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Tensor Algebras, Induced Representations, and the Wold Decomposition

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Abstract. Our objective in this sequel to [18] is to develop extensions, to representations of tensor algebras over C^*-correspondences, of two fundamental facts about isometries on Hilbert space: The Wold decomposition theorem and Beurling's theorem, and to apply these to the analysis of the invariant subspace structure of certain subalgebras of Cuntz-Krieger algebras.

1 Introduction

Throughout, E will be a C^*-correspondence over a fixed C^*-algebra A. This means that E is a Hilbert C^*-module over A and that E is endowed with a left module structure over A. That is, there is a $*$-homomorphism φ from A to the bounded, adjointable operators on E, $\mathcal{L}(E)$, such that the left module structure is given in terms of φ: $a\xi = \varphi(a)\xi$, for all $a \in A$ and $\xi \in E$. We denote the A-valued inner product on E by $\langle \cdot\,, \cdot \rangle$. The *full Fock space* over E will be denoted by $\mathcal{F}(E)$, so $\mathcal{F}(E) = A \oplus E \oplus E^{\otimes 2} \oplus \cdots$. (The tensor products here are internal tensor products, see [18] or [19].) The space $\mathcal{F}(E)$ is evidently a Hilbert C^*-module over A, being the direct sum of Hilbert C^*-modules, and it is also a C^*-correspondence with left action φ_∞ given by the formula $\varphi_\infty(a) = \operatorname{diag}\bigl(a, \varphi(a), \varphi^{(2)}(a), \dots\bigr)$, where $\varphi^{(n)}(a)(\xi_1 \otimes \xi_2 \otimes \cdots \xi_n) = \bigl(\varphi(a)\xi_1\bigr) \otimes \xi_2 \otimes \cdots \xi_n$. For $\xi \in E$, we write T_ξ for the creation operator on $\mathcal{F}(E)$: $T_\xi \eta = \xi \otimes \eta$, $\eta \in \mathcal{F}(E)$. Then T_ξ is a continuous, adjointable operator on $\mathcal{F}(E)$. The norm closed subalgebra of $\mathcal{L}\bigl(\mathcal{F}(E)\bigr)$ generated by all the T_ξ's and $\varphi_\infty(A)$ is called *the tensor algebra over E* and is denoted $\mathcal{T}_+(E)$. The C^*-algebra generated by $\mathcal{T}_+(E)$ is called the *Toeplitz algebra of the correspondence* and is denoted by $\mathcal{T}(E)$.

A certain quotient of $\mathcal{T}(E)$, called the Cuntz-Pimsner algebra and denoted $\mathcal{O}(E)$, plays an important role in this work. To define it, let $\mathfrak{J}$ denote the C^*-subalgebra of $\mathcal{L}\bigl(\mathcal{F}(E)\bigr)$ generated by operators of the form $\mathbf{Q}_n T \mathbf{Q}_m$, $n, m = 0, 1, \dots$, where $\mathbf{Q}_n$ denotes the projection of $\mathcal{F}(E)$ onto the summand $E^{\otimes n}$ (by definition, $E^{\otimes 0} = A$), and $T \in \mathcal{L}\bigl(\mathcal{F}(E)\bigr)$. Then $\mathcal{T}(E)$ is contained in the multiplier algebra of $\mathfrak{J}$, $M(\mathfrak{J})$, and by definition $\mathcal{O}(E)$ is the image of $\mathcal{T}(E)$ in the corona algebra $M(\mathfrak{J})/\mathfrak{J}$. It is not hard to see that $\mathcal{O}(E)$ may reduce to 0. However, if φ is injective, then $\mathcal{O}(E) \neq 0$, as was shown by Pimsner in [19].

In the special case when $A = E = \mathbb{C}$ and φ has its only possible meaning, one finds easily that $\mathcal{F}(E)$ may be identified with $\ell^2(\mathbb{Z}_+)$; $\mathcal{T}_+(E)$ is (completely) isometrically isomorphic to

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the disc algebra $A(\mathbb{D})$, viewed as represented by analytic Toeplitz matrices; $\mathcal{T}(E)$, then, is the C^*-algebra generated by all Toeplitz operators with continuous symbols; and $\mathcal{O}(E)$ is naturally C^*-isomorphic to $C(\mathbb{T})$.

Coburn's celebrated theorem [6] says that when $A = E = \mathbb{C}$, C^*-representations of $\mathcal{T}(E)$ are in bijective correspondence with Hilbert space isometries. Two representations of $\mathcal{T}(E)$ are unitarily equivalent if and only if the associated isometries are unitarily equivalent. Wold's decomposition theorem asserts that every isometry decomposes into a direct sum of a pure isometry, *i.e.*, a shift of a suitable multiplicity, and a unitary operator. This decomposition decomposes the associated C^*-representation, of course. Shifts are determined up to unitary equivalence by their multiplicities. Unitary operators, in turn, correspond bijectively with C^*-representations of $C(\mathbb{T})$; spectral theory parameterizes their unitary equivalence classes.

In the setting of general C^*-correspondences, the C^*-representations of $\mathcal{T}(E)$ are in bijective correspondence with so-called isometric covariant representations of E (see below). Our primary objective in the next section, Theorem 2.9, is to show that each isometric covariant representation decomposes uniquely into the direct sum of representations of two kinds: the first is "induced" by a representation of A, in the sense of Rieffel [25], while the second yields a representation of the Cuntz-Pimsner algebra $\mathcal{O}(E)$. The induced isometric covariant representations of E have many of the properties of shifts. In fact, close inspection of the proofs of the classical facts about shifts reveals that to a large extent they reflect the representation theory of $\mathbb{C}$ and the fact that shifts are really induced by representations of $\mathbb{C}$. In the general setting, however, things are more complicated than in the restricted setting of shifts. Some induced representations can represent $\mathcal{O}(E)$, and we identify which ones in Corollary 2.15. Also, of course, while the representation theory of $C(\mathbb{T})$ is well understood, there is no hope in general of parameterizing the representations of $\mathcal{O}(E)$.

Our Wold decomposition theorem, Theorem 2.9, generalizes Popescu's Wold decomposition theorem [20] which may be obtained from our result by specializing to the case when $A = \mathbb{C}$, E is a Hilbert space over $\mathbb{C}$, and $\varphi(c) = cI_E$, $c \in \mathbb{C}$.

If one thinks in terms of Hilbert modules over general operator algebras (see [10] and [15] for the basic theory), then isometries correspond to so-called Shilov Hilbert modules over $A(\mathbb{D})$—which, as we have said, may be viewed as $\mathcal{T}_+(E)$ when $A = E = \mathbb{C}$. In the setting of tensor algebras in general and in particular in the setting of $A(\mathbb{D})$, Shilov modules coincide with the so-called orthoprojective Hilbert modules [18, Proposition 4.5]. The classical Wold decomposition theorem may be viewed as the assertion that every orthoprojective Hilbert module over $A(\mathbb{D})$ decomposes into the direct sum of boundary Hilbert module (essentially) in the sense of Arveson [1] and an orthoprojective Hilbert module that is pure in the sense that it contains no boundary summands. The boundary representations for $A(\mathbb{D})$ correspond to C^*-representations of its C^*-envelope in the sense of [1], $C(\mathbb{T})$, and the pure orthoprojective Hilbert modules correspond to shifts. In Theorem 3.2, we prove that a similar decomposition theorem holds for orthoprojective Hilbert modules over *arbitrary* operator algebras. This result, coupled with Rieffel's imprimitivity theorem [25, Theorem 6.29], is then applied in Theorem 3.5 to refine Theorem 2.9 under hypotheses on the correspondence E that allow us to assert that the C^*-envelope of $\mathcal{T}_+(E)$ is $\mathcal{O}(E)$ (see [18, Theorem 6.4 and corollaries]).

