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Dirichlet-type spaces of the unit bidisc and toral 2-isometries*

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Abstract. **We introduce and study Dirichlet-type spaces $\mathcal{D}(\mu_1, \mu_2)$ of the unit bidisc $\mathbb{D}^2$, where μ_1, μ_2 are finite positive Borel measures on the unit circle. We show that the coordinate functions z_1 and z_2 are multipliers for $\mathcal{D}(\mu_1, \mu_2)$ and the complex polynomials are dense in $\mathcal{D}(\mu_1, \mu_2)$. Further, we obtain the division property and solve Gleason's problem for $\mathcal{D}(\mu_1, \mu_2)$ over a bidisc centered at the origin. In particular, we show that the commuting pair $\mathscr{M}_z$ of the multiplication operators $\mathscr{M}_{z_1}$, $\mathscr{M}_{z_2}$ on $\mathcal{D}(\mu_1, \mu_2)$ defines a cyclic toral 2-isometry and $\mathscr{M}_z^*$ belongs to the Cowen-Douglas class $\mathbf{B}_1(\mathbb{D}^2_r)$ for some $r > 0$. Moreover, we formulate a notion of wandering subspace for commuting tuples and use it to obtain a bidisc analog of Richter's representation theorem for cyclic analytic 2-isometries. In particular, we show that a cyclic analytic toral 2-isometric pair T with cyclic vector f_0 is unitarily equivalent to $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ for some μ_1, μ_2 if and only if $\ker T^*$, spanned by f_0, is a wandering subspace for T.**

1 Introduction and preliminaries

The aim of this paper is to obtain a bidisc counter-part of the theory of Dirichlet-type spaces of the open unit disc as presented in [26] (see [8] for a ball counter-part of this theory). Throughout this paper, $\mathbb{D}$ denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Recall that Dirichlet-type spaces of $\mathbb{D}$ are model spaces for the class of cyclic analytic 2-isometries (see [26]). Thus to arrive at an appropriate notion of the Dirichlet-type spaces of the unit bidisc $\mathbb{D}^2$, it is helpful to look for function spaces which support the class of 2-isometries naturally associated with $\mathbb{D}^2$. Let us first recall the definition of such 2-isometries.

For a complex Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. For a positive integer d, a *commuting d-tuple T on $\mathcal{H}$* is the d-tuple $(T_1, \ldots, T_d)$ of operators $T_1, \ldots, T_d \in \mathcal{B}(\mathcal{H})$ satisfying $T_iT_j = T_jT_i$, $1 \leqslant i \neq j \leqslant d$. Let $T = (T_1, \ldots, T_d)$ be a commuting d-tuple on $\mathcal{H}$. We say that $T = (T_1, \ldots, T_d)$ is a *toral isometry* if $T_1, \ldots, T_d$ are isometries. Following [1, 5, 26], T is said to be a *toral 2-isometry* if

$$I - T_i^*T_i - T_j^*T_j + T_j^*T_i^*T_iT_j = 0, \quad i, j = 1, \ldots, d. \tag{1.1}$$

A toral isometry is necessarily a toral 2-isometry, but the converse is not true (see [5, Example 1]).

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To propose a successful analog of Dirichlet-type spaces on $\mathbb{D}^2$, it is helpful to examine examples of toral 2-isometries arising from function spaces. Since the operator of multiplication by the coordinate function on the classical Dirichlet space $\mathcal{D}(\mathbb{D})$ is a 2-isometry, it is natural to seek the classical Dirichlet space of the unit bidisc. Recall that the *Dirichlet space* $\mathcal{D}(\mathbb{D}) \otimes \mathcal{D}(\mathbb{D})$ *of* $\mathbb{D}^2$ is given by

$$\Big\{f \in \mathcal{O}(\mathbb{D}^2) : \|f\|^2_{\mathcal{D}(\mathbb{D})\otimes\mathcal{D}(\mathbb{D})} := \sum_{(m,n)\in\mathbb{Z}^2_+} |\hat{f}(m,n)|^2(m+1)(n+1) < \infty\Big\},$$

where $\mathcal{O}(\Omega)$ denotes the space of holomorphic functions on a domain Ω, $\mathbb{Z}_+$ denotes the set of nonnegative integers and $\hat{f}$ denotes the Fourier transform of f. It turns out that if $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are the operators of multiplication by the coordinate functions z_1 and z_2, respectively, on $\mathcal{D}(\mathbb{D}) \otimes \mathcal{D}(\mathbb{D})$, then the commuting pair $(\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ satisfies (1.1) for $1 \leqslant i = j \leqslant 2$, but it fails to satisfy (1.1) for $1 \leqslant i \neq j \leqslant 2$. This failure may be attributed to the fact that the mapping $(m,n) \mapsto \|z_1^m z_2^n\|^2$ is a polynomial of bi-degree $(1,1)$. Interestingly, there is a "natural" choice $\mathcal{D}(\mathbb{D}^2)$ of the Dirichlet space containing $\mathcal{D}(\mathbb{D}) \otimes \mathcal{D}(\mathbb{D})$ for which the associated pair $(\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a toral 2-isometry:

$$\mathcal{D}(\mathbb{D}^2) = \Big\{f \in \mathcal{O}(\mathbb{D}^2) : \|f\|^2_{\mathcal{D}(\mathbb{D}^2)} := \sum_{(m,n)\in\mathbb{Z}^2_+} |\hat{f}(m,n)|^2(m+n+1) < \infty\Big\}.$$

The norm $\|\cdot\|_{\mathcal{D}(\mathbb{D}^2)}$ can also be written as follows:

$$\begin{aligned}\|f\|^2_{\mathcal{D}(\mathbb{D}^2)} &= \|f\|^2_{H^2(\mathbb{D}^2)} + \sup_{0<r<1}\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2\, dA(z_1)d\theta \\ &\quad + \sup_{0<r<1}\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2\, dA(z_2)d\theta, \qquad (1.2)\end{aligned}$$

where $d\theta$ (resp. dA) denotes the *normalized Lebesgue arc-length* (*resp. area*) *measure* on $\mathbb{T}$ (resp. $\mathbb{D}$). Recall that the *Hardy space* $H^2(\mathbb{D}^2)$ of the unit bidisc $\mathbb{D}^2$ is the reproducing kernel Hilbert space (see [23] for the definition of the reproducing kernel Hilbert space) associated with the *Cauchy kernel*

$$\kappa(z,w) = \prod_{j=1}^{2}(1 - z_j\overline{w}_j)^{-1}, \quad z = (z_1, z_2),\ w = (w_1, w_2) \in \mathbb{D}^2.$$

It is worth noting that for any $f \in H^2(\mathbb{D}^2)$,

$$\begin{aligned}\|f\|^2_{H^2(\mathbb{D}^2)} &= \sum_{\alpha\in\mathbb{Z}^2_+} |\hat{f}(\alpha)|^2 \qquad (1.3)\\ &= \sup_{0<r<1}\int_{[0,2\pi]^2} |f(re^{i\theta_1}, re^{i\theta_2})|^2 d\theta_1 d\theta_2 \qquad (1.4)\end{aligned}$$

(see [27, Section 3.4]).

For a nonempty subset Ω of $\mathbb{C}$, let $M_+(\Omega)$ denote the set of finite positive Borel measures on Ω. Let $P_\mu(w)$ denote the *Poisson integral* $\int_{\mathbb{T}} \frac{1-|w|^2}{|w-\zeta|^2} d\mu(\zeta)$ of the measure $\mu \in M_+(\mathbb{T})$. For future reference, we record the following consequence of the Fubini

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theorem (see [28, Theorem 8.8]) and the fact that the mapping $r \mapsto \int_{\mathbb{T}} |f(z, re^{i\theta})|^2 d\theta$ is increasing.

***Lemma* 1.1** *For $f \in \mathcal{O}(\mathbb{D}^2)$ and $\mu \in M_+(\mathbb{D})$, the extended real-valued mapping $\phi(r) = \int_{\mathbb{T}} \int_{\mathbb{D}} |f(z, re^{i\theta})|^2 d\mu(z) d\theta$, $r \in (0,1)$, is increasing.*

The formula (1.2) together with Richter's notion of Dirichlet-type spaces (see [26, Sect. 3]) motivates us to the following:

***Definition* 1.1** For $\mu_1, \mu_2 \in M_+(\mathbb{T})$ and $f \in \mathcal{O}(\mathbb{D}^2)$, *the Dirichlet integral $D_{\mu_1,\mu_2}(f)$* of f is given by

$$\begin{aligned} D_{\mu_1,\mu_2}(f) = \sup_{0<r<1} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ + \sup_{0<r<1} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta. \end{aligned}$$

If either μ_1 or μ_2 is 0, then the *Dirichlet-type space* $\mathcal{D}(\mu_1, \mu_2)$ is the space of functions $f \in H^2(\mathbb{D}^2)$ satisfying $D_{\mu_1,\mu_2}(f) < \infty$. Otherwise, we set $\mathcal{D}(\mu_1, \mu_2) = \{f \in \mathcal{O}(\mathbb{D}^2) : D_{\mu_1,\mu_2}(f) < \infty\}$.

Before we define a norm on the Dirichlet-type space $\mathcal{D}(\mu_1, \mu_2)$, we present a 2-variable analog of [26, Lemma 3.1].

***Lemma* 1.2** *For $\mu_1, \mu_2 \in M_+(\mathbb{T})$, $\mathcal{D}(\mu_1, \mu_2) \subseteq H^2(\mathbb{D}^2)$.*

Proof By the definition of $\mathcal{D}(\mu_1, \mu_2)$, we may assume that both measures μ_1 and μ_2 are non-zero. Note that

$$P_\mu(w) \geqslant \frac{\mu(\mathbb{T})}{4}(1 - |w|^2), \quad \mu \in M_+(\mathbb{T}),\ w \in \mathbb{D}. \tag{1.5}$$

Thus, for any $f(z_1, z_2) = \sum_{m,n=0}^\infty a_{m,n} z_1^m z_2^n \in \mathcal{D}(\mu_1, \mu_2)$,

$$\begin{aligned} &\int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &\overset{(1.5)}{\geqslant} \frac{\mu_1(\mathbb{T})}{4} \sum_{m=1}^\infty \sum_{n=0}^\infty |a_{m,n}|^2 m^2 r^{2n} \int_{\mathbb{D}} |z_1^{m-1}|^2 (1 - |z_1|^2) dA(z_1) \\ &= \frac{\mu_1(\mathbb{T})}{4} \sum_{m=1}^\infty \sum_{n=0}^\infty |a_{m,n}|^2 \frac{m r^{2n}}{m+1}. \end{aligned}$$

A similar estimate using (1.5) gives

$$\int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta \geqslant \frac{\mu_2(\mathbb{T})}{4} \sum_{m=0}^\infty \sum_{n=1}^\infty |a_{m,n}|^2 \frac{n r^{2m}}{n+1}.$$

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Since $f \in \mathcal{D}(\mu_1, \mu_2)$,

$$\sup_{0<r<1} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} |a_{m,n}|^2 \frac{mr^{2n}}{m+1} < \infty, \quad \sup_{0<r<1} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}|^2 \frac{nr^{2m}}{n+1} < \infty.$$

It is now easy to see using the monotone convergence theorem (see [28, Theorem 1.26]) that f belongs to $H^2(\mathbb{D}^2)$. ∎

In view of Lemmas 1.1 and 1.2, the Dirichlet-type space $\mathcal{D}(\mu_1, \mu_2)$ can be endowed with the norm

$$\begin{aligned}\|f\|^2_{\mathcal{D}(\mu_1,\mu_2)} &= \|f\|^2_{H^2(\mathbb{D}^2)} + \lim_{r \to 1^-} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &\quad + \lim_{r \to 1^-} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta.\end{aligned}$$

We see that $\mathcal{D}(\mu_1, \mu_2)$ is a reproducing kernel Hilbert space (see Lemma 3.1).

