_{\alpha,\tau}f(x) \leq C\{M_\lambda f(x) + \gamma(x)\} \leq C\gamma(x)$$

for $\delta = \frac{1}{4}$.

Hence,

$$\gamma(x)^{-1}I_{\alpha,\tau}f(x) \leq C\{(\gamma(x)^{-1}M_\lambda f(x))^{1-\alpha p(x)/Q} + 1\} \leq C\{\{M_\lambda f(x)\}^{p(x)/p^*(x)} + 1\}$$

since $\gamma(x)^{-1/p^*(x)} \leq C$, so that there exists a constant $c > 0$ such that

$$\{c^{-1}\gamma(x)^{-1}I_{\alpha,\tau}f(x)\}^{p^*(x)} \leq \frac{1}{2C_0}\{\{M_\lambda f(x)\}^{p(x)} + 1\},$$

where $C_0 = \max\{c_M, \mu(X)\}$. By Theorem 2.4,

$$\int_X \{c^{-1}\gamma(x)^{-1}I_{\alpha,\tau}f(x)\}^{p^*(x)}\,d\mu(x) \leq \frac{1}{2C_0}\int_X \{\{M_\lambda f(x)\}^{p(x)} + 1\}\,d\mu(x) \leq 1,$$

which completes the proof. □

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