Beurling's theorem [2], as generalized by Lax [14] and Halmos [12], asserts that if S is a shift on a Hilbert space $\mathfrak{H}$ and if $\mathfrak{M}$ is a subspace of $\mathfrak{H}$ that is invariant under S, then

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there is an essentially unique partial isometry V in the commutant of S such that $\mathfrak{M} = V\mathfrak{H}$. Such operators V are called inner operators to remind one of inner functions. On the face of it, there appears to be no hope of generalizing this result to the full context of tensor algebras over C^*-correspondences. After all, this would contain the setting of free semigroup algebras studied by Popescu in [21], [22] and Davidson and Pitts in [5]. They showed emphatically that such a representation of invariant subspaces in terms of single operators in the commutant is not possible. They showed, instead, that in their setting every invariant subspace $\mathfrak{M}$ may be expressed as $\mathfrak{M} = \sum V_i\mathfrak{H}$, where the V_i are partial isometries in the commutant of the representation having orthogonal ranges. We show in Theorem 4.7 that this sort of representation is valid for induced representations of tensor algebras over arbitrary correspondences E, provided the representation π of A from which the representation of the tensor algebra is induced is what we call *quasi-invariant* with respect to E. In fact, the validity of this extension of Beurling's theorem is equivalent to the quasi-invariance of π.

Another way to formulate the Beurling-Lax-Halmos theorem is the assertion that every submodule of a pure Shilov module over the disc algebra $A(\mathbb{D})$ is unitarily equivalent to a reducing submodule. If one replaces "unitary equivalence" by "quasi-equivalence", then our result may be viewed as asserting that a Shilov module $\mathfrak{S}$ over $\mathcal{T}_+(E)$, that is induced by a representation π of A, has the property that every submodule is quasi-equivalent to a reducing submodule of $\mathfrak{S}$ if and only if E is quasi-invariant with respect to π.

Still another formulation of Beurling's original theorem from 1949 [2] is the assertion that the space $H^\infty(\mathbb{T})$ has the property that every weak-$*$ closed ideal is generated by an inner function. Thus, in a very specific sense, $H^\infty(\mathbb{T})$ may be viewed as a Hilbert space analogue of the polynomial ring in one variable $\mathbb{C}[X]$; both are principal ideal domains. This, of course, breaks down in the setting of general tensor algebras over C^*-correspondences: The free semi-group algebras of Davidson and Pitts have weak-$*$ closed ideals that fail to be singly generated. However, they were able to show that in the Fock representation (*i.e.*, the left regular representation) of a free semi-group algebra, the invariant subspaces are in bijective correspondence with σ-weakly closed right ideals of the weakly closed algebra generated by the representation. In the last section of this paper, we extend their analysis to tensor algebras over correspondences determined by finite quivers. These, essentially, are those correspondences that parameterize Cuntz-Krieger algebras.

2 The Wold Decomposition

An *isometric covariant representation* of E on a Hilbert space $\mathfrak{H}$ is a pair (V, σ) where σ is a C^*-representation[1] of A on $\mathfrak{H}$ and where V is a linear map from E into $B(\mathfrak{H})$, the bounded linear operators on $\mathfrak{H}$, satisfying the two conditions:

1) $V(\varphi(a)\xi b) = \sigma(a)V(\xi)\sigma(b)$, $\xi \in E$ and $a, b \in A$.
2) $V(\xi)^*V(\eta) = \sigma(\langle \xi, \eta\rangle)$, $\xi, \eta \in E$.

Note that condition 2) implies that V is continuous; in fact, $\|V(\xi)\| \leq \|\xi\|$. The reason for the adjective "isometric" will be clear in a moment.

[1] All representations of C^*-algebras are assumed to be nondegenerate, unless otherwise specified.

The map that sends $\varphi_\infty(a) \in \mathcal{T}(E)$ to $\sigma(a)$ and T_ξ to $V(\xi)$ extends to a C^*-representation of $\mathcal{T}(E)$ on $\mathfrak{H}$, denoted $\sigma \times V$ and called the *integrated form* of (V, σ). Conversely, given a C^*-representation $\pi\colon \mathcal{T}(E) \to B(\mathfrak{H})$, defining σ by the formula, $\sigma(a) := \pi\big(\varphi_\infty(a)\big)$, and V by the formula, $V(\xi) := \pi(T_\xi)$, yields an isometric covariant representation of E on $\mathfrak{H}$. This was proved by Pimsner in Theorem 3.4 of [19] under the assumptions that φ is injective and E is full, meaning that A is generated by the scalar products $\langle \xi, \eta \rangle$, $\xi, \eta \in E$. These assumptions are not necessary, as we remarked in Theorem 2.12 of [18]. This result, without any hypotheses on E, is proved in full detail in [11], where $\mathcal{T}(E)$ is presented as a universal object for covariant representations of E.

An important role in our work will be played by *induced representations* in the sense of Rieffel [25]. If F is a Hilbert C^*-module over A and if $\pi\colon A \to B(\mathfrak{H})$ is a C^*-representation, then the representation of $\mathcal{L}(F)$ induced by F, π^F, is defined as follows. The Hilbert space of π^F, denoted $F \otimes_\pi \mathfrak{H}$, is the Hausdorff completion of the algebraic tensor product of F and $\mathfrak{H}$ in the inner product

$$(\xi \otimes h, \eta \otimes k) = \big(\pi(\langle \eta, \xi \rangle)h, k\big).$$

The representation π^F is given by the formula

$$\pi^F(T) = T \otimes I_{\mathfrak{H}},$$

$T \in \mathcal{L}(F)$. It should be noted that the restriction of π^F to $\mathcal{K}(F)$ is nondegenerate. It is important to keep in mind that $\mathcal{L}(F)$ is the multiplier algebra of $\mathcal{K}(F)$, so a representation of $\mathcal{L}(F)$ is completely determined by its behavior on $\mathcal{K}(F)$ if and only if its restriction to $\mathcal{K}(F)$ acts nondegenerately.

It is important for the present investigation to know that the commutant of π^F may be expressed in terms of the commutant of π as follows: An operator X on $F \otimes_\pi \mathfrak{H}$ commutes with $\pi^F\big(\mathcal{L}(F)\big)$ if and only if X is of the form $I \otimes X_0$, where X_0 is an operator in the commutant of $\pi(A)$ [25, Theorem 6.23]; *i.e.*, $\pi^F\big(\mathcal{L}(F)\big)' = \mathbb{C} \otimes \big(\pi(A)'\big)$. For the sake of completeness, we outline a proof here.