The present paper is devoted to the study of Dirichlet-type spaces with efforts to understand the bidisc counter-part of the work carried out in [26]. Before we state the main results of this paper, we need some definitions.

For a positive integer d, let Ω be a domain in $\mathbb{C}^d$ and let $\mathscr{H}$ be a Hilbert space such that $\mathscr{H} \subseteq \mathcal{O}(\Omega)$. A function $\varphi : \Omega \to \mathbb{C}$ is said to be a *multiplier* of $\mathscr{H}$ if $\varphi f \in \mathscr{H}$ for every $f \in \mathscr{H}$. For a nonempty subset U of Ω, we say that *Gleason's problem can be solved for* $\mathscr{H}$ *over* U if for every $f \in \mathscr{H}$ and $\lambda = (\lambda_1, \ldots, \lambda_d) \in U$, there exist functions $g_1, \ldots, g_d$ in $\mathscr{H}$ such that

$$f(z) = f(\lambda) + \sum_{j=1}^{d} (z_j - \lambda_j) g_j(z), \quad z = (z_1, \ldots, z_d) \in \Omega.$$

We say that *Gleason's problem can be solved for* $\mathscr{H}$ if Gleason's problem can be solved for $\mathscr{H}$ over Ω (the reader is referred to [31] for a solution of Gleason's problem for Bergman and Bloch spaces of the unit ball). It turns out that Gleason's problem can be solved for $H^2(\mathbb{D}^d)$ (see Remark 5.2).

***Definition* 1.2** Let Ω be a domain in $\mathbb{C}^d$ and let $\mathscr{H}$ be a Hilbert space such that $\mathscr{H} \subseteq \mathcal{O}(\Omega)$. We say that $\mathscr{H}$ has the *j-division property*, $j = 1, \ldots, d$, if $\frac{f(z)}{z_j - \lambda_j}$ defines a function in $\mathscr{H}$ whenever $\lambda \in \Omega$, $f \in \mathscr{H}$ and $\{z \in \Omega : z_j = \lambda_j\}$ is contained in $Z(f)$, the zero set of f. If $\mathscr{H}$ has j-division property for every $j = 1, \ldots, d$, then we say that $\mathscr{H}$ has the *division property*.

In case of $d = 1$, this property appeared in [3, Definition 1.1]. One of the main results of this paper shows that $\mathcal{D}(\mu_1, \mu_2)$ has the division property. In what follows, we require a generalization of the notion of the wandering subspace introduced by Halmos (see [20, P. 103]).

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Definition 1.3 Let $T = (T_1, \ldots, T_d)$ be a commuting d-tuple on $\mathcal{H}$. A closed subspace $\mathcal{W}$ of $\mathcal{H}$ is said to be *wandering* for T if for every $i = 1, \ldots, d$,

$$\prod_{j=1}^{d} T_j^{\alpha_j}\mathcal{W} \perp \prod_{j=1}^{d} T_j^{\beta_j}\mathcal{W}, \quad \alpha_j, \beta_j \in \mathbb{Z}_+,\ j = 1, \ldots, d,\ \alpha_i = 0,\ \beta_i \neq 0.$$

Remark 1.3 If $d = 1$, then $\mathcal{W}$ is a wandering subspace for T if and only if $\mathcal{W} \perp T^k(\mathcal{W})$ for every integer $k \geqslant 1$. In particular, $\ker T^*$ is a wandering subspace for any $T \in \mathcal{B}(\mathcal{H})$. Moreover, if $T = (T_1, \ldots, T_d)$ is a commuting d-tuple such that $T_j^* T_i = T_i T_j^*$, $1 \leqslant i \neq j \leqslant d$, then $\ker T^* = \cap_{j=1}^d \ker T_j^*$ is a wandering subspace for T.

It follows from Remark 1.3 that the space spanned by the constant function 1 is a wandering subspace for the multiplication 2-tuple $\mathscr{M}_z$ on $H^2(\mathbb{D}^2)$. Interestingly, this fact extends to the multiplication 2-tuple $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ (see Corollary 3.12).

Recall that a commuting d-tuple $T = (T_1, \ldots, T_d)$ on $\mathcal{H}$ is *cyclic* with *cyclic vector* $f_0 \in \mathcal{H}$ if $\bigvee \left\{T^\alpha f_0 : \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d\right\} = \mathcal{H}$, where $\bigvee$ denotes the closed linear span and $T^\alpha = \prod_{j=1}^d T_j^{\alpha_j}$. For later purpose, we state the following property of cyclic tuples (see [4, Proposition 1.1]):

$$\textit{If } T \textit{ is cyclic, then for any } \omega \in \mathbb{C}^d,\ \dim\ker(T^* - \omega) \textit{ is at most } 1, \tag{1.6}$$

where $\ker S = \cap_{j=1}^d \ker S_j$ for the d-tuple $S = (S_1, \ldots, S_d)$ and dim stands for the Hilbert space dimension. A commuting d-tuple T on $\mathcal{H}$ has the *wandering subspace property* if $\mathcal{H} = \bigvee_{\alpha \in \mathbb{Z}_+} T^\alpha(\ker T^*)$. Following [15, P. 56], we say that a commuting d-tuple $T = (T_1, \ldots, T_d)$ on $\mathcal{H}$ is *analytic* if

$$\bigcap_{k=0}^{\infty} \sum_{\alpha \in \Gamma_k} T^\alpha \mathcal{H} = \{0\},$$

where, for $k \in \mathbb{Z}_+$, $\Gamma_k := \{\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d : \alpha_1 + \cdots + \alpha_d = k\}$. Note that if T is analytic, then $T_1, \ldots, T_d$ are analytic.

Let Ω be a domain in $\mathbb{C}^d$. For a positive integer n, let $\mathbf{B}_n(\Omega)$ denote the set of all commuting d-tuples T on $\mathcal{H}$ satisfying the following conditions:

- for every $\omega = (\omega_1, \ldots, \omega_d) \in \Omega$, the map $D_{T-\omega}(x) = ((T_j - \omega_j)x)_{j=1}^d$ from $\mathcal{H}$ into $\mathcal{H}^{\oplus d}$ has closed range and $\dim\ker(T - \omega) = n$,
- the subspace $\bigvee_{\omega \in \Omega} \ker(T - \omega)$ of $\mathcal{H}$ equals $\mathcal{H}$.

We call the set $\mathbf{B}_n(\Omega)$ the *Cowen-Douglas class of rank n with respect to* Ω (refer to [10, 13] for the basic theory of Cowen-Douglas class).

2 Statements of main theorems

The following three theorems collect several basic properties of Dirichlet-type spaces $\mathcal{D}(\mu_1, \mu_2)$.

Theorem 2.1 *For $\mu_1, \mu_2 \in M_+(\mathbb{T})$, we have the following statements*:

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(i) *the coordinate functions* z_1, z_2 *are multipliers of* $\mathcal{D}(\mu_1, \mu_2)$,
(ii) *the polynomials are dense in* $\mathcal{D}(\mu_1, \mu_2)$,
(iii) *for non-negative integers* k, l *and a polynomial* p *in* z_1 *and* z_2,

$$\|z_1^k z_2^l p\|^2_{\mathcal{D}(\mu_1,\mu_2)} = \|p\|^2_{\mathcal{D}(\mu_1,\mu_2)} + k \int_{\mathbb{T}^2} |p(e^{i\eta}, e^{i\theta})|^2 d\mu_1(\eta) d\theta + l \int_{\mathbb{T}^2} |p(e^{i\theta}, e^{i\eta})|^2 d\mu_2(\eta) d\theta. \tag{2.1}$$

***Theorem* 2.2** *For* $\mu_1, \mu_2 \in M_+(\mathbb{T})$, $\mathcal{D}(\mu_1, \mu_2)$ *has the division property.*

***Theorem* 2.3** *For* $\mu_1, \mu_2 \in M_+(\mathbb{T})$, *Gleason's problem can be solved for* $\mathcal{D}(\mu_1, \mu_2)$ *over* $\mathbb{D}_r^2$ *for some* $r \in (0, 1]$.

Here $\mathbb{D}_r^2$ denotes the bidisc $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < r, |z_2| < r\}$, where r is a positive real number. Unlike the one variable situation, we do not know whether Gleason's problem can be solved for $\mathcal{D}(\mu_1, \mu_2)$ over the unit bidisc. It is worth noting that not all facts about Dirichlet-type spaces of the unit disc have successful counterparts in the bidisc case. For example, the commuting pair $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ on $\mathcal{D}(\mu_1, \mu_2)$ fails to be essentially normal (see Corollary 3.13). Moreover, the verbatim analog of the model theorem [26, Theorem 5.1] does not hold true (see Remark 2.5).

The following result asserts that $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ is a canonical model for analytic 2-isometries T for which $\ker T^*$ is a cyclic wandering subspace.

***Theorem* 2.4 (*A representation theorem*)** *Let* $T = (T_1, T_2)$ *be a commuting pair on* $\mathcal{H}$. *Then the following statements are equivalent:*

(i) T *is a cyclic analytic toral* 2*-isometry with cyclic vector* $f_0 \in \ker T^*$ *and* $\ker T^*$ *is a wandering subspace for* T,
(ii) T *is a cyclic toral* 2*-isometry with cyclic vector* $f_0 \in \ker T^*$, T^* *belongs to* $\mathbf{B}_1(\mathbb{D}_r^2)$ *for some* $r \in (0, 1]$ *and* $\ker T^*$ *is a wandering subspace for* T,
(iii) *there exist* $\mu_1, \mu_2 \in M_+(\mathbb{T})$ *such that* T *is unitarily equivalent to* $\mathscr{M}_z$ *on* $\mathcal{D}(\mu_1, \mu_2)$.

***Remark* 2.5** By [25, Theorem 1], any analytic 2-isometry T on $\mathcal{H}$ has the wandering subspace property. This result fails even for analytic toral isometric d-tuples if $d > 1$. Indeed, if $a \in \mathbb{D}^2 \setminus \{(0, 0)\}$, then the restriction of $\mathscr{M}_z$ to $\{f \in H^2(\mathbb{D}^2) : f(a) = 0\}$ is a toral isometry without the wandering subspace property. This may be seen by imitating the argument of [6, Example 6.8] with the only change that the application of [19, Theorem 4.3] is replaced by that of [19, Corollary 4.6]. This example also shows that the assumption that the cyclic vector f_0 belongs to $\ker T^*$ in (i) can not be dropped from Theorem 2.4. Also, by Theorem 2.1(ii), the cyclicity of T in (ii) of Theorem 2.4 can not be relaxed.

