PARALLEL LINES ASSOCIATED WITH TWO SETS

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What conditions determine when a collection of points A lies on a collection of parallel lines each member of which intersects a set B? In order to describe these conditions the following notations and definitions are used. Also for earlier results see Robkin and Valentine **(2)**.

Notations. We use the following abbreviations where E_n is n-dimensional Euclidean space and where $S \subset E_n$, $x \in E_n$, $y \in E_n$:

$$\begin{aligned} \text{cl}\, S &= \text{closure of } S, \ \text{int}\, S = \text{interior of } S, \\ \text{bd}\, S &= \text{boundary of } S, \qquad \text{conv}\, S = \text{convex hull of } S, \\ xy &= \text{closed line segment joining } x \text{ and } y \text{ when } x \neq y, \\ L(xy) &= \text{line determined by } x \text{ and } y \text{ when } x \neq y, \\ \emptyset &= \text{empty set}, \qquad O = \text{origin of } S. \end{aligned}$$

The symbols $\cup$, $\cap$, and $\sim$ are used for set union, set intersection, and set difference respectively.

DEFINITION 1. *A set of points A in E_n has the m-point parallel line intersection property $P(m)$ relative to a set B in E_n if every collection of m or fewer points of A lies on a collection of parallel lines each member of which intersects B.*

The set A in E_n is said to have the parallel line intersection property $P(A)$ relative to the set B in E_n if all the points of A lie on a collection of parallel lines each member of which intersects B.

In this treatment we shall characterize those compact convex sets B in the plane E_2 such that if A is a closed connected set in E_2 which is disjoint from B and which has property $P(m)$ relative to B, then A also has property $P(A)$ relative to B (m is an integer).

The concepts of "exposed point" and "antipodal points" play a crucial role in this development.

DEFINITION 2. *A point x in the boundary of a closed convex set $B \subset E_2$ is called an exposed point of S if there exists a line of support L to B such that $L \cap B = x$.*

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DEFINITION 3. *A pair of distinct points x and y are antipodal points of a plane convex set S if there exist two parallel lines of support to S, say L_1 and L_2, such that $L_1 \cap S = x$, $L_2 \cap S = y$.*

The following theorem contains the most general results for compact convex sets B in E_2. The reader is advised to glance ahead to Corollaries 1 and 2 in order to appreciate the content of the theorem more fully. In the following, m is a fixed positive integer.

THEOREM 1. *Let B be a compact convex set in the Euclidean plane E_2.*

(a) *If $m \geqslant 2$ and if each closed arc in* bd B *whose end points are antipodal points of B contains at most m exposed points of B, then each closed connected set $A \subset E_2$ which is disjoint from B and which has the m-point parallel line intersection property $P(m)$ relative to B also has the parallel line intersection property $P(A)$ relative to B.*

(b) *On the other hand, if there exists a pair of antipodal points which are the end points of a closed arc in* bd B *containing at least $m + 1$ exposed points of B, then there exists a closed connected set A which is disjoint from B, which has property $P(m)$ relative to B, but which does not have property $P(A)$ relative to B.*

Proof. To prove (a) *when $m > 2$*, first observe that if A lies in the closed strip bounded by two parallel lines of support to B, then A obviously has property $P(A)$ relative to B. Hence, suppose A does not lie in any such strip. In this case, however, *there exist two parallel lines of support, say $L(x)$ and $L(y)$, such that x and y are antipodal points of B with $L(x) \cap B = x$, $L(y) \cap B = y$, and such that A has points in each of the two components of the complement of the strip* conv $(L(x) \cup L(y))$. This is easy to prove as follows. First, since the two end points of a diameter of B are a pair of antipodal points of S, the hypotheses in (a) imply that B has a finite number of exposed points, so that bd B is a convex polygon. Since we are assuming that A is not contained in a closed strip bounded by a pair of parallel supporting lines to B, there exists a pair of parallel supporting lines of B, say $L(u)$ and $L(v)$, such that points of A lie in at least one of the two open components of the complement of conv $(L(u) \cup L(v))$. Now rotate the lines $L(u)$ and $L(v)$ about bd B in such a way that they remain at all stages parallel lines of support to B. Since a rotation through π radians interchanges $L(u)$ and $L(v)$, and since A is a closed connected set disjoint from B, and since we have assumed that no parallel strip of support contains A, there must exist a pair of parallel lines of support $L(x)$ and $L(y)$ to B such that A intersects both components of

$$E_2 \sim \text{conv}\ (L(x) \cup L(y)).$$

If x and y are not both antipodal, then a sufficiently small rotation of $L(x)$ and $L(y)$ in the appropriate direction will yield two parallel lines of support of

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the type described in the above italicized sentence. This is illustrated in Fig. 1.