First, let $\tilde{F}$ be the adjoint or dual module of F [25, Definition 6.17]. It may be identified with F, but with "conjugated and reversed" operations; alternatively, it may be identified with $\mathcal{K}(F, A)$. Then $\tilde{F}$ is naturally a left A-, right $\mathcal{K}(F)$-imprimitivity bimodule. In particular, $\tilde{F} \otimes_{\mathcal{K}(F)} F$ is an A, A-bimodule and the map from $\tilde{F} \otimes_{\mathcal{K}(F)} F$ to A that sends $\tilde{\eta} \otimes \xi$ to $\langle \eta, \xi \rangle$ is an A, A-bimodule, isometric, isomorphism. It follows that the map $W\colon \tilde{F} \otimes_{\mathcal{K}(F)} F \otimes_\pi \mathfrak{H} \to \mathfrak{H}$ defined by the equation $W(\tilde{\eta} \otimes \xi \otimes \zeta) = \pi(\langle \eta, \xi \rangle)\zeta$ is a Hilbert space isomorphism that implements an equivalence between $\pi^{\tilde{F} \otimes_{\mathcal{K}(F)} F}$ and π. In a similar fashion, one sees that $\pi^{F \otimes_A \tilde{F} \otimes_{\mathcal{K}(F)} F}$ is naturally unitarily equivalent to π^F. Now, to prove that $\pi^F\big(\mathcal{L}(F)\big)' = \mathbb{C} \otimes \big(\pi(A)'\big)$, note that we really need only prove that the left hand side is contained in the right, since the other inclusion is evident. Therefore, suppose X commutes with $\pi^F\big(\mathcal{L}(F)\big)$. Then $I \otimes X$ commutes with $\pi^{\tilde{F} \otimes_{\mathcal{K}(F)} F}\big(\mathcal{L}(\tilde{F} \otimes_{\mathcal{K}(F)} F)\big) \simeq \pi\big(M(A)\big)$. Since $\pi(A)$ and $\pi\big(M(A)\big)$ have the same commutants, there is an $X_0 \in \pi(A)'$ such that $W(I \otimes X)W^* = X_0$. With the identification of $\pi^{F \otimes_A \tilde{F} \otimes_{\mathcal{K}(F)} F}$ with π^F, it is easy to see that $X = I \otimes X_0$.

The following lemma extends Lemmas 3.5 and 3.6 of [18]. The first assertion was also proved in [11]. It describes all the isometric covariant representations in terms of more familiar constructs and helps to explain the choice of terminology.

Lemma 2.1 *Given an isometric covariant representation (V,σ) of the correspondence E on a Hilbert space $\mathfrak{H}$, define $\tilde{V}: E\otimes_\sigma \mathfrak{H} \to \mathfrak{H}$ by the formula $\tilde{V}(\xi\otimes h) = V(\xi)h$. Then $\tilde{V}$ is an isometry, with range equal to the closed linear span of $\{V(\xi)h \mid \xi \in E, h \in \mathfrak{H}\}$, that satisfies the equation*

$$\tilde{V}\sigma^E \circ \varphi = \sigma\tilde{V}. \tag{2.1}$$

Conversely, given an isometry $\tilde{V}$ from $E\otimes_\sigma \mathfrak{H}$ to $\mathfrak{H}$ that satisfies equation (2.1), the map $V: E \to B(\mathfrak{H})$, defined by the formula $V(\xi)h = \tilde{V}(\xi\otimes h)$, $\xi\in E$, $h\in\mathfrak{H}$, together with σ, constitutes an isometric covariant representation of E on $\mathfrak{H}$.

Proof The "direct" part of the lemma is proved in Lemmas 3.5 and 3.6 of [18], and in [11]. For the converse, simply observe that given $\tilde{V}$, V is a well defined linear map from E to $B(\mathfrak{H})$, by the properties of tensor products. It satisfies the bimodule condition, $V\big(\varphi(a)\xi b\big) = \sigma(a)V(\xi)\sigma(b)$, by virtue of equation (2.1) and the definition of σ^E. Finally, to see that $V(\xi)^*V(\eta) = \sigma(\langle\xi,\eta\rangle)$, simply note that for all $h,k\in\mathfrak{H}$, and $\xi,\eta\in E$, $\big(V(\xi)^*V(\eta)h,k\big) = \big(V(\eta)h, V(\xi)k\big) = \big(\tilde{V}(\eta\otimes h), \tilde{V}(\xi\otimes k)\big) = (\eta\otimes h, \xi\otimes k) = \big(\sigma(\langle\xi,\eta\rangle)h,k\big)$, by definition of the inner product on $E\otimes_\sigma\mathfrak{H}$. ■

Inducing a representation π of A on a Hilbert space $\mathfrak{H}$ up to $\mathcal{L}\big(\mathcal{F}(E)\big)$ and then restricting to $\mathcal{T}(E)$ yields a representation of $\mathcal{T}(E)$ on $\mathcal{F}(E)\otimes_\pi\mathfrak{H}$, which in turn gives a covariant representation (V,σ) of E on $\mathcal{F}(E)\otimes_\pi\mathfrak{H}$ defined by the formulae

$$\sigma(a) = \pi^{\mathcal{F}(E)}\circ\varphi_\infty(a) = \varphi_\infty(a)\otimes I_{\mathfrak{H}},$$
$$V(\xi) = \pi^{\mathcal{F}(E)}(T_\xi) = T_\xi\otimes I_{\mathfrak{H}}.$$

We call this (V,σ) the isometric covariant representation *induced* by π. By abuse of language, we shall say that any isometric covariant representation that is unitarily equivalent to an induced isometric covariant representation is also induced.

As we indicated in the introduction, one may profitably think of induced isometric covariant representations as generalizations of unilateral shifts. Indeed, when $A = \mathbb{C} = E$ and E is given the obvious structure of a correspondence over A, then $\mathcal{T}(E)$ is the usual Toeplitz algebra, the C^*-algebra generated by the unilateral shift. There is, of course, but one irreducible representation of A and every representation is a multiple of it. If (V,σ) is the covariant representation of E induced by a representation π of A on $\mathfrak{H}_0$, then after identifying $E\otimes_\sigma\mathcal{F}(E)\otimes_\pi\mathfrak{H}_0$ with $\mathcal{F}(E)\otimes_\pi\mathfrak{H}_0$, as we may in this case, the isometry $\tilde{V}$ is simply the unilateral shift of multiplicity equal to the dimension of $\mathfrak{H}_0$.

It is also worthwhile to note that when (V,σ) is induced by a representation π of A on $\mathfrak{H}_0$, then the map $\tilde{V}$ is tautological, reflecting the associativity of the various tensor products involved: $\tilde{V}$ maps from $E\otimes_\sigma(\mathcal{F}(E)\otimes_\pi\mathfrak{H}_0)$ to $\mathcal{F}(E)\otimes_\pi\mathfrak{H}_0$ via the formula, $\tilde{V}\big(\xi\otimes_\sigma(\eta\otimes_\pi h)\big) = (\xi\otimes_{\varphi_\infty}\eta)\otimes_\pi h$, $\xi\in E$, $\eta\in\mathcal{F}(E)$, $h\in\mathfrak{H}_0$, where, recall, $\sigma = \varphi_\infty\otimes I_{\mathfrak{H}_0}$.