Theorems 2.1, 2.3 and 2.4 provide bidisc analogs of [26, Theorems 3.6, 3.7 and 5.1], respectively. Also, Theorem 2.2 presents a counterpart of the fact that Dirichlet-type spaces on the unit disc have the division property (see [26, Corollary 3.8] and

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[24, Lemma 2.1]). The proofs of these results and their consequences are presented in Sections 3-6 (see Corollaries 3.8, 3.9, 3.12, 3.13, 4.6, 5.5, 5.6, 6.2, 6.6). In the final short section, we discuss the spectral picture of the multiplication 2-tuple $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ and raise some related questions.

3 Proof of Theorem 2.1 and its consequences

We need several lemmas to prove Theorem 2.1.

***Lemma* 3.1** *The Dirichlet-type space $\mathcal{D}(\mu_1, \mu_2)$ is a reproducing kernel Hilbert space. If $\kappa : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ is the reproducing kernel of $\mathcal{D}(\mu_1, \mu_2)$, then for any $r \in (0, 1)$, $\bigvee\{\kappa(\cdot, w) : |w| < r\} = \mathcal{D}(\mu_1, \mu_2)$ and $\kappa(\cdot, 0) = 1$.*

Proof We borrow an argument from the proof of [14, Theorem 1.6.3]. Let $\{f_n\}_{n\geqslant 0}$ be a Cauchy sequence in $\mathcal{D}(\mu_1, \mu_2)$. Since $H^2(\mathbb{D}^2)$ is complete (see [27, p 53]), there exists a $f \in H^2(\mathbb{D}^2)$ such that $\|f_n - f\|^2_{H^2(\mathbb{D}^2)} \to 0$ as $n \to \infty$. Moreover, since $H^2(\mathbb{D}^2)$ is a reproducing kernel Hilbert space, for every $j = 1, 2$, $\partial_j f_n$ converges compactly to $\partial_j f$ on $\mathbb{D}^2$. Also, since $\{f_n\}_{n\geqslant 0}$ is bounded in $\mathcal{D}(\mu_1, \mu_2)$, by Lemma 1.1, there exists an $M > 0$ such that for every integer $n \geqslant 0$ and $r \in (0, 1)$,

$$\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 f_n(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta < M,$$
$$\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_2 f_n(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta < M.$$

By Fatou's lemma (see [28, Lemma 1.28]), for any $r \in (0, 1)$,

$$\begin{aligned}&\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &\leqslant \liminf_n \int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 f_n(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \leqslant M.\end{aligned} \tag{3.1}$$

Similarly, one can see that

$$\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta \leqslant M, \quad r \in (0, 1).$$

This shows that $f \in \mathcal{D}(\mu_1, \mu_2)$. We may now argue as in (3.1) (with f replaced by $f_n - f$ and f_n replaced by $f_n - f_m$) and use Fatou's lemma to conclude that

$$\begin{aligned}&\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 (f_n - f)(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &\leqslant \liminf_m \int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 (f_n - f_m)(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta.\end{aligned}$$

Similarly, we obtain

$$\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_2 (f_n - f)(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta$$

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$$\leqslant \liminf_m \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 (f_n - f_m)(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta.$$

These two estimates combined with Lemma 1.1 yield

$$D_{\mu_1,\mu_2}(f_n - f) \leqslant \liminf_m D_{\mu_1,\mu_2}(f_n - f_m), \quad n \geqslant 0.$$

This shows that $\{f_n\}_{n\geqslant 0}$ converges to f in $\mathcal{D}(\mu_1, \mu_2)$. Finally, since $H^2(\mathbb{D}^2)$ is a reproducing kernel Hilbert space, so is $\mathcal{D}(\mu_1, \mu_2)$ (see Lemma 1.2).

To see the 'moreover' part, note that for any $f \in \mathcal{D}(\mu_1, \mu_2)$, by the reproducing property of $\mathcal{D}(\mu_1, \mu_2)$,

$$\langle f,\, 1\rangle_{\mathcal{D}(\mu_1,\mu_2)} = \langle f,\, 1\rangle_{H^2(\mathbb{D}^2)} = f(0) = \langle f,\, \kappa(\cdot, 0)\rangle_{\mathcal{D}(\mu_1,\mu_2)},$$

and hence $\kappa(\cdot, 0) = 1$. The rest follows from the reproducing property of $\mathcal{D}(\mu_1, \mu_2)$ together with an application of the identity theorem. ■

Although we do not need in this section the full strength of the following lemma (cf. [18, Theorem 4.2]), we include it for later usage:

***Lemma* 3.2** *Let $f : \mathbb{D}^2 \to \mathbb{C}$ be a holomorphic function. For $r \in (0, 1)$ and $\theta \in [0, 2\pi]$, consider the holomorphic function $f_{r,\theta}(w) = f(w, re^{i\theta})$, $w \in \mathbb{D}$. If $f \in H^2(\mathbb{D}^2)$, then $f_{r,\theta} \in H^2(\mathbb{D})$ for every $r \in (0, 1)$ and $\theta \in [0, 2\pi]$. Moreover,*

$$\sup_{0<r<1} \int_0^{2\pi} \|f_{r,\theta}\|^2_{H^2(\mathbb{D})} d\theta = \|f\|^2_{H^2(\mathbb{D}^2)}, \quad f \in H^2(\mathbb{D}^2). \tag{3.2}$$

Proof The proof relies on the formula (1.3). First, note that

$$\sum_{m=0}^{\infty} |\hat{f}_{r,\theta}(m)|^2 = \sum_{m=0}^{\infty} \Big| \sum_{n=0}^{\infty} \hat{f}(m, n) r^n e^{in\theta} \Big|^2. \tag{3.3}$$

If $f \in H^2(\mathbb{D}^2)$, then applying the Cauchy-Schwarz inequality to (3.3) gives that $f_{r,\theta} \in H^2(\mathbb{D})$ for every $r \in (0, 1)$ and $\theta \in [0, 2\pi]$. Moreover, integrating both sides of (3.3) with respect to θ over $[0, 2\pi]$ and taking supremum over $r \in (0, 1)$ yields (3.2). ■

Remark 3.3 We note that

$$\begin{aligned} &\text{if } f \in \mathcal{D}(\mu_1, \mu_2), \text{ then for every } r \in (0, 1),\ f(\cdot, re^{i\theta}) \in \mathcal{D}(\mu_1) \\ &\text{and } f(re^{i\theta}, \cdot) \in \mathcal{D}(\mu_2) \text{ for almost every } \theta \in [0, 2\pi]. \end{aligned} \tag{3.4}$$

To see this, note that for any holomorphic function $f : \mathbb{D}^2 \to \mathbb{C}$,

$$D_{\mu_1,\mu_2}(f) = \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta})) d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot)) d\theta, \tag{3.5}$$

and hence, if $f \in \mathcal{D}(\mu_1, \mu_2)$, then by Lemma 1.1, $\int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta})) d\theta$ and $\int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot)) d\theta$ are finite for every $r \in (0, 1)$. One may now apply Lemma 3.2 to complete the verification of (3.4).

It turns out that the operator $\mathscr{M}_{z_j}$ of multiplication by the coordinate functions z_j, $j = 1, 2$, defines a bounded linear operator on $\mathcal{D}(\mu_1, \mu_2)$.

***Lemma* 3.4** *The coordinate functions z_1, z_2 are multipliers of $\mathcal{D}(\mu_1, \mu_2)$.*

Proof By (3.4), for any $f \in \mathcal{D}(\mu_1, \mu_2)$ and $r \in (0, 1)$, $f(\cdot, re^{i\theta}) \in \mathcal{D}(\mu_1)$ for a.e. $\theta \in [0, 2\pi]$. By [26, Theorem 3.6], the operator $\mathscr{M}_w$ of multiplication by the coordinate function w on $\mathcal{D}(\mu_1)$ is bounded and satisfies

$$\|\mathscr{M}_w f(\cdot, re^{i\theta})\|_{\mathcal{D}(\mu_1)} \leqslant \|\mathscr{M}_w\| \|f(\cdot, re^{i\theta})\|_{\mathcal{D}(\mu_1)} \text{ for a.e. } \theta \in [0, 2\pi]. \tag{3.6}$$

Since $\mathscr{M}_w^* \mathscr{M}_w \geqslant I$, $\|\mathscr{M}_w\| \geqslant 1$. Fix now $f \in \mathcal{D}(\mu_1, \mu_2)$. By Lemma 1.2, $f \in H^2(\mathbb{D}^2)$, and hence $z_1 f \in H^2(\mathbb{D}^2)$. By (3.5) (two applications),

$$\begin{aligned}
& D_{\mu_1,\mu_2}(z_1 f) \\
= \;& \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_1}((z_1 f)(\cdot, re^{i\theta}))d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}((z_1 f)(re^{i\theta}, \cdot))d\theta \\
\leqslant \;& \lim_{r\to 1^-} \int_{\mathbb{T}} \|\mathscr{M}_w f(\cdot, re^{i\theta})\|^2_{\mathcal{D}(\mu_1)} d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot))d\theta \\
\overset{(3.6)}{\leqslant} \;& \|\mathscr{M}_w\|^2 \lim_{r\to 1^-} \int_{\mathbb{T}} \|f(\cdot, re^{i\theta})\|^2_{\mathcal{D}(\mu_1)} d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot))d\theta \\
\leqslant \;& \|\mathscr{M}_w\|^2 \lim_{r\to 1^-} \int_{\mathbb{T}} \|f(\cdot, re^{i\theta})\|^2_{H^2(\mathbb{D})} d\theta + \|\mathscr{M}_w\|^2 D_{\mu_1,\mu_2}(f) \\
\overset{(3.2)}{=} \;& \|\mathscr{M}_w\|^2 \|f\|^2_{\mathcal{D}(\mu_1,\mu_2)}.
\end{aligned}$$

Similarly, one can see that for some $c_2 \geqslant 1$,

$$D_{\mu_1,\mu_2}(z_2 f) \leqslant c_2 \|f\|^2_{\mathcal{D}(\mu_1,\mu_2)}, \quad f \in \mathcal{D}(\mu_1, \mu_2).$$

This completes the proof. ∎

The following is a bidisc-analog of Richter's formula (see [26, Proof of Theorem 4.1], [8, Theorem 1.3]).