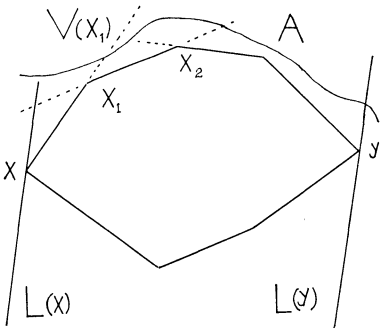

FIGURE 1

Now, to complete the proof, the lines determined by the edges of bd B divide E_2 into convex parts. If x_i is a vertex of B, let $V(x_i)$ be the closed V-shaped region abutting B externally at x_i, and if $x_i x_{i+1}$ is an edge of B, let $V(x_i x_{i+1})$ be the open half-plane that abuts B externally along $L(x_i x_{i+1})$. Return to the two parallel lines of support described above and illustrated in Fig. 1. There exists an arc of bd B joining x and y, denoted by arc xy, with consecutive vertices (relative to an order on *bd* B)

$$x, x_1, x_2, \ldots, x_s, y,$$

such that if we define A_i $(i = 1, \ldots, r)$ as follows:

$$(1)\quad \begin{array}{lll} A_0 \equiv A \cap V(xx_1), & A_1 \equiv A \cap V(x_1), & A_2 \equiv A \cap V(x_1 x_2), \\ A_3 \equiv A \cap V(x_2), & \ldots, & A_r \equiv A \cap V(x_s y), \end{array}$$

then

$$A \cap A_i \neq \emptyset \qquad (i = 0, \ldots, r),$$

where r depends on s. Also since arc xy contains at most m exposed points of B, we have

$$(2)\qquad s \leqslant m - 2.$$

To continue, for each point $x \in A$, let $C(x)$ denote the union of all rays that emanate from x and intersect B. Also let $D(x)$ be that translate of $C(x)$ such that x goes to the origin O of E_2. Clearly $D(x)$ is a closed convex cone having O as its vertex, and $D(x) \sim O$ is contained in an open half-space bounded by a line through O since $A \cap B = \emptyset$. We shall prove that there exists a ray R emanating from O such that

$$(3)\qquad R \subset \cap_{x \in A} D(x).$$

At this point it should be noted that although the hypotheses of Theorem 1 imply that every m members of the two-napped cones determined by $D(x)$, $x \in A$, and their reflections through O have a line in common, this, in itself, does not imply (3) without additional argument. Next, observe that if

$$a \in A_i, \qquad b \in A_{i+1}, \qquad x_0 \in A_{i+2}$$

(see (1)), then property $P(m)$ $(m > 2)$ implies that

$$D(a) \cap D(b) \cap D(x_0) \neq \emptyset. \tag{4}$$

By a simple induction, property $P(m)$, $m > 2$, and the condition (2), namely $s \leqslant m - 2$, imply that condition (4) also holds for every three points a, b, x_0 in the set

$$Q \equiv \bigcup_{i=0}^{r} A_i. \tag{5}$$

Let $C \equiv [x : ||x|| = 1]$ be the unit circle with centre at O and fix a point $x_0 \in Q$. Each of the elements of the collection

$$\mathfrak{M} \equiv \{C \cap D(a) \cap D(x_0), a \in Q\}$$

is a compact arc of C which is less than a semicircle. Condition (4) implies that every two members in $\mathfrak{M}$ have at least one point in common with a semicircular arc of C. Hence Helly's theorem for 1-dimensional space **(1)** implies that there exists a point u in common to all the members of $\mathfrak{M}$. (*Helly's theorem.* Let $\mathfrak{F}$ be a family of compact convex sets in E_n containing at least $n + 1$ members. If every $n + 1$ members of $\mathfrak{F}$ have a point in common, then all the members of $\mathfrak{F}$ have a point in common.) Let L be the line determined by O and u, and for $x \in A$ let $L(x)$ be that line through x which is parallel to L. Clearly the definition of $C(x)$, $x \in A$ implies $L(x) \cap B \neq \emptyset$ for $x \in Q$. This implies that the set Q given by (5) lies between two parallel lines of support to B, denoted by $L(c)$ and $L(d)$. However, since Q intersects both components of the complement of conv $(L(x) \cup L(y))$ (see Fig. 1), then at least one of the two parallel lines $L(c)$ and $L(d)$ will support B in such a way that the set A is not connected. Thus we have arrived at a contradiction, and therefore A does lie between two parallel lines of support to B, and statement (a) has been proved when $m > 2$.

When $m = 2$, the hypotheses in (a) imply that B is either a point or a closed line segment. In this case condition (3) follows immediately from Helly's theorem, and statement (a) is also true. Hence (a) has been proved.