An isometric covariant representation (V,σ) of a general correspondence E on the Hilbert space $\mathfrak{H}$ gives rise to an isometric covariant representation $(V^{\otimes n},\sigma)$ of $E^{\otimes n}$ on $\mathfrak{H}$

defined by the formula $V^{\otimes n}(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = V(\xi_1)V(\xi_2)\cdots V(\xi_n)$, $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in E^{\otimes n}$. This may be proved easily from the basic facts about tensor products. We write $\tilde{V}_n$ for the isometry from $E^{\otimes n} \otimes_\sigma \mathfrak{H}$ to $\mathfrak{H}$ determined by $V^{\otimes n}$ in Lemma 2.1. Henceforth, we shall write $\tilde{V}_1$ for $\tilde{V}$. The next lemma describes the fundamental relations among the $\tilde{V}_n$ that we will need.

Lemma 2.2 *For positive integers n and m, the expression $I_n \otimes \tilde{V}_m$ gives a well defined isometry from $E^{\otimes n} \otimes E^{\otimes m} \otimes_\sigma \mathfrak{H}$ to $E^{\otimes n} \otimes_\sigma \mathfrak{H}$. Its adjoint, $(I_n \otimes \tilde{V}_m)^*$ is $I_n \otimes \tilde{V}_m^*$. Furthermore, $\tilde{V}_{n+m} = \tilde{V}_n(I_n \otimes \tilde{V}_m) = \tilde{V}_m(I_m \otimes \tilde{V}_n)$.*

Proof Given its formal looking nature, the lemma may seem like a triviality; and indeed, it is easy to prove. However, there is a key observation that is necessary to prove it, to which we wish to call attention, namely equation (2.1). At the level of $E^{\otimes n}$ it reads

$$\tilde{V}_n \sigma^{E^{\otimes n}} \circ \varphi^{(n)} = \sigma \tilde{V}_n.$$

This condition guarantees (and probably is necessary to prove) that $I_n \otimes \tilde{V}_m$ is a well-defined bounded operator. Once this is noted, the calculations needed to complete the proof are straightforward and are left to the reader. ■

The following lemma is an easy consequence of the preceding two lemmas. Variants of it seem to be known in more abstract settings (see, *e.g.*, [8] and [9]). We learned the first assertion in it from Neal Fowler and Iain Raeburn (see [11]).

Lemma 2.3 *If (V, σ) is an isometric covariant representation of E on a Hilbert space $\mathfrak{H}$, then the formula*

$$L(x) = \tilde{V}_1(I_1 \otimes x)\tilde{V}_1^*,$$

$x \in \sigma(A)'$, defines a normal endomorphism of the commutant of $\sigma(A)$, $\sigma(A)'$. The n-th iterate of L, $n \geq 1$, is given by the formula

$$L^n(x) = \tilde{V}_n(I_n \otimes x)\tilde{V}_n^*,$$

$x \in \sigma(A)'$. Finally, if P_n denotes the projection onto the range of $\tilde{V}_n$, which is the closed linear span of $\{V^{\otimes n}(\xi)h \mid \xi \in E^{\otimes n}, h \in \mathfrak{H}\}$, then $P_n = L^n(I)$ and $P_{n+m} = \tilde{V}_n(I_n \otimes P_m)\tilde{V}_n^* = \tilde{V}_m(I_m \otimes P_n)\tilde{V}_m^*$, for all positive integers n and m.

Proof For the first assertion, we note that the expression defining L makes sense, since x is assumed to be in $\sigma(A)'$, *i.e.*, $I \otimes x$ is a bounded operator on $E \otimes_\sigma \mathfrak{H}$. The fact that L maps $\sigma(A)'$ into $\sigma(A)'$ is a consequence of formula (2.1). Indeed, if $x \in \sigma(A)'$, then $I \otimes x \in \sigma^E(\mathcal{L}(E))'$. Consequently, by Lemma 2.1 we have the following equation that

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proves the assertion:

$$\begin{aligned}
\sigma(a)\tilde{V}(I\otimes x)\tilde{V}^* &= \tilde{V}\big(\sigma^E\circ\varphi(a)\big)(I\otimes x)\tilde{V}^*\\
&= \tilde{V}(I\otimes x)\big(\sigma^E\circ\varphi(a)\big)\tilde{V}^*\\
&= \tilde{V}(I\otimes x)\tilde{V}^*\tilde{V}\big(\sigma^E\circ\varphi(a)\big)\tilde{V}^*\\
&= \tilde{V}(I\otimes x)\tilde{V}^*\sigma(a)\tilde{V}\tilde{V}^*\\
&= \tilde{V}(I\otimes x)\tilde{V}^*\tilde{V}\tilde{V}^*\sigma(a)\\
&= \tilde{V}(I\otimes x)\tilde{V}^*\sigma(a).
\end{aligned}$$

The normality of L results from the evident normality of the map $x \to I\otimes x$ from the commutant of $\sigma(A)$ to the commutant of $\sigma^E\big(\mathcal{L}(E)\big)$. The formula for the iterates of L is immediate from Lemma 2.2. The final assertion is just the observation that $P_n = \tilde{V}_n\tilde{V}_n^*$ coupled with the formulas of Lemma 2.2. ∎

This lemma guarantees that the projections P_n associated to an isometric covariant representation (V,σ) of E on a Hilbert space $\mathfrak{H}$ decrease with increasing n. That is, $P_1 \geq P_2 \geq P_3 \geq \cdots$. We write $Q_0 = I - P_1$, and for $k \geq 1$, we write $Q_k = P_k - P_{k+1}$. Since the P_k all commute with $\sigma(A)$ by virtue Lemma 2.3, so do all the Q_k. By definition, the Q_k's are orthogonal and their direct sum is the complement of the infimum of the projections P_k. We denote this infimum by P_∞; *i.e.*, $\bigoplus_{k=0}^{\infty} Q_k = I - P_\infty$. The following corollary is an immediate consequence of Lemma 2.3. It is the generalization to our setting of the well known fact that if V is an isometry on a Hilbert space $\mathfrak{H}$ and if Q is the defect projection $I-VV^*$, then Q is a wandering projection in the sense that $\{V^nQV^{*n}\}_{n=0}^{\infty}$ is an orthogonal family, each term of which has the same dimension. For this reason, we refer to Q_0 as the *wandering projection* associated with (V,σ); its range is the *wandering subspace* associated with (V,σ); and vectors in the range of Q_0 will be called *wandering vectors*.

Corollary 2.4 *For $l \geq 0$ and $k > 0$, $L^k(Q_l) = \tilde{V}_k(I_k\otimes Q_l)\tilde{V}_k^* = Q_{k+l}$.*

The relation between the commutant of $\sigma\times V\big(\mathcal{T}_+(E)\big)$ and L will play a role in our analysis. The following easy lemma and its corollary summarize what we need.