***Lemma* 3.5** *For nonnegative integers k, l and a polynomial p in the complex variables z_1 and z_2, we have the formula (2.1).*

Proof By (3.4) (see also (3.5)) and [26, Proof of Theorem 4.1],

$$\begin{aligned}
& \|z_1^k z_2^l p\|^2_{\mathcal{D}(\mu_1,\mu_2)} \\
= \;& \|z_1^k z_2^l p\|^2_{H^2(\mathbb{D}^2)} + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_1}(w^k p(w, re^{i\theta}))d\theta \\
+ \;& \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(w^l p(re^{i\theta}, w))d\theta \\
= \;& \|p\|^2_{H^2(\mathbb{D}^2)} + \lim_{r\to 1^-} \int_{\mathbb{T}} \Big(D_{\mu_1}(p(w, re^{i\theta})) + k \int_{\mathbb{T}} |p(e^{i\eta}, re^{i\theta})|^2 d\mu_1(\eta)\Big) d\theta
\end{aligned}$$

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$$+ \lim_{r\to 1^-} \int_{\mathbb{T}} \Big(D_{\mu_2}(p(re^{i\theta}, w)) + l \int_{\mathbb{T}} |p(re^{i\theta}, e^{i\eta})|^2 d\mu_2(\eta)\Big) d\theta$$
$$= \|p\|^2_{\mathcal{D}(\mu_1,\mu_2)} + k \int_{\mathbb{T}^2} |p(e^{i\eta}, e^{i\theta})|^2 d\mu_1(\eta) d\theta + l \int_{\mathbb{T}^2} |p(e^{i\theta}, e^{i\eta})|^2 d\mu_2(\eta) d\theta,$$

where we used Lemma 1.1 and the monotone convergence theorem. ■

For $R = (R_1, R_2) \in (0, 1)^2$ and $f \in \mathcal{O}(\mathbb{D}^2)$, let $f_R(z) = f(R_1 z_1, R_2 z_2)$. To get the polynomial density in $\mathcal{D}(\mu_1, \mu_2)$, we need the following inequality.

***Lemma* 3.6** *For any $R = (R_1, R_2) \in (0, 1)^2$ and $f \in \mathcal{D}(\mu_1, \mu_2)$,*

$$D_{\mu_1,\mu_2}(f_R) \leqslant D_{\mu_1,\mu_2}(f).$$

Proof By (3.5) and [29, Proposition 3],

$$D_{\mu_1,\mu_2}(f_R) = \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_1}(f_R(\cdot, re^{i\theta})) d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(f_R(re^{i\theta}, \cdot)) d\theta$$
$$\leqslant \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_1}(f(\cdot, rR_2 e^{i\theta})) d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(f(rR_1 e^{i\theta}, \cdot)) d\theta.$$

This, combined with Lemma 1.1, yields

$$D_{\mu_1,\mu_2}(f_R) \leqslant \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta})) d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot)) d\theta.$$

An application of (3.5) now completes the proof. ■

Here is a key step in deducing the density of polynomials in $\mathcal{D}(\mu_1, \mu_2)$.

***Lemma* 3.7** *For any $f \in \mathcal{D}(\mu_1, \mu_2)$,*

$$\lim_{R_1, R_2 \to 1^-} D_{\mu_1,\mu_2}(f - f_R) = 0.$$

Proof The proof is an adaptation of that of [14, Theorem 7.3.1] to the present situation. For $R = (R_1, R_2) \in (0, 1)^2$, by the Parallelogram law (which holds for any seminorm) and Lemma 3.6,

$$\begin{aligned} D_{\mu_1,\mu_2}(f - f_R) + D_{\mu_1,\mu_2}(f + f_R) &= 2(D_{\mu_1,\mu_2}(f) + D_{\mu_1,\mu_2}(f_R)) \\ &\leqslant 4D_{\mu_1,\mu_2}(f). \end{aligned} \tag{3.7}$$

We claim that

$$\liminf_{R_1, R_2 \to 1^-} D_{\mu_1,\mu_2}(f + f_R) \geqslant 4D_{\mu_1,\mu_2}(f). \tag{3.8}$$

To see this, fix $r \in (0, 1)$. By Fatou's lemma,

$$\begin{aligned} &\liminf_{R_1, R_2 \to 1^-} \Big(\int_{\mathbb{T}} D_{\mu_1}((f + f_R)(\cdot, re^{i\theta})) d\theta + \int_{\mathbb{T}} D_{\mu_2}((f + f_R)(re^{i\theta}, \cdot)) d\theta \Big) \\ &\geqslant 4\Big(\int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta})) d\theta + \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot)) d\theta \Big). \end{aligned} \tag{3.9}$$

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On the other hand, by Lemma 1.1,

$$\begin{aligned} &D_{\mu_1,\mu_2}(f+f_R) \\ &\geqslant \int_{\mathbb{T}} D_{\mu_1}((f+f_R)(\cdot, re^{i\theta}))d\theta + \int_{\mathbb{T}} D_{\mu_2}((f+f_R)(re^{i\theta},\cdot))d\theta. \end{aligned}$$

After taking lim inf on both sides (one by one) and applying (3.9), we get

$$\begin{aligned} &\liminf_{R_1,R_2\to 1^-} D_{\mu_1,\mu_2}(f+f_R) \\ &\geqslant 4\Big(\int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta}))d\theta + \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta},\cdot))d\theta\Big). \end{aligned}$$

Letting $r \to 1^-$ on the right-hand side now yields (3.8) (see (3.5)). Finally, note that by (3.7),

$$\limsup_{R_1,R_2\to 1^-} D_{\mu_1,\mu_2}(f-f_R) \leqslant 4D_{\mu_1,\mu_2}(f) - \liminf_{R_1,R_2\to 1^-} D_{\mu_1,\mu_2}(f+f_R),$$

and hence by (3.8), we get

$$\limsup_{R_1,R_2\to 1^-} D_{\mu_1,\mu_2}(f-f_R) = 0,$$

which completes the proof. ∎

We now complete the proof of Theorem 2.1.

Proof (Proof of Theorem 2.1) Parts (i) and (iii) are Lemmas 3.4 and 3.5, respectively. To see (ii), let $f \in \mathcal{D}(\mu_1,\mu_2)$ and $\epsilon > 0$. It suffices to check that there exists a polynomial p in z_1 and z_2 such that $\|f-p\|_{\mathcal{D}(\mu_1,\mu_2)} < \epsilon$. It is easy to see using Lemma 3.7 that there exist an $R = (R_1, R_2) \in (0,1)^2$ such that

$$\|f - f_R\|_{\mathcal{D}(\mu_1,\mu_2)} < \epsilon/2. \tag{3.10}$$

Since f_R is holomorphic in an open neighborhood of $\overline{\mathbb{D}}^2$, there exists a polynomial p such that

$$\|\partial^\alpha f_R - \partial^\alpha p\|_{\infty,\overline{\mathbb{D}}^2} < \frac{\sqrt{\epsilon}}{4\sqrt{M}}, \quad \alpha \in \{(0,0),(1,0),(0,1)\},$$

where $M = \max\Big\{\int_{\mathbb{D}} P_{\mu_j}(w)dA(w) : j = 1,2\Big\} + 1$. This together with the fact that the norm on $H^2(\mathbb{D}^2)$ is dominated by the $\|\cdot\|_{\infty,\overline{\mathbb{D}}^2}$ shows that $\|f_R - p\|_{\mathcal{D}(\mu_1,\mu_2)} < \epsilon/2$. Combining this with (3.10) yields $\|f-p\|_{\mathcal{D}(\mu_1,\mu_2)} < \epsilon$, which completes the proof. ∎

The following provides a ground to discuss operator theory on $\mathcal{D}(\mu_1,\mu_2)$.

Corollary 3.8 *For $j = 1,2$, let $\mathscr{M}_{z_j}$ denote the operator of multiplication by the coordinate function z_j. Then the commuting pair $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ on $\mathcal{D}(\mu_1,\mu_2)$ is a cyclic toral 2-isometry with cyclic vector 1.*

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Proof Note that by Theorem 2.1(i) and the closed graph theorem, $\mathscr{M}_z$ defines a pair of bounded linear operators $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ on $\mathcal{D}(\mu_1, \mu_2)$. By Theorem 2.1(ii), $\mathscr{M}_z$ is cyclic with cyclic vector 1. Finally, the fact that $\mathscr{M}_z$ is a toral 2-isometry may be derived from (ii) and (iii) of Theorem 2.1. ■

Let $\kappa : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ denote the reproducing kernel of $\mathcal{D}(\mu_1, \mu_2)$ (see Lemma 3.1).

Corollary 3.9 *For any $w \in \mathbb{D}^2$, $\ker(\mathscr{M}_z - w) = \{0\}$ and $\ker(\mathscr{M}_z^* - w)$ is the one-dimensional space spanned by $\kappa(\cdot, \overline{w})$.*

Proof Since $\mathcal{D}(\mu_1, \mu_2)$ is contained in the space of complex-valued holomorphic functions on $\mathbb{D}^2$, the pair $\mathscr{M}_z$ has no eigenvalue. By Theorem 2.1, $\mathscr{M}_z$ is cyclic, and hence, for any $w \in \mathbb{C}^2$, the dimension of $\ker(\mathscr{M}_z^* - w)$ is at most 1 (see (1.6)). If $w \in \mathbb{D}^2$, then by the reproducing property of $\mathcal{D}(\mu_1, \mu_2)$ (see Lemma 3.1), $\kappa(\cdot, \overline{w}) \in \ker(\mathscr{M}_z^* - w)$. Since $1 \in \mathcal{D}(\mu_1, \mu_2)$, once again by the reproducing property of $\mathcal{D}(\mu_1, \mu_2)$, $\kappa(\cdot, \overline{w}) \neq 0$. ■

Before we state the next application of Theorem 2.1, we need a formula for the inner-product of monomials in $\mathcal{D}(\mu_1, \mu_2)$.

Lemma 3.10 *For $\mu \in M_+(\mathbb{T})$ and $j \geqslant 0$, let $\hat{\mu}(j) = \int_{\mathbb{T}} \zeta^{-j} d\mu(\zeta)$. Then*

$$\langle z_1^m z_2^n,\, z_1^p z_2^q \rangle_{\mathcal{D}(\mu_1,\mu_2)} = \begin{cases} 0 & \text{if } m \neq p,\ n \neq q, \\ \min\{n, q\}\hat{\mu}_2(q-n) & \text{if } m = p,\ n \neq q, \\ \min\{m, p\}\hat{\mu}_1(p-m) & \text{if } m \neq p,\ n = q, \\ 1 + m\hat{\mu}_1(0) + n\hat{\mu}_2(0) & \text{if } m = p,\ n = q. \end{cases} \tag{3.11}$$

In particular, the monomials are orthogonal in $\mathcal{D}(\mu_1, \mu_2)$ if and only if μ_1 and μ_2 are nonnegative multiples of the Lebesgue measure on $\mathbb{T}$.