To prove statement (b) for $m > 1$, let x and y be a pair of antipodal points such that an arc xy in bd B contains at least $m + 1$ exposed points of B. Let $x_1, x_2, \ldots, x_{m-1}$ designate $m - 1$ consecutive exposed points on arc xy, ordered from x to y between x and y. There exist parallel lines of support $L(x)$ and $L(y)$ to B such that $L(x) \cap B = x$, $L(y) \cap B = y$. Let $L(x_i)$ be a line such that $L(x_i) \cap B = x_i$ $(i = 1, \ldots, m - 1)$. Then there exist points x_{ij}, y_1, y_{m-1} such that

$$\begin{aligned} &L(x_i) \cap L(x_j) \equiv x_{ij} \qquad (i \neq j, i, j = 1, \ldots, m-1),\\ &L(x) \cap L(x_1) \equiv y_1,\\ &L(y) \cap L(x_{m-1}) \equiv y_{m-1}.\end{aligned}$$

Extend the segment $x_1 y_1$ to $x_1 x_0$ and extend $x_{m-1} y_{m-1}$ to $x_{m-1} x_m$ so that $x_0 \in \operatorname{int} H$, $x_m \in \operatorname{int} H$, and so that

$$\begin{cases} L(x_0 x) \cap L(yx_{m-1}) \cap H \neq \emptyset, \\ L(x_m y) \cap L(xx_1) \cap H \neq \emptyset, \end{cases} \tag{6}$$

where H is the half-plane bounded by $L(xy)$ which contains arc xy. Define

$$A_1 \equiv x_0 x_{12} \cup x_{12} x_{23} \cup \ldots \cup x_{m-2,m-1} x_m.$$

We shall modify A_1 to obtain A as follows. At the vertex x_1 replace a segment α of $x_0 x_{12}$ with mid-point x_1 by a semicircular arc C_1 which misses B and which has its end points at the extremities of α, as illustrated in Fig. 2. Perform the

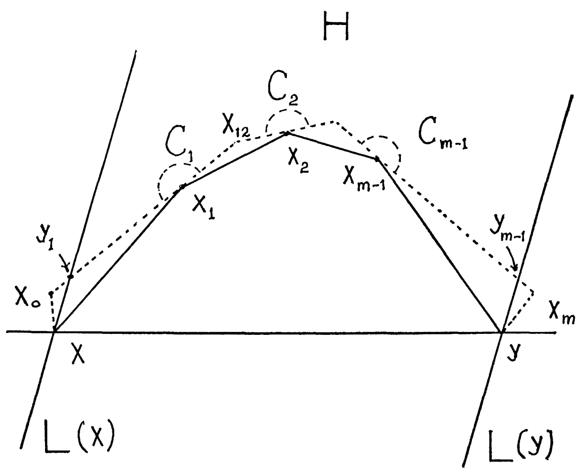

FIGURE 2

corresponding construction at each of the vertices x_i $(i = 1, \ldots, m - 1)$. The resulting connected set consisting of line segments and semicircular arcs is the set A. Clearly $A \cap B = \emptyset$. Now we shall prove that the radii $r_i = r$ of the arcs C_i $(i = 1, \ldots, m - 1)$ may be chosen sufficiently small so that A will have property $P(A)$ relative to B. First, the radii $r_i = r$ can be chosen sufficiently small so that each of the sets $A \sim C_i$ $(i = 1, \ldots, m - 1)$ has property $P(A)$ relative to B. Furthermore, if $z_i \in C_i$ $(i = 1, \ldots, m - 1)$ and if z_m is any other point of A_1, then we may choose the radii $r_i = r$ smaller, if necessary, so that the set of points $\{z_1, z_2, \ldots, z_m\}$ have property $P(m)$

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relative to B because of condition (6). To see this, suppose, without loss of generality, that $z_m \in x_0 y_1$, $z_m \neq y_1$. In this case if r is sufficiently small the m lines $L(z_i)$ $(i = 1, \ldots, m)$, where $z_i \in L(z_i)$, which are parallel to $L(z_m x)$ all intersect B because of condition (6). Combining both evaluations for the radii $r_i = r$, we see that A has property $P(m)$ relative to B. However, the set A does not have property $P(A)$ relative to B obviously (see Fig. 1). Hence, statement (b) has been proved when $m > 1$. When $m = 1$, the proof is trivial.

COROLLARY 1. *Let B be a compact convex set in the Euclidean plane E_2 which contains at most m exposed points with $m \geqslant 2$.* (Hence bd B is a polygon.) *Then each closed connected set A in E_2 which is disjoint from B and which has the m-point parallel line intersection property $P(m)$ relative to B also has the parallel line intersection property $P(A)$ relative to B* (see Definition 1).

COROLLARY 2. *Suppose that B is a compact convex set in E_2 and suppose that B contains at least $2m - 1$ exposed points* (m is a positive integer). *Then there exists a closed connected set A in E_2 which is disjoint from B, which has the m-point parallel line intersection property $P(m)$ relative to B, but which does not have the parallel line intersection property $P(A)$ relative to B.*

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