Lemma 2.5 *Let (V,σ) be an isometric representation of E on the Hilbert space $\mathfrak{H}$ and let x be an operator in the commutant of $\sigma(A)$. Then the following assertions are equivalent:*

1. *x commutes with $\sigma\times V\big(\mathcal{T}_+(E)\big)$.*
2. *$\tilde{V}_1(I\otimes x) = x\tilde{V}_1$.*
3. *$L(x) = xP_1$.*

Proof If x commutes with $\sigma\times V\big(\mathcal{T}_+(E)\big)$, then for $\xi\otimes h\in E\otimes_\sigma\mathfrak{H}$,

$$\begin{aligned}
x\tilde{V}_1(\xi\otimes h) &= xV(\xi)h = V(\xi)xh\\
&= \tilde{V}_1(\xi\otimes xh) = \tilde{V}_1(I\otimes x)(\xi\otimes h).
\end{aligned}$$

So $x\tilde{V}_1 = \tilde{V}_1(I \otimes x)$. On the other hand, if this equation is satisfied, then $xV(\xi)h = x\tilde{V}_1(\xi \otimes h) = \tilde{V}_1(I \otimes x)(\xi \otimes h) = V(\xi)xh$. Thus x commutes with the $V(\xi)$. Since x commutes with $\sigma(A)$ by hypothesis, x commutes with $\sigma \times V\big(\mathcal{T}_+(E)\big)$. This shows the equivalence of the first two assertions.

If $\tilde{V}_1(I \otimes x) = x\tilde{V}_1$, then $L(x) = \tilde{V}_1(I \otimes x)\tilde{V}_1^* = x\tilde{V}_1\tilde{V}_1^* = xP_1$. Conversely, if $L(x) = xP_1$, then $\tilde{V}_1(I \otimes x) = L(x)\tilde{V}_1 = xP_1\tilde{V}_1 = x\tilde{V}_1$, since P_1 is the projection onto the range of $\tilde{V}_1$. Thus, the second two assertions are equivalent. ■

Corollary 2.6 *With the notation of Lemma 2.5, if x commutes with $\sigma \times V\big(\mathcal{T}_+(E)\big)$, then the following assertions are equivalent:*

1. *x^* commutes with $\sigma \times V\big(\mathcal{T}_+(E)\big)$.*
2. *x commutes with $\sigma \times V\big(\mathcal{T}(E)\big)$.*
3. *x commutes with P_1.*

Proof Since $\mathcal{T}(E)$ is the C^*-algebra generated by $\mathcal{T}_+(E)$, the first two assertions are clearly equivalent. Observe that x commutes with $\sigma \times V\big(\mathcal{T}(E)\big)$ if and only if both x and x^* commute with $\sigma \times V\big(\mathcal{T}_+(E)\big)$. Also, the second assertion implies the third, since P_1 is in the strong closure of $\sigma \times V\big(\mathcal{T}(E)\big)$. Conversely, if x commutes with P_1, then so does x^*, and, therefore, since x commutes with $\sigma \times V\big(\mathcal{T}_+(E)\big)$, $L(x^*) = L(x)^* = (xP_1)^* = P_1x^* = x^*P_1$, proving that so does x^* by Lemma 2.5. ■

In the classical case, when (pure) isometries are represented by multiplication by z on vector-valued Hardy spaces, operators in their commutants are represented by operator-valued, bounded analytic functions. The $*$-commutant is represented by constant operator-valued functions. Consequently, we shall call operators that commute with $\sigma \times V\big(\mathcal{T}(E)\big)$ *constant operators.*

It is helpful to relate the restriction of an isometric covariant representation (V, σ) to an invariant subspace in terms of the operators $\tilde{V}_k$. To say that a subspace $\mathfrak{M}$ is invariant for (V, σ), we mean, of course, that $\mathfrak{M}$ is invariant under $\sigma(A)$ (so that in fact the projection onto $\mathfrak{M}$, $P_{\mathfrak{M}}$, commutes with $\sigma(A)$) and that $\mathfrak{M}$ is invariant for all the operators $V(\xi)$, $\xi \in E$. This is the same as saying that $\mathfrak{M}$ is invariant under $\sigma \times V\big(\mathcal{T}_+(E)\big)$. We say that $\mathfrak{M}$ *reduces* (V, σ) in case $\mathfrak{M}$ and $\mathfrak{M}^\perp$ are invariant under (V, σ). This is equivalent, of course, to saying that $P_{\mathfrak{M}}$ commutes with $\sigma \times V\big(\mathcal{T}(E)\big)$.

Lemma 2.7 *Let (V, σ) be an isometric covariant representation of E on the Hilbert space $\mathfrak{H}$ and let $\mathfrak{M} \subseteq \mathfrak{H}$ be a subspace that reduces $\sigma(A)$. If $P_{\mathfrak{M}}$ denotes the projection of $\mathfrak{H}$ onto $\mathfrak{M}$, then $\mathfrak{M}$ is invariant for (V, σ) if and only if $L(P_{\mathfrak{M}}) \le P_{\mathfrak{M}}$, and $\mathfrak{M}$ reduces (V, σ) if and only if $L(P_{\mathfrak{M}}) = P_{\mathfrak{M}}P_1 = P_1P_{\mathfrak{M}}$. Further, if $\mathfrak{M}$ is invariant for (V, σ), then $\tilde{V}_k(I_k \otimes P_{\mathfrak{M}}) = P_{\mathfrak{M}}\tilde{V}_k(I_k \otimes P_{\mathfrak{M}})$ for all $k \ge 1$; while if $\mathfrak{M}$ reduces (V, σ), then $\tilde{V}_k(I_k \otimes P_{\mathfrak{M}}) = P_{\mathfrak{M}}\tilde{V}_k$ for all $k \ge 1$.*

Proof The assertions about reducing subspaces follow immediately from Corollary 2.6 and so we attend only to invariant subspaces. The fact that $P_{\mathfrak{M}}$ commutes with $\sigma(A)$ guarantees that $I_k \otimes P_{\mathfrak{M}}$ is a projection in the commutant of $\sigma^{E^{\otimes k}}\big(\mathcal{L}(E^{\otimes k})\big)$ by general properties

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of induced representations [25, Theorem 6.23] as we noted above. If $\mathfrak{M}$ is invariant under all the operators $V(\xi)$, then $\mathfrak{M}$ is invariant under all the operators $V^{\otimes k}(\xi)$, $\xi \in E^{\otimes k}$. Consequently, for all such ξ, $V^{\otimes k}(\xi)P_{\mathfrak{M}} = P_{\mathfrak{M}}V^{\otimes k}(\xi)P_{\mathfrak{M}}$. Therefore, for all $h \in \mathfrak{H}$ and $\xi \in E^{\otimes k}$,

$$\tilde{V}_k(I_k \otimes P_{\mathfrak{M}})(\xi \otimes h) = V^{\otimes k}(\xi)P_{\mathfrak{M}}h = P_{\mathfrak{M}}V^{\otimes k}(\xi)P_{\mathfrak{M}}h = P_{\mathfrak{M}}\tilde{V}_k(I_k \otimes P_{\mathfrak{M}})(\xi \otimes h). \tag{2.2}$$

Thus, $\tilde{V}_k(I_k \otimes P_{\mathfrak{M}}) = P_{\mathfrak{M}}\tilde{V}_k(I_k \otimes P_{\mathfrak{M}})$. But if $\tilde{V}_k(I_k \otimes P_{\mathfrak{M}}) = P_{\mathfrak{M}}\tilde{V}_k(I_k \otimes P_{\mathfrak{M}})$ then $L^k(P_{\mathfrak{M}}) = \tilde{V}_k(I_k \otimes P_{\mathfrak{M}})\tilde{V}_k^* = P_{\mathfrak{M}}\tilde{V}_k(I_k \otimes P_{\mathfrak{M}})\tilde{V}_k^*$, proving that $L^k(P_{\mathfrak{M}}) \leq P_{\mathfrak{M}}$.