Proof Fix non-negative integers m, n, p, q. By the polarization identity,

$$\begin{aligned} \langle z_1^m z_2^n,\, z_1^p z_2^q \rangle_{\mathcal{D}(\mu_1,\mu_2)} &= \langle z_1^m z_2^n,\, z_1^p z_2^q \rangle_{H^2(\mathbb{D}^2)} \\ &+ \lim_{r\to 1^-} \int_{\mathbb{T}} r^{n+q} e^{i(n-q)\theta} \int_{\mathbb{D}} \partial_1(z_1^m)\, \overline{\partial_1(z_1^p)} P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &+ \lim_{r\to 1^-} \int_{\mathbb{T}} r^{m+p} e^{i(m-p)\theta} \int_{\mathbb{D}} \partial_2(z_2^n)\, \overline{\partial_2(z_2^q)} P_{\mu_2}(z_2)\, dA(z_2) d\theta. \end{aligned}$$

Since $\langle z_1^m z_2^n,\, z_1^p z_2^q \rangle_{H^2(\mathbb{D}^2)} = \delta(m, p)\delta(n, q)$ with $\delta(\cdot, \cdot)$ denoting the Kronecker delta of two variables, (3.11) may be deduced from the following formula for the inner-product of the Dirichlet-type space $\mathcal{D}(\mu)$ (see [22, Equation (3.2)]):

$$\langle z^r, z^s \rangle_{\mathcal{D}(\mu)} = \delta(r, s) + \min\{r, s\}\hat{\mu}(s-r), \quad r, s \in \mathbb{Z}_+ \tag{3.12}$$

(this formula may also be derived directly using [28, Theorem 11.9]). The "In particular" part follows from the Weierstrass approximation theorem and Riesz representation theorem. ■

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Remark 3.11 Assume that μ_1, μ_2 are non-zero. It is easy to see using (3.11) and (3.12) that $\|f\|_{\mathcal{D}(\mu_1,\mu_2)} = \|f\|_{\mathcal{D}(\mu_1)\otimes\mathcal{D}(\mu_2)}$ holds for all monomials f if and only if at least one of μ_1 and μ_2 is the zero measure. In particular, $\mathcal{D}(\mu_1, \mu_2) \neq \mathcal{D}(\mu_1) \otimes \mathcal{D}(\mu_2)$, in general.

The following is a consequence of (3.11) (see Definition 1.3).

***Corollary* 3.12** *For $\mu_1, \mu_2 \in M_+(\mathbb{T})$, the subspace of $\mathcal{D}(\mu_1, \mu_2)$ spanned by the constant function 1 is a wandering subspace for $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$.*

A bounded linear operator T on a Hilbert space is *essentially normal* if $T^*T - TT^*$ is a compact operator. An essentially normal operator is said to be *essentially unitary* if $T^*T - I$ is compact. Unlike the case of one variable Dirichlet-type spaces (see [7, Proposition 2.21]), $\mathcal{D}(\mu_1, \mu_2)$ does not support essentially normal multiplication 2-tuple $\mathscr{M}_z$.

***Corollary* 3.13** *The multiplication operators $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ on $\mathcal{D}(\mu_1, \mu_2)$ are never essentially normal.*

Proof By Corollary 3.8, the multiplication 2-tuple $\mathscr{M}_z$ is a toral 2-isometry. In particular, $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are 2-isometries. If these are essentially normal, then the image of $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ in the Calkin algebra is a normal 2-isometry, and hence $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are essentially unitary (since a normal 2-isometry, being invertible, is a unitary). It follows that $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are Fredholm. In view of Atkinson's theorem (see [9, Theorem XI.2.3]), it suffices to check that the kernels of $\mathscr{M}^*_{z_1}$ and $\mathscr{M}^*_{z_2}$ are of infinite dimension. To see this, fix a nonnegative integer j. Note that by (3.11),

$$\langle \mathscr{M}^*_{z_1} z_2^j,\, z_1^p z_2^q \rangle = \langle z_2^j,\, z_1^{p+1} z_2^q \rangle = 0, \quad p, q \in \mathbb{Z}_+,$$

and hence by the linearity of the inner-product and the density of the polynomials in $\mathcal{D}(\mu_1, \mu_2)$ (see Theorem 2.1(ii)), we obtain $\mathscr{M}^*_{z_1} z_2^j = 0$. Similarly, one can check that $z_1^j \in \ker \mathscr{M}^*_{z_2}$, completing the proof. ■

4 Proof of Theorem 2.2 and a consequence

We begin the proof of Theorem 2.2 with the following special case.

***Lemma* 4.1** *The Hardy space $H^2(\mathbb{D}^2)$ has the division property.*

Proof For $j = 1, 2$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$, let $f \in H^2(\mathbb{D}^2)$ be such that $\{z \in \mathbb{D}^2 : z_j = \lambda_j\} \subseteq Z(f)$. Let $w = (w_1, w_2) \in \mathbb{D}^2$. If $w_j \neq \lambda_j$, then clearly $g_j(z) = \frac{f(z)}{z_j - \lambda_j}$ defines a holomorphic function in a neighborhood of w. If $w_j = \lambda_j$, then since $\{z \in \mathbb{D}^2 : z_j = \lambda_j\} \subseteq Z(f)$, f as a function of z_j has a removable singularity at w_j, and hence by Hartogs' separate analyticity theorem (see [27, pp 1-2]), g_j as above extends holomorphically in a neighborhood of w. This shows that g_j is holomorphic on $\mathbb{D}^2$. To see that $g_j \in H^2(\mathbb{D}^2)$, note that for $r \in (|\lambda_j|, 1)$ and $\theta_1, \theta_2 \in [0, 2\pi]$,

$$\frac{|f(re^{i\theta_1}, re^{i\theta_2})|}{|re^{i\theta_j} - \lambda_j|} \leqslant \frac{|f(re^{i\theta_1}, re^{i\theta_2})|}{r - |\lambda_j|}.$$

Since $f \in H^2(\mathbb{D}^2)$, it may now be deduced from (1.4) that $g_j \in H^2(\mathbb{D}^2)$. ∎

Remark 4.2 One may argue as above to see that for any positive integer d, the Hardy space $H^2(\mathbb{D}^d)$ has the division property.

We also need the following fact essentially noticed in [26].

Lemma 4.3 *For any $\mu \in M_+(\mathbb{T})$, $\mathcal{D}(\mu)$ has the division property.*

Proof For $\lambda \in \mathbb{D}$, let $g \in \mathcal{D}(\mu)$ be such that $g(\lambda) = 0$. Note that g is orthogonal to $\kappa(\cdot, \lambda)$. Since $\ker(\mathscr{M}_z^* - \overline{\lambda})$ is spanned by $\kappa(\cdot, \lambda)$ and the range of $\mathscr{M}_z - \lambda$ is closed (see [26, Corollary 3.8]), there exists $f \in \mathcal{D}(\mu)$ such that $g = (z - \lambda)f$, which completes the proof. ∎

Proof (Proof of Theorem 2.2) For $\lambda \in \mathbb{D}$, assume that $(z_j - \lambda)h \in \mathcal{D}(\mu_1, \mu_2)$ for some $j = 1, 2$. Thus

$$(z_j - \lambda)h \in H^2(\mathbb{D}^2) \tag{4.1}$$

$$D_{\mu_1,\mu_2}((z_j - \lambda)h) < \infty. \tag{4.2}$$

Since the arguments for the cases $j = 1, 2$ are similar, we only treat the case when $j = 1$. It follows from Lemma 4.1 and (4.1) that $h \in H^2(\mathbb{D}^2)$. Applying (3.4) to (4.2) gives

$$D_{\mu_1}((z_1 - \lambda)h(\cdot, re^{i\theta})) < \infty, \quad r \in (0, 1),\ \theta \in \Omega_r, \tag{4.3}$$

where Ω_r is a Lebesgue measurable subset of $[0, 2\pi]$ such that $[0, 2\pi] \backslash \Omega_r$ is of measure 0. For $r \in (0, 1)$ and $\theta \in \Omega_r$, consider $f_{r,\theta} : \mathbb{D} \to \mathbb{C}$ defined by

$$f_{r,\theta}(w) = (w - \lambda)h(w, re^{i\theta}), \quad w \in \mathbb{D}.$$

By (4.1) and Lemma 3.2, $f_{r,\theta}$ belongs to $H^2(\mathbb{D})$. Hence, by (4.3), $f_{r,\theta}$ belongs to $\mathcal{D}(\mu_1)$. Hence, by Lemma 4.3, $h(\cdot, re^{i\theta}) \in \mathcal{D}(\mu_1)$. Since

$$\|h(\cdot, re^{i\theta})\|_{\mathcal{D}(\mu_1)} \leqslant \|wh(\cdot, re^{i\theta})\|_{\mathcal{D}(\mu_1)}$$

(see [26, Theorem 3.6]), by the reverse triangle inequality,

$$\|f_{r,\theta}\|^2_{\mathcal{D}(\mu_1)} \geqslant (1 - |\lambda|)^2 \|h(\cdot, re^{i\theta})\|^2_{\mathcal{D}(\mu_1)} \geqslant (1 - |\lambda|)^2 D_{\mu_1}(h(\cdot, re^{i\theta})).$$

Integrating both sides over $[0, 2\pi]$ yields

$$\begin{aligned}(1 - |\lambda|)^2 \int_0^{2\pi} D_{\mu_1}(h(\cdot, re^{i\theta}))d\theta &\leqslant \int_0^{2\pi} \|f_{r,\theta}\|^2_{H^2(\mathbb{D})} d\theta + \int_0^{2\pi} D_{\mu_1}(f_{r,\theta})d\theta. \\ &\leqslant \|(z_1 - \lambda)h\|^2_{H^2(\mathbb{D}^2)} + D_{\mu_1,\mu_2}((z_1 - \lambda)h),\end{aligned}$$

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where we used (3.2). Taking supremum over $r \in (0,1)$ gives now

$$\sup_{0<r<1} \int_0^{2\pi} D_{\mu_1}(h(\cdot, re^{i\theta}))d\theta < \infty.$$

Also, since $h \in H^2(\mathbb{D}^2)$, it now suffices to check that

$$\sup_{0<r<1} \int_0^{2\pi} D_{\mu_2}(h(re^{i\theta}, \cdot))d\theta < \infty. \tag{4.4}$$

Note that by (4.2),

$$\sup_{0<r<1} \int_0^{2\pi} D_{\mu_2}\Big(((z_1-\lambda)h)(re^{i\theta}, \cdot)\Big)d\theta < \infty.$$

However, for any $s \in (|\lambda|, 1)$,

$$\begin{aligned}
&\sup_{0<r<1} \int_0^{2\pi} D_{\mu_2}\Big(((z_1-\lambda)h)(re^{i\theta}, \cdot)\Big)d\theta \\
&\geqslant \int_0^{2\pi} D_{\mu_2}\Big(((z_1-\lambda)h)(se^{i\theta}, \cdot)\Big)d\theta \\
&\geqslant (s-|\lambda|)^2 \int_0^{2\pi} \int_{\mathbb{D}} |\partial_2 h(se^{i\theta}, w)|^2 P_{\mu_2}(w)dA(w)d\theta.
\end{aligned}$$

Applying Lemma 1.1 and letting $s \uparrow 1^-$ now yields (4.4). ∎

Before we present an application of Theorem 2.2, let us recall some facts from the multivariate spectral theory (see [11, 12, 30]). Let $T = (T_1, T_2)$ be a commuting pair on $\mathcal{H}$ and set $D_T(x) = (T_1x, T_2x)$, $x \in \mathcal{H}$. Note that

$$\text{if } D_T^* D_T \text{ is Fredholm, then } D_T \text{ has closed range.} \tag{4.5}$$