For the converse, observe that if $L(P_{\mathfrak{M}}) \leq P_{\mathfrak{M}}$, then $\tilde{V}_1(I \otimes P_{\mathfrak{M}})\tilde{V}_1^*$ is a projection Q, say, smaller than $P_{\mathfrak{M}}$. Consequently, $\tilde{V}_1(I \otimes P_{\mathfrak{M}}) = Q\tilde{V}_1 = P_{\mathfrak{M}}Q\tilde{V}_1$. But then, for all $h \in \mathfrak{M}$ and $\xi \in E$, we have $V(\xi)h = \tilde{V}_1(I \otimes P_{\mathfrak{M}})(\xi \otimes h) = P_{\mathfrak{M}}Q\tilde{V}_1(\xi \otimes h)$, showing that $V(\xi)$ leaves $\mathfrak{M}$ invariant. ∎

The following terminology comes from [18].

Definition 2.8 Let (V, σ) be an isometric covariant representation of E on the Hilbert space $\mathfrak{H}$, let $P_1 = \tilde{V}_1\tilde{V}_1^*$—the projection onto closed linear span of $\{V(\xi)h \mid \xi \in E, h \in \mathfrak{H}\}$, and let J be the (possibly degenerate) ideal $\varphi^{-1}\big(\mathcal{K}(E)\big)$ in A. Then (V, σ) is called *J-coisometric*, or simply *coisometric*, in case $P_1\mathfrak{H}$ contains the essential subspace of $\sigma|J$, $\sigma(J)\mathfrak{H}$. We say that (V, σ) is *fully coisometric* in case the projection $P_1 = I$.

Under the assumption that E is full and φ is injective, the coisometric, isometric covariant representations (V, σ) are precisely those isometric covariant representations of E such that the integrated representation of $\mathcal{T}(E)$, $\sigma \times V$, passes to the quotient, $\mathcal{O}(E)$ [19, Theorem 3.13]. They are thus analogues of unitary operators. Fully coisometric (V, σ)'s form a proper subclass of the coisometric (V, σ)'s. However, it is not possible to see the distinction at the level of the classical Toeplitz algebra. We will have more to say about the distinction shortly. At this point, we are able to prove the first version of a Wold decomposition theorem for isometric covariant representations of C^*-correspondences.

Theorem 2.9 *If (V, σ) is an isometric covariant representation of E on a Hilbert space $\mathfrak{H}$, then (V, σ) decomposes into the direct sum of two such representations, $(V, \sigma) = (V_1, \sigma_1) \oplus (V_2, \sigma_2)$ on $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where (V_1, σ_1) is an induced isometric covariant representation and (V_2, σ_2) is fully coisometric. The decomposition is unique in the sense that if $\mathfrak{K}$ is a subspace of $\mathfrak{H}$ which reduces (V, σ) (i.e., if $\mathfrak{K}$ reduces $\sigma \times V\big(\mathcal{T}(E)\big)$), and if the restriction of (V, σ) to $\mathfrak{K}$ is induced (resp. is fully coisometric), then $\mathfrak{K} \subseteq \mathfrak{H}_1$ (resp. $\mathfrak{K} \subseteq \mathfrak{H}_2$).*

Proof We follow the notation already established, and write P_k for the projections $L^k(I) = \tilde{V}_k\tilde{V}_k^*$, so that $P_1 \geq P_2 \geq \cdots$. We let $P_\infty = \bigwedge_{k=1}^\infty P_k$, we set $\mathfrak{H}_2 = P_\infty\mathfrak{H}$, and we set $\mathfrak{H}_1 = P_\infty^\perp\mathfrak{H}$ $(= \mathfrak{H} \ominus \mathfrak{H}_2)$. Recall from Lemma 2.3 $P_k = L^k(I)$ so that $L(P_\infty) = P_\infty = P_\infty P_1$. Thus, by that lemma, the spaces $\mathfrak{H}_i$, $i = 1, 2$, reduce (V, σ). Further, the lemma shows that if $V_i(\xi)$ is the restriction of $V(\xi)$ to $\mathfrak{H}_i$, $i = 1, 2$, then $\tilde{V}_{2,1} = P_\infty\tilde{V}_1 = \tilde{V}_1(I \otimes P_\infty) = P_\infty\tilde{V}_1(I \otimes P_\infty)$, while $\tilde{V}_{1,1} = P_\infty^\perp\tilde{V}_1 = \tilde{V}_1(I \otimes P_\infty^\perp) = P_\infty^\perp\tilde{V}_1(I \otimes P_\infty^\perp)$. (Here, of course, $\tilde{V}_{i,1}$ denotes the isometry from $E \otimes_\sigma \mathfrak{H}_i$ to $\mathfrak{H}_i$ determined by V_i, $i = 1, 2$.) Consequently, $\tilde{V}_{2,1}\tilde{V}_{2,1}^* = P_\infty\tilde{V}_1\tilde{V}_1^*P_\infty = P_\infty P_1 P_\infty = P_\infty$, proving that (V_2, σ_2) is fully coisometric.

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Similarly, if $P_{1,k} = \tilde{V}_{1,k}\tilde{V}_{1,k}^*$, $k = 1, 2, \ldots$, then $P_{1,k} = P_k - P_\infty$, so $\bigwedge_{k=1}^\infty P_{1,k} = 0$. Thus to complete the first half of the proof, *i.e.*, to show that (V_1, σ_1) is induced, we need only show that if $P_\infty = 0$, then (V, σ) is induced.

To this end, form $Q_k = P_k - P_{k+1}$, $k = 0, 1, \ldots$, where $P_0 = I$. Set $\mathfrak{H}_0 = Q_0\mathfrak{H}$, and set $\sigma_0(\cdot) = \sigma(\cdot)|\mathfrak{H}_0$. By Corollary 2.4, $L^k(Q_0) = \tilde{V}_k(I \otimes Q_0)\tilde{V}_k^* = Q_k$, $k = 1, 2, \ldots$. Since $P_\infty = 0$, by hypothesis, $I = \sum_{k\geq 0}^{\oplus} Q_0$. So, if $U\colon \mathcal{F}(E) \otimes_{\sigma_0} \mathfrak{H}_0$ $(= \sum_{k\geq 0}^{\oplus} E^{\otimes n} \otimes_{\sigma_0} \mathfrak{H}_0) \to \mathfrak{H}$ is defined by the formula

$$U\Big(\sum_{k\geq 0} h_n\Big) = \sum_{k\geq 0} \tilde{V}_n h_n,$$

where $h_n \in E^{\otimes n} \otimes_{\sigma_0} \mathfrak{H}_0 = (I_n \otimes Q_0)\mathcal{F}(E) \otimes_\sigma \mathfrak{H}$, then since $\tilde{V}_n$ maps $E^{\otimes n} \otimes_{\sigma_0} \mathfrak{H}_0$ onto $Q_n\mathfrak{H}$, a straightforward calculation shows that U is a Hilbert space isomorphism from $\mathcal{F}(E) \otimes_{\sigma_0} \mathfrak{H}_0$ onto $\mathfrak{H}$ such that $U\big(\varphi_\infty(a) \otimes I_{\mathfrak{H}_0}\big) = \sigma(a)U$ and $U(T_\xi \otimes I_{\mathfrak{H}_0}) = V(\xi)U$, for all $a \in A$ and $\xi \in E$. This shows that (V, σ) is induced if $P_\infty = 0$.

The proof of the uniqueness could be given here, but we defer it until after Proposition 2.11, which enables us to say somewhat more. ■

For the sake of emphasis, we state the following immediate corollary of our proof of Theorem 2.9.

Corollary 2.10 *Let (V, σ) be an isometric covariant representation of E on the Hilbert space $\mathfrak{H}$. Then (V, σ) is induced if and only if $\bigwedge_{k\geq 1} \tilde{V}_k\tilde{V}_k^*$ $(= P_\infty) = 0$.*

Proposition 2.11 *Let (V, σ) be an induced isometric covariant representation of E on a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}$ be invariant for (V, σ). Then the restriction of (V, σ) to $\mathfrak{M}$ is an induced isometric covariant representation.*

Proof Observe that the restriction of a (not-necessarily-induced) isometric covariant representation to an invariant subspace gives a covariant representation that is still isometric. Thus, the issue is whether the restriction is induced, if the original representation is induced.

Write (W, τ) for the restriction of (V, σ) to $\mathfrak{M}$ and write $P_{\mathfrak{M}}$ for the projection of $\mathfrak{H}$ onto $\mathfrak{M}$. By Corollary 2.10, we need to show that $\bigwedge_{k\geq 0} \tilde{W}_k\tilde{W}_k^* = 0$. Observe that $W(\cdot)$ may be viewed as $P_{\mathfrak{M}}V(\cdot)P_{\mathfrak{M}}$. Consequently, for each $k \geq 1$, $\tilde{W}_k$ may be viewed as $P_{\mathfrak{M}}\tilde{V}_k(I_k \otimes P_{\mathfrak{M}}) = \tilde{V}_k(I_k \otimes P_{\mathfrak{M}})$ by Lemma 2.7. Hence $\tilde{W}_k\tilde{W}_k^* = P_{\mathfrak{M}}\tilde{V}_k(I_k \otimes P_{\mathfrak{M}})\tilde{V}_k^*P_{\mathfrak{M}} = \tilde{V}_k(I_k \otimes P_{\mathfrak{M}})\tilde{V}_k^* \leq \tilde{V}_k\tilde{V}_k^*$. Since $\bigwedge_{k\geq 0} \tilde{V}_k\tilde{V}_k^* = 0$, we conclude from this inequality that $\bigwedge_{k\geq 0} \tilde{W}_k\tilde{W}_k^* = 0$. ■

Completion of the proof of Theorem 2.9 We prove a bit more than is claimed. Let $\mathfrak{K}$ be a reducing subspace for (V, σ) with projection $P_{\mathfrak{K}}$. By Lemma 2.5, we know that $P_{\mathfrak{K}}$ commutes with $\sigma(A)$ and satisfies the equation $\tilde{V}_k(I_k \otimes P_{\mathfrak{K}}) = P_{\mathfrak{K}}\tilde{V}_k$ for all $k \geq 1$. This easily implies that $P_{\mathfrak{K}}$ commutes with P_∞, so that $\mathfrak{K} = \mathfrak{K} \cap \mathfrak{H}_1 \oplus \mathfrak{K} \cap \mathfrak{H}_2$. Now $\mathfrak{K} \cap \mathfrak{H}_1$ is a reducing subspace for (V_1, σ_1) and so the restriction of (V_1, σ_1) to $\mathfrak{K} \cap \mathfrak{H}_1$ is induced by Proposition 2.11. On the other hand, since V_2 may be identified with $P_\infty V = VP_\infty = P_\infty VP_\infty$, an easy calculation shows that $\tilde{V}_{2,1}(I \otimes P_{\mathfrak{K}\cap\mathfrak{H}_2}) = P_{\mathfrak{K}\cap\mathfrak{H}_2}\tilde{V}_{2,1}$ from which it follows that if (W, τ) is the restriction of (V_2, σ_2) to $\mathfrak{K} \cap \mathfrak{H}_2$, then $\tilde{W}_1$ may be identified with

$\tilde{V}_{2,1}(I \otimes P_{\mathfrak{K}\cap\mathfrak{H}_2}) = P_{\mathfrak{K}\cap\mathfrak{H}_2}\tilde{V}_{2,1} = P_{\mathfrak{K}\cap\mathfrak{H}_2}\tilde{V}_{2,1}(I \otimes P_{\mathfrak{K}\cap\mathfrak{H}_2})$, showing that $\tilde{W}_1$ maps $E \otimes_{\sigma|\mathfrak{K}\cap\mathfrak{H}_2} \mathfrak{K} \cap \mathfrak{H}_2$ onto $\mathfrak{K} \cap \mathfrak{H}_2$. Thus, (W, τ) is fully coisometric. From these two calculations, the proof of the uniqueness assertion in Theorem 2.9 is immediate. ∎

Definition 2.12 If (V, σ) is an isometric covariant representation of E on the Hilbert space $\mathfrak{H}$, and if (V, σ) is decomposed into the direct sum, $(V, \sigma) = (V_1, \sigma_1) \oplus (V_2, \sigma_2)$ on $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, as in Theorem 2.9, where (V_1, σ_1) is an induced isometric covariant representation and (V_2, σ_2) is fully coisometric, then we call $\mathfrak{H}_1$ (or (V_1, σ_1)) the *induced summand*, or *part*, of $\mathfrak{H}$ (or (V, σ)), and $\mathfrak{H}_2$ (or (V_2, σ_2)) is called the *fully coisometric summand*, or *part*, of $\mathfrak{H}$.

The following corollary of Proposition 2.11 should be noted.

Corollary 2.13 *If (V, σ) is an isometric covariant representation of E on a Hilbert space $\mathfrak{H}$ that is induced by a representation π of A on a Hilbert space $\mathfrak{H}_0$, then the weakly closed algebra generated by $\sigma \times V\big(\mathcal{T}(E)\big)$ is $\pi^{\mathcal{F}(E)}\Big(\mathcal{L}\big(\mathcal{F}(E)\big)\Big)''$ and the commutant of $\sigma \times V\big(\mathcal{T}(E)\big)$ consists of all operators of the form $I \otimes x$, where $x \in \sigma(A)'$. Thus, in particular, the map*

$$\mathfrak{M} \to \mathcal{F}(E) \otimes_\pi \mathfrak{M}$$

from the subspaces of $\mathfrak{H}_0$ that reduce $\pi(A)$ to the subspaces of $\mathcal{F}(E) \otimes_\pi \mathfrak{H}_0$, is a bijection between all the subspaces of $\mathfrak{H}_0$ that reduce $\pi(A)$ and the subspaces of $\mathcal{F}(E) \otimes_\pi \mathfrak{H}_0$ that reduce (V, σ).