Indeed, if $D_T^* D_T$ is Fredholm, then D_T is left-Fredholm, and hence we obtain (4.5). To define the Taylor spectrum, we consider the following complex:

$$K(T, \mathcal{H}) : \{0\} \xrightarrow{0} \mathcal{H} \xrightarrow{B_2} \mathcal{H} \oplus \mathcal{H} \xrightarrow{B_1} \mathcal{H} \xrightarrow{0} \{0\}, \tag{4.6}$$

where the *boundary maps* B_1 and B_2 are given by

$$B_2(h) := (T_2h, -T_1h), \quad B_1(h_1, h_2) := T_1h_1 + T_2h_2.$$

Note that $K(T, \mathcal{H})$ is a complex, that is, $B_1 \circ B_2 = 0$. Let $H^k(T)$ denote the k-th cohomology group in $K(T, \mathcal{H})$, $k = 0, 1, 2$. Following [30] (resp. [11]), we say that T is *Taylor-invertible* (resp. *Fredholm*) if $H^k(T) = \{0\}$ (resp. $\dim H^k(T) < \infty$) for $k = 0, 1, 2$. The *Taylor spectrum* $\sigma(T)$ and the *essential spectrum* $\sigma_e(T)$ are given by

$$\begin{aligned}
\sigma(T) &= \{\lambda \in \mathbb{C}^2 : T - \lambda \text{ is not Taylor-invertible}\}, \\
\sigma_e(T) &= \{\lambda \in \mathbb{C}^2 : T - \lambda \text{ is not Fredholm}\}.
\end{aligned}$$

The *Fredholm index* $\text{ind}(T)$ of a commuting 2-tuple T on $\mathcal{H}$ is the *Euler characteristic* of the Koszul complex $K(T, \mathcal{H})$, that is,

$$\text{ind}(T) := \dim H^0(T) - \dim H^1(T) + \dim H^2(T). \tag{4.7}$$

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As an application of the division property, we now show that we always have exactness at the middle stage of the Koszul complex of the multiplication 2-tuple $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$. First a general fact.

***Lemma* 4.4** *Let $\mathscr{H}$ be a reproducing kernel Hilbert space of complex-valued holomorphic functions on the unit bidisc $\mathbb{D}^2$. Assume that $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a commuting pair on $\mathscr{H}$. If $\mathscr{H}$ has the division property, then for every $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$, the Koszul complex of $\mathscr{M}_z - \lambda = (\mathscr{M}_{z_1} - \lambda_1, \mathscr{M}_{z_2} - \lambda_2)$ is exact at the middle stage (see (4.6)).*

Proof Note that $\mathscr{H}$ has the division property if and only if for $j = 1, 2$, we have the following property:

$$\begin{aligned}&\text{for any holomorphic function } h : \mathbb{D}^2 \to \mathbb{C} \text{ and } \lambda \in \mathbb{D}^2,\\ &\text{if } (z_j - \lambda_j)h \in \mathscr{H}, \text{ then } h \in \mathscr{H}.\end{aligned} \tag{4.8}$$

We first assume that (4.8) holds for $j = 2$. To see that the Koszul complex of $\mathscr{M}_z - \lambda$ is exact at the middle stage, let $g, h \in \mathscr{H}$ be such that

$$(z_2 - \lambda_2)g(z_1, z_2) = (z_1 - \lambda_1)h(z_1, z_2), \quad (z_1, z_2) \in \mathbb{D}^2. \tag{4.9}$$

Letting $z_2 = \lambda_2$, we obtain $(w - \lambda_1)h(w, \lambda_2) = 0$ for every $w \in \mathbb{D}$. It follows that $h(\cdot, \lambda_2) = 0$ on $\mathbb{D}$. Since $h : \mathbb{D}^2 \to \mathbb{C}$ is holomorphic, there exists a holomorphic function $k : \mathbb{D}^2 \to \mathbb{C}$ such that

$$h(z_1, z_2) = (z_2 - \lambda_2)k(z_1, z_2), \quad (z_1, z_2) \in \mathbb{D}^2 \tag{4.10}$$

(in case of $\lambda_2 = 0$, this can be seen using the power series for h; the general case can be dealt now by replacing $h(z_1, z_2)$ by $h(z_1, \varphi(z_2))$, where φ is the automorphism of $\mathbb{D}$ which takes λ_2 to 0). Since $h \in \mathscr{H}$, by (4.8), $k \in \mathscr{H}$. We now combine (4.9) with (4.10) to obtain

$$\begin{aligned}(z_2 - \lambda_2)g(z_1, z_2) &= (z_1 - \lambda_1)h(z_1, z_2)\\ &= (z_1 - \lambda_1)(z_2 - \lambda_2)k(z_1, z_2), \quad z \in \mathbb{D}^2.\end{aligned}$$

This gives $g(z_1, z_2) = (z_1 - \lambda_1)k(z_1, z_2)$, $z \in \mathbb{D}^2$. This together with (4.10) shows that $\mathscr{M}_z - \lambda$ is exact at the middle stage. We may also obtain the same conclusion in case (4.8) holds for $j = 1$. Indeed, one may proceed as above with the only change that the roles of λ_1 and λ_2 are interchanged (e.g. (4.9) is evaluated at $z_1 = \lambda_1$). ∎

Remark 4.5 Let Ω be a bounded domain in $\mathbb{C}^2$. One may imitate the first part of the proof of Lemma 4.1 to show that there exists a holomorphic function $k : \Omega \to \mathbb{C}$ satisfying (4.10). This gives an analog of Lemma 4.4 for arbitrary bounded domains.

The following is a consequence of Theorem 2.2 and Lemma 4.4.

***Corollary* 4.6** *For every $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$, the Koszul complex of the 2-tuple $\mathscr{M}_z - \lambda = (\mathscr{M}_{z_1} - \lambda_1, \mathscr{M}_{z_2} - \lambda_2)$ on $\mathcal{D}(\mu_1, \mu_2)$ is exact at the middle stage (see (4.6)).*

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5 Proof of Theorem 2.3 and its consequences

We begin with a lemma, which is a variant of [17, Lemma 4.14]. We include its proof for the sake of completeness.

***Lemma* 5.1** *For a domain Ω of $\mathbb{C}^2$, let $\mathscr{H}$ be the reproducing kernel Hilbert space of complex-valued holomorphic functions associated with the kernel $\kappa : \Omega \times \Omega \to \mathbb{C}$. Assume that the constant function 1 belongs to $\mathscr{H}$, the multiplication operators $\mathscr{M}_{z_1}, \mathscr{M}_{z_2}$ are bounded on $\mathscr{H}$ and the commuting 2-tuple $\mathscr{M}_z$ is cyclic. For $w \in \Omega$, Gleason's problem can be solved for $\mathscr{H}$ over $\{w\}$ if and only if*

$$D^*_{\mathscr{M}_z^* - \overline{w}} \text{ has closed range.} \tag{5.1}$$

In particular, Gleason's problem can be solved for $\mathscr{H}$ over $\Omega \backslash \sigma_e(\mathscr{M}_z)$.

Proof Let $w \in \Omega$ and let $f \in \mathscr{H}$. By the reproducing kernel property of $\mathscr{H}$,

$$f - f(w) \in \{c\kappa(\cdot, w) : c \in \mathbb{C}\}^{\perp}. \tag{5.2}$$

However, since $\mathscr{M}_z$ is cyclic, $\dim \ker(\mathscr{M}_z^* - \overline{w}) \leqslant 1$ for every $w \in \mathbb{C}^2$ (see (1.6)). As $1 \in \mathscr{H}$, we have $\kappa(\cdot, w) \neq 0$, and hence

$$\{c\kappa(\cdot, w) : c \in \mathbb{C}\} = \ker(\mathscr{M}_z^* - \overline{w}) = \ker D_{\mathscr{M}_z^* - \overline{w}}.$$

It now follows from (5.2) that

$$f - f(w) \in (\ker D_{\mathscr{M}_z^* - \overline{w}})^{\perp} = \overline{\operatorname{ran}(D^*_{\mathscr{M}_z^* - \overline{w}})}. \tag{5.3}$$

Also, it is easy to see that

$$\operatorname{ran}(D^*_{\mathscr{M}_z^* - \overline{w}}) = \{(z_1 - w_1)g_1 + (z_2 - w_2)g_2 : g_1, g_2 \in \mathscr{H}\}. \tag{5.4}$$

If (5.1) holds, then it now follows from (5.3) that

$$f - f(w) \in \{(z_1 - w_1)g_1 + (z_2 - w_2)g_2 : g_1, g_2 \in \mathscr{H}\},$$

and hence Gleason's problem can be solved for $\mathscr{H}$ over $\{w\}$. Conversely, if Gleason's problem can be solved for $\mathscr{H}$ over $\{w\}$, then by (5.3), any function in $\overline{\operatorname{ran}(D^*_{\mathscr{M}_z^* - \overline{w}})}$ is of the form $f - f(w)$ for some $f \in \mathscr{H}$, and hence by (5.4), it belongs to $\operatorname{ran}(D^*_{\mathscr{M}_z^* - \overline{w}})$. This completes the proof of the equivalence.

To see the remaining part, let $w = (w_1, w_2) \in \Omega \backslash \sigma_e(\mathscr{M}_z)$. Since $D_S^* D_S = S_1^* S_1 + S_2^* S_2$ for any commuting pair $S = (S_1, S_2)$, by [11, Corollary 3.6], the operator $D^*_{\mathscr{M}_z^* - \overline{w}} D_{\mathscr{M}_z^* - \overline{w}}$ is Fredholm, and hence by (4.5), $D_{\mathscr{M}_z^* - \overline{w}}$ has closed range. Hence, by the closed-range theorem (see [9, Theorem VI.1.10]), we obtain (5.1) completing the proof. ∎

Remark 5.2 Let $\mathscr{M}_z$ be the multiplication 2-tuple on the Hardy space $H^2(\mathbb{D}^2)$ of the unit bidisc $\mathbb{D}^2$. Since $\sigma_e(\mathscr{M}_z) \cap \mathbb{D}^2 = \emptyset$ (see [11, Theorem 5(c)]), by Lemma 5.1, Gleason's problem can be solved for $H^2(\mathbb{D}^2)$.

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The following lemma provides a situation in which the division property ensures a solution to Gleason's problem.

Lemma **5.3** *Let $\mathscr{H}$ be a reproducing kernel Hilbert space of complex-valued holomorphic functions on the unit bidisc $\mathbb{D}^2$ and let $w = (w_1, w_2) \in \mathbb{D}^2$. Assume that $\mathscr{H}$ has the division property and $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a commuting pair on $\mathscr{H}$. If, for every $f \in \mathscr{H}$, either $f(\cdot, w_2)$ or $f(w_1, \cdot)$ belongs to $\mathscr{H}$, then Gleason's problem can be solved for $\mathscr{H}$ over $\{w\}$.*

Proof For $f \in \mathscr{H}$, assume that $f(w_1, \cdot) \in \mathscr{H}$. Thus $f - f(w_1, \cdot) \in \mathscr{H}$. Hence, if $h : \mathbb{D}^2 \to \mathbb{C}$ is a holomorphic function such that

$$f(z_1, z_2) - f(w_1, z_2) = (z_1 - w_1)h(z_1, z_2), \quad z_1, z_2 \in \mathbb{D}, \tag{5.5}$$

by the division property for $\mathscr{H}$, we have $h \in \mathscr{H}$. Also, since $f(w_1, \cdot) \in \mathscr{H}$, one may argue as above to see that there exists $k \in \mathscr{H}$ satisfying

$$f(w_1, z_2) - f(w_1, w_2) = (z_2 - w_2)k(z_1, z_2), \quad z_1, z_2 \in \mathbb{D}..$$

This, combined with (5.5), completes the proof in this case. Similarly, one can deal with the case in which $f(\cdot, w_2) \in \mathscr{H}$. ■

We also need the following fact of independent interest:

Lemma **5.4** *For every $f \in \mathcal{D}(\mu_1, \mu_2)$, the slice functions $f(\cdot, 0)$ and $f(0, \cdot)$ belong to $\mathcal{D}(\mu_1, \mu_2)$. Moreover, the mappings $f \mapsto f(\cdot, 0)$ and $f \mapsto f(0, \cdot)$ from $\mathcal{D}(\mu_1, \mu_2)$ into itself are contractive homomorphisms.*

Proof If $f \in \mathcal{D}(\mu_1, \mu_2)$, then

$$\begin{aligned} &\int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_1 f(z_1, 0)|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &+ \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 f(0, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta \leqslant D_{\mu_1, \mu_2}(f). \end{aligned}$$

Since the mappings $f \mapsto f(\cdot, 0)$ and $f \mapsto f(0, \cdot)$ from $H^2(\mathbb{D}^2)$ into itself are contractive homomorphisms, the desired conclusions may be deduced from the estimate above. ■

Proof (Proof of Theorem 2.3) Since $\mathcal{D}(\mu_1, \mu_2)$ has the division property (see Theorem 2.2), by Lemmas 5.3 and 5.4, Gleason's problem can be solved for $\mathscr{H}$ over $\{(0, 0)\}$. Hence, by Lemma 5.1 (which is applicable since $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ is cyclic; see Theorem 2.1), $D^*_{\mathscr{M}_z^*}$ has closed range (see (5.1)). It is now easy to see using Corollaries 3.9 and 4.6 that $\mathscr{M}_z$ is Fredholm. Since the essential spectrum is a closed subset of $\mathbb{C}^2$ not containing $(0, 0)$, there exists $r > 0$ such that $\mathbb{D}^2_r \subseteq \mathbb{C}^2 \backslash \sigma_e(\mathscr{M}_z)$. Another application of Lemma 5.1 now completes the proof. ■

The following fact is implicit in the proof of Theorem 2.3.

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***Corollary* 5.5** *The commuting 2-tuple $\mathscr{M}_z^*$ on $\mathcal{D}(\mu_1, \mu_2)$ belongs to $\mathbf{B}_1(\mathbb{D}_r^2)$ for some $r \in (0, 1]$.*

Proof This may be deduced from Theorem 2.3, Lemma 3.1, Corollary 3.9 and Lemma 5.1 (see (5.1)). ■

We conclude this section with the following corollary describing the cokernels of the multiplication operators $\mathscr{M}_{z_j}$, $j = 1, 2$, on $\mathcal{D}(\mu_1, \mu_2)$.

***Corollary* 5.6** *For $1 \leqslant i \neq j \leqslant 2$,* $\ker \mathscr{M}_{z_j}^* = \bigvee \left\{z_i^k : k \geqslant 0\right\}$.

Proof As observed in the proof of Corollary 3.13,

$$\{p(z_i) : p \in \mathbb{C}[w]\} \subseteq \ker \mathscr{M}_{z_j}^*, \quad 1 \leqslant i \neq j \leqslant 2. \tag{5.6}$$

To see the reverse inclusion, let $f \in \ker \mathscr{M}_{z_1}^*$. By Theorem 2.1, there exists a sequence $\{p_n\}_{n \geqslant 1}$ of complex polynomials in z_1, z_2 converging to f. By Lemma 5.4, $f(0, \cdot) \in \mathcal{D}(\mu_1, \mu_2)$, and $\{p_n(0, \cdot)\}_{n \geqslant 1}$ converges to $f(0, \cdot)$. Hence, by (5.6), $f(0, \cdot) \in \ker \mathscr{M}_{z_1}^*$. Thus $f - f(0, \cdot) \in \ker \mathscr{M}_{z_1}^*$. However, there exists a holomorphic function $h : \mathbb{D}^2 \to \mathbb{C}$ such that

$$f(z_1, z_2) - f(0, z_2) = z_1 h(z_1, z_2), \quad z_1, z_2 \in \mathbb{D}. \tag{5.7}$$

By Theorem 2.2, we have $h \in \mathcal{D}(\mu_1, \mu_2)$. It now follows from (5.7) that $\mathscr{M}_{z_1} h \in \ker \mathscr{M}_{z_1}^*$. Since $\mathscr{M}_{z_1}^* \mathscr{M}_{z_1}$ is invertible, we must have $h = 0$, and hence, by (5.7), $f = f(0, \cdot)$, or equivalently, f belongs to the closure of $\{p(z_2) : p \in \mathbb{C}[w]\}$. Similarly, one can check that $\ker \mathscr{M}_{z_2}^*$ is equal to the closure of $\{p(z_1) : p \in \mathbb{C}[w]\}$. ■

6 Proof of Theorem 2.4 and its consequences

The proof of Theorem 2.4 relies on revealing the structure of toral 2-isometries T with $\ker T^*$ as a wandering subspace (see Definition 1.3).

***Lemma* 6.1** *Let $T = (T_1, T_2)$ be a toral 2-isometry. Then the following statements are true:*

(i) *for any integers $k, l \geqslant 0$,*

$$\begin{aligned} T_1^{*k} T_2^{*l} T_2^l T_1^k &= T_1^{*k} T_1^k + T_2^{*l} T_2^l - I \\ &= k T_1^* T_1 + l T_2^* T_2 - (k + l - 1) I, \end{aligned} \tag{6.1}$$

(ii) *for $f_0 \in \ker T^*$, assume that*

$$\langle T_1^m f_0, T_1^p T_2^q f_0 \rangle = 0, \quad q \geqslant 1, m, p \geqslant 0, \tag{6.2}$$

$$\langle T_2^n f_0, T_1^p T_2^q f_0 \rangle = 0, \quad p \geqslant 1, n, q \geqslant 0. \tag{6.3}$$

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Then we have the following:

$$\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle = \begin{cases} 0 & \textit{if } m \neq p,\ n \neq q, \\ \langle T_2^n f_0,\ T_2^q f_0\rangle & \textit{if } m = p,\ n \neq q, \\ \langle T_1^m f_0,\ T_1^p f_0\rangle & \textit{if } m \neq p,\ n = q, \\ \|T_1^m f_0\|^2 + \|T_2^n f_0\|^2 - \|f_0\|^2 & \textit{if } m = p,\ n = q. \end{cases}$$

Proof (i) To see (6.1), we proceed by strong induction on $k+l$, $k, l \geqslant 0$. Clearly, (6.1) holds for $0 \leqslant k+l \leqslant 1$. Assume that (6.1) holds for integers $k, l \geqslant 0$ such that $0 \leqslant k+l \leqslant n$. Note that for $k \geqslant 1$ and $l \leqslant n-1$, by the induction hypothesis,

$$\begin{aligned} T_1^*(T_1^{*k}T_2^{*l}T_2^lT_1^k)T_1 &= T_1^{*k+1}T_1^{k+1} + T_1^*T_2^{*l}T_2^lT_1 - T_1^*T_1 \\ &= T_1^{*k+1}T_1^{k+1} + T_2^{*l}T_2^l - I. \end{aligned}$$

Similarly, for $k \leqslant n-1$ and $l \geqslant 1$, (6.1) holds. This completes the induction argument. The remaining identity in (i) now follows from the known fact that for any 2-isometry S, we have

$$S^{*k}S^k = k(S^*S - I) + I, \quad k \geqslant 0 \tag{6.4}$$

(this known fact can be seen by induction on $k \geqslant 1$).

(ii) Let m, n, p, q be integers such that $m \neq p$ and $n \neq q$. Consider the case when $m < p$ and $n < q$. Since $f_0 \in \ker T^*$, we have

$$\begin{aligned} &\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle \\ =\ &\langle T_2^{*n}T_1^{*m}T_1^mT_2^n f_0,\ T_1^{p-m}T_2^{q-n} f_0\rangle \\ \overset{(6.1)}{=}\ &\langle T_1^{*m}T_1^m f_0,\ T_1^{p-m}T_2^{q-n} f_0\rangle + \langle T_2^{*n}T_2^n f_0,\ T_1^{p-m}T_2^{q-n} f_0\rangle \\ =\ &\langle T_1^m f_0,\ T_1^pT_2^{q-n} f_0\rangle + \langle T_2^n f_0,\ T_2^qT_1^{p-m} f_0\rangle, \end{aligned}$$

which, by (6.2) and (6.3), is equal to 0. Since the inner-product is conjugate linear, $\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle = 0$ when $p < m$ and $q < n$. Consider the case when $m < p$ and $q < n$. Arguing as above, we have

$$\begin{aligned} &\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle \\ =\ &\langle T_2^{*q}T_1^{*m}T_1^mT_2^qT_2^{n-q} f_0,\ T_1^{p-m} f_0\rangle \\ \overset{(6.1)\&(6.3)}{=}\ &\langle T_1^{*m}T_1^mT_2^{n-q} f_0,\ T_1^{p-m} f_0\rangle + \langle T_2^{*q}T_2^qT_2^{n-q} f_0,\ T_1^{p-m} f_0\rangle \\ =\ &\langle T_1^mT_2^{n-q} f_0,\ T_1^p f_0\rangle + \langle T_2^n f_0,\ T_2^qT_1^{p-m} f_0\rangle, \end{aligned}$$

which, by (6.2) and (6.3), is equal to 0. Once again, by the conjugate-symmetry, $\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle = 0$ when $p < m$ and $n < q$.

If $m = p$ and $n \neq q$, then one may argue as above using (6.4) (by making cases $n < q$ and $q < n$) to show that $\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle = \langle T_2^n f_0,\ T_2^q f_0\rangle$. Similarly, one may derived the formula in case of $m \neq p$ and $n = q$. Finally, if $m = p$ and $n = q$, then the formula follows from (6.1). ∎

Proof (Proof of Theorem 2.4) (i)⇒(iii) Fix $j \in \{1, 2\}$. Since T is analytic, so is T_j. Thus T_j is an analytic 2-isometry. Consider the T_j-invariant subspace $\mathcal{H}_j := \bigvee\{T_j^k f_0 :$

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$k \geqslant 0\}$ and note that $T_j|_{\mathcal{H}_j}$ is a cyclic analytic 2-isometry. Hence, by [26, Theorem 5.1], there exist a finite positive Borel measure μ_j on $\mathbb{T}$ and a unitary map $V_j : \mathcal{H}_j \to \mathcal{D}(\mu_j)$ such that

$$V_j f_0 = 1, \quad V_j T_j = \mathscr{M}_w^{(j)} V_j, \tag{6.5}$$

where $\mathscr{M}_w^{(j)}$ denotes the operator of multiplication by the coordinate function w on $\mathcal{D}(\mu_j)$. We contend that the map is given by

$$U(T_1^k T_2^l f_0) = z_1^k z_2^l, \quad k, l \geqslant 0$$

extends to an unitary from $\mathcal{H}$ onto $\mathcal{D}(\mu_1, \mu_2)$. Since $\mathcal{H} = \vee\{T_1^k T_2^l f_0 : k, l \geqslant 0\}$ and $\mathcal{D}(\mu_1, \mu_2) = \vee\{z_1^k z_2^l : k, l \geqslant 0\}$, it suffices to check that

$$\langle T_1^m T_2^n f_0, T_1^p T_2^q f_0\rangle = \langle z_1^m z_2^n, z_1^p z_2^q\rangle_{\mathcal{D}(\mu_1,\mu_2)}, \quad m, n, p, q \geqslant 0. \tag{6.6}$$

Note that for any integers $m, n \geqslant 0$, by (6.5),

$$\begin{aligned}
\langle T_j^m f_0, T_j^n f_0\rangle &= \langle V_j T_j^m f_0, V_j T_j^n f_0\rangle_{\mathcal{D}(\mu_j)} \\
&= \langle (\mathscr{M}_w^{(j)})^m V_j f_0, (\mathscr{M}_w^{(j)})^n V_j f_0\rangle_{\mathcal{D}(\mu_j)} \\
&= \langle w^m, w^n\rangle_{\mathcal{D}(\mu_j)} \\
&= \langle z_j^m, z_j^n\rangle_{\mathcal{D}(\mu_1,\mu_2)}.
\end{aligned}$$

Since (6.2) and (6.3) hold (as $\ker T^*$ is a wandering subspace for T), combining this with Lemma 6.1(ii) yields (6.6), which completes the proof.

(iii)⇒(ii) This follows from Corollaries 3.8, 3.12 and 5.5.

(ii)⇒(i) It suffices to check that T is analytic. By Oka-Grauert's theorem (see [21, P. 71, Corollary 2.17], [16, P. 3]), every holomorphic vector bundle on a bidisc is holomorphically trivial. Combining this with the proof of [16, Theorem 4.5] shows that if $T^* \in \mathbf{B}_1(\mathbb{D}_r^2)$, then T is unitarily equivalent to the multiplication 2-tuple $\mathscr{M}_z$ on a reproducing kernel Hilbert space of scalar-valued holomorphic functions on $\mathbb{D}_r^2$. Since $\mathscr{M}_z$ is analytic, T is analytic. ■

The conclusion of Theorem 2.4 can be rephrased as follows:

***Corollary* 6.2** *A cyclic analytic toral 2-isometric 2-tuple on $\mathcal{H}$ is unitarily equivalent to the multiplication pair $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ if and only if* $\ker T^*$ *is wandering subspace for T spanned by a cyclic vector for T.*

Remark 6.3 Let $\mathcal{D}$ denote the Dirichlet space (that is, the Dirichlet-type space associated with the Lebesgue measure on the unit circle) and let $\mathscr{M}_w$ be the operator of multiplication by w on $\mathcal{D}$. It is easy to see that the commuting pair $T = (\mathscr{M}_w, \mathscr{M}_w)$ is a cyclic analytic toral 2-isometry on $\mathcal{D}$. Note that $\ker T^* = \ker \mathscr{M}_w^*$ is spanned by 1 and it is not a wandering subspace for T. It is evident that T is not unitarily equivalent to the multiplication pair $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ for any $\mu_1, \mu_2 \in M_+(\mathbb{T})$.

The following is a 2-variable analog of [26, Theorem 5.2].

Theorem 6.4 *For $j = 1, 2$, let $\mu_1^{(j)}, \mu_2^{(j)} \in M_+(\mathbb{T})$. Then the multiplication 2-tuple $\mathscr{M}_z^{(1)}$ on $\mathcal{D}(\mu_1^{(1)}, \mu_2^{(1)})$ is unitarily equivalent to the multiplication 2-tuple $\mathscr{M}_z^{(2)}$ on $\mathcal{D}(\mu_1^{(2)}, \mu_2^{(2)})$ if and only if $\mu_j^{(1)} = \mu_j^{(2)}$, $j = 1, 2$.*

Proof Suppose there is a unitary operator $U : \mathcal{D}(\mu_1^{(1)}, \mu_2^{(1)}) \to \mathcal{D}(\mu_1^{(2)}, \mu_2^{(2)})$ such that

$$\mathscr{M}_{z_j}^{(2)} U = U \mathscr{M}_{z_j}^{(1)}, \quad j = 1, 2. \tag{6.7}$$

Since the joint kernel of the adjoint of multiplication tuples is spanned by 1, by (6.7), U must map 1 to some constant of modulus 1. After multiplying U by a unimodular constant, if required, we may assume that $U1 = 1$. It now follows from (6.7) that U is identity on polynomials. By Lemma 3.5 (applied twice), we obtain for any polynomial p in two variables,

$$\int_{\mathbb{T}^2} |p(e^{i\eta}, e^{i\theta})|^2 d\mu_1^{(1)}(\eta) d\theta = \int_{\mathbb{T}^2} |p(e^{i\eta}, e^{i\theta})|^2 d\mu_1^{(2)}(\eta) d\theta,$$
$$\int_{\mathbb{T}^2} |p(e^{i\theta}, e^{i\eta})|^2 d\mu_2^{(1)}(\eta) d\theta = \int_{\mathbb{T}^2} |p(e^{i\theta}, e^{i\eta})|^2 d\mu_2^{(2)}(\eta) d\theta.$$

It is easy to see that for any polynomial p in one variable,

$$\int_{\mathbb{T}} |p(e^{i\eta})|^2 d\mu_j^{(1)}(\eta) = \int_{\mathbb{T}} |p(e^{i\eta})|^2 d\mu_j^{(2)}(\eta), \quad j = 1, 2.$$

Combining polarization identity with the uniqueness of the trigonometric moment problem yields the desired uniqueness. ■

Remark 6.5 One may use Lemma 3.5 and argue as in [26, Theorem 6.2] to obtain the following fact: *For $j = 1, 2$, let $\mu_1^{(j)}, \mu_2^{(j)} \in M_+(\mathbb{T})$. Then*

$$\mathcal{D}(\mu_1^{(1)}, \mu_2^{(1)}) \subseteq \mathcal{D}(\mu_1^{(2)}, \mu_2^{(2)})$$

if and only if $\mu_j^{(2)} \ll \mu_j^{(1)}$ and the Radon-Nikodým derivative $d\mu_j^{(2)}/d\mu_j^{(1)} \in L^\infty(\mathbb{T})$, $j = 1, 2$. We leave the details to the reader.

We conclude this section with an application to toral isometries.

Corollary 6.6 *Let $T = (T_1, T_2)$ be a cyclic analytic toral isometry with cyclic vector $f_0 \in \ker T^*$. Then the following statements are equivalent:*

(i) $\ker T^*$ *is a wandering subspace for T,*
(ii) *T is unitarily equivalent to $\mathscr{M}_z$ on $H^2(\mathbb{D}^2)$,*
(iii) *T is doubly commuting, that is, $T_j^* T_i = T_i T_j^*$, $1 \leqslant i \neq j \leqslant 2$.*

Proof (i)⟹(ii) By Theorem 2.4, there exist $\mu_1, \mu_2 \in M_+(\mathbb{T})$ such that T is unitarily equivalent to $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$. Since T is a toral isometry, $\mathscr{M}_z$ is also a toral isometry.

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It now follows from (2.1) that for every polynomial p in two variables,

$$\int_{\mathbb{T}^2} |p(e^{i\eta}, e^{i\theta})|^2 d\mu_1(\eta) d\theta = 0, \quad \int_{\mathbb{T}^2} |p(e^{i\theta}, e^{i\eta})|^2 d\mu_2(\eta) d\theta = 0.$$

One may now argue as the proof of Theorem 6.4 to conclude that $\mu_1 = 0$ and $\mu_2 = 0$. This yields (ii).

The implication (ii)⇒(iii) is a routine verification, while the implication (iii)⇒(i) is recorded in Remark 1.3. ■

7 Concluding remarks

We conclude the paper with a brief discussion on the spectral picture of the multiplication 2-tuple $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$. We claim that

$$\sigma(\mathscr{M}_z) = \overline{\mathbb{D}}^2, \tag{7.1}$$

$$\sigma_e(\mathscr{M}_z) \subseteq \overline{\mathbb{D}}^2 \setminus \Omega \tag{7.2}$$

for some open set Ω in $\mathbb{C}^2$ containing $(\mathbb{D}\times\{0\})\cup(\{0\}\times\mathbb{D})$. To see (7.1), note that by [12, Theorem 4.9], for any commuting pair $T = (T_1, T_2)$, $\sigma(T) \subseteq \sigma(T_1) \times \sigma(T_2)$. Since the spectrum of any 2-isometry is contained in $\overline{\mathbb{D}}$ (see [2, Lemma 1.21]) and both $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are 2-isometries (see Corollary 3.8), we obtain $\sigma(\mathscr{M}_z) \subseteq \overline{\mathbb{D}}^2$. Also, by Corollary 3.9, $\mathbb{D}^2 \subseteq \sigma_p(\mathscr{M}_z^*) \subseteq \sigma(\mathscr{M}_z^*)$. Since $\sigma(\mathscr{M}_z^*) = \{\overline{z} : z \in \sigma(\mathscr{M}_z)\}$, we have the inclusion $\mathbb{D}^2 \subseteq \sigma(\mathscr{M}_z)$. Finally, since the Taylor spectrum is closed (see [12, Corollary 4.2]), we obtain (7.1). On the other hand, an examination of the proof of Theorem 2.3 (using the full power of Lemma 5.3 together with Lemma 5.4) shows that

$$(\mathbb{D} \times \{0\}) \cup (\{0\} \times \mathbb{D}) \subseteq \mathbb{C}^2 \setminus \sigma_e(\mathscr{M}_z).$$

Since the essential spectrum is a closed subset of the Taylor spectrum, (7.2) now follows from (7.1). The natural question arises *whether the unit bidisc lies in the complement of the essential spectrum of* $\mathscr{M}_z$ (there are interesting examples of toral 2-isometries supporting this possibility; see [5, Proposition 5(iii)]). If this question has an affirmative answer, then $\sigma_e(\mathscr{M}_z) = \partial(\mathbb{D}^2)$. Indeed, if $\lambda \in \partial(\mathbb{D}^2)\setminus\sigma_e(\mathscr{M}_z)$, then there exist two sequences in $\mathbb{D}^2$ and $\mathbb{C}^2\setminus\overline{\mathbb{D}}^2$ converging to λ, which together with the continuity of the Fredholm index (see (4.7)) leads to a contradiction. This in turn leads to an improvement of Corollary 5.5 providing a bidisc analog of [26, Corollary 3.8] and also solves Gleason's problem for $\mathcal{D}(\mu_1, \mu_2)$ (see Lemma 5.1).

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