Proof Of course, the weakly closed algebra generated by $\sigma \times V\big(\mathcal{T}(E)\big)$ is contained in $\pi^{\mathcal{F}(E)}\Big(\mathcal{L}\big(\mathcal{F}(E)\big)\Big)''$. So to show equality, one needs only to show that any projection P that commutes with $\sigma \times V\big(\mathcal{T}(E)\big)$ also commutes with $\pi^{\mathcal{F}(E)}\Big(\mathcal{L}\big(\mathcal{F}(E)\big)\Big)''$. Lemma 2.7 implies that P commutes with all the projections P_k and hence with the projections Q_k, $k \geq 0$. Since (V, σ) is induced, $\mathfrak{H} = \mathcal{F}(E) \otimes_\pi \mathfrak{H}_0$ and $Q_k\mathfrak{H} = E^{\otimes k} \otimes_\pi \mathfrak{H}_0$. The map that sends $\xi_1 \otimes \xi_2 \otimes \cdots \xi_k \otimes \eta_k^* \otimes \eta_{k-1}^* \otimes \cdots \eta_1^* \in \mathcal{K}(E^{\otimes k})$ to $T_{\xi_1}T_{\xi_2}\cdots T_{\xi_k}T_{\eta_k}^*T_{\eta_{k-1}}^*\cdots T_{\eta_1}^*$ extends to a C^*-homomorphism of $\mathcal{K}(E^{\otimes k})$ into $\mathcal{T}(E)$ [19, p. 199]. A moment's reflection directed toward the matricial form of the operators in $\sigma \times V\big(\mathcal{T}(E)\big)$ (corresponding to the decomposition of $\mathcal{F}(E) \otimes_\pi \mathfrak{H}_0$ as $\sum_{k\geq 0}^{\oplus} E^{\otimes k} \otimes_\pi \mathfrak{H}_0$) reveals that in $Q_k\Big(\sigma \times V\big(\mathcal{T}(E)\big)\Big)Q_k$ the image of this homomorphism may be identified with $\pi^{E^{\otimes k}}\big(\mathcal{K}(E^{\otimes k})\big)$. Since PQ_k commutes with $\pi^{E^{\otimes k}}\big(\mathcal{K}(E^{\otimes k})\big)$ and $\mathcal{L}(E^{\otimes k})$ is the multiplier algebra of $\mathcal{K}(E^{\otimes k})$, PQ_k commutes with $\pi^{E^{\otimes k}}\big(\mathcal{L}(E^{\otimes k})\big)$. Consequently, PQ_k is of the form $I_k \otimes R_k$ for a projection R_k commuting with $\pi(A)$ by [25, Theorem 6.23]. Thus, $P = \sum_{k\geq 0}^{\oplus}(I_k \otimes R_k)$. However, from Lemma 2.7 and the tautological form of $\tilde{V}_k$ in the case of induced, isometric, covariant representations, it is immediate that the R_k are all equal. Thus, $P = I_{\mathcal{F}(E)} \otimes R_0$, and by [25, Theorem 6.23] again, P commutes with $\pi^{\mathcal{F}(E)}(\mathcal{L}\big(\mathcal{F}(E)\big)$. ∎

Remark 2.14 It is worthwhile to note that if E were full, if φ were injective, and if $\varphi(A)$ were contained in $\mathcal{K}(E)$, then Corollary 2.13 would follow easily from the fact that $\mathcal{K}\big(\mathcal{F}(E)\big) \subseteq \mathcal{T}(E)$ [19, Theorem 3.13]. In light of this, it is somewhat surprising that Corollary 2.13 holds even when $\mathcal{K}\big(\mathcal{F}(E)\big) \cap \mathcal{T}(E) = \{0\}$.

In the special case when $A = E = \mathbb{C}$, Corollary 2.13 gives the structure of the von Neumann algebra (and its commutant) generated by a unilateral shift and establishes the well-known fact that the reducing subspaces of a unilateral shift are "slices" [12].

We conclude this section with one more corollary of Theorem 2.9, a result that identifies isometric covariant representations that are also coisometric.

Corollary 2.15 *Let (V, σ) be an isometric covariant representation of E on the Hilbert space $\mathfrak{H}$ and let $(V, \sigma) = (V_i, \sigma_i) \oplus (V_f, \sigma_f)$ on $\mathfrak{H} = \mathfrak{H}_i \oplus \mathfrak{H}_f$ be its Wold decomposition (where (V_i, σ_i) is induced and (V_f, σ_f) is fully coisometric). Then (V, σ) is coisometric if and only if (V_i, σ_i) is induced by a representation π of A on the Hilbert space $\mathfrak{H}_0$ with the property that $\pi(J) = 0$, where J is the ideal $\varphi^{-1}\big(\mathcal{K}(E)\big)$ in A. Further, every isometric covariant representation (V, σ) of E on a Hilbert space decomposes as $(V, \sigma) = (V_p, \sigma_p) \oplus (V_c, \sigma_c)$, where (V_c, σ_c) is coisometric and (V_p, σ_p) is pure in the sense that it contains no coisometric summands.*

Proof For the first assertion of the corollary, note that since (V_f, σ_f) is fully coisometric, (V, σ) is coisometric if and only if (V_i, σ_i) is coisometric. So we may as well assume at the outset that (V, σ) is induced, say by $\pi \colon A \to B(\mathfrak{H}_0)$. Consider the matrices of elements in $\sigma \times V\big(\mathcal{T}(E)\big)$ with respect to the direct sum decomposition of $\mathfrak{H} = \mathcal{F}(E) \otimes_\pi \mathfrak{H}_0$ as $\sum_{n\geq 0}^{\oplus} E^{\otimes n} \otimes_\pi \mathfrak{H}_0$. Then for $a \in A$,

$$\sigma(a) = \operatorname{diag}\big(\pi(a), \varphi(a) \otimes I_H, \ldots, \varphi(a) \otimes I_{n-1} \otimes I_H, \ldots\big),$$

where I_{n-1} denotes the identity operator on $E^{\otimes n-1}$. On the other hand, if $\sigma^{(1)}$ denotes the natural representation of $\mathcal{K}(E)$ determined by (V, σ), so that $\sigma^{(1)}(\xi \otimes \eta^*) = V(\xi)V(\eta)^*$ (see [18], [19]), then for $k \in \mathcal{K}(E)$,

$$\sigma^{(1)}(k) = \operatorname{diag}(0, k \otimes I_H, \ldots, k \otimes I_{n-1} \otimes I_H, \ldots).$$

Now recall that by Theorem 2.19 and Lemma 5.4 of [18], (V, σ) is coisometric if and only if $\sigma^{(1)} \circ \varphi(a) = \sigma(a)$ for all $a \in J$ (see [19] also). Applying this criteria with the matrix representations for σ and $\sigma^{(1)}$, we see that (V, σ) is coisometric if and only if $\pi(a) = 0$ for all $a \in J$.

For the second assertion, it again suffices to assume at the outset that (V, σ) is induced, say by $\pi \colon A \to B(\mathfrak{H}_0)$. Restrict π to J and decompose π in the usual way into $\pi_p \oplus \pi_c$ on $\mathfrak{H}_{0p} \oplus \mathfrak{H}_{0c}$, where $\mathfrak{H}_{0p} = \pi(J)\mathfrak{H}_0$ and $\mathfrak{H}_{0c} = \mathfrak{H}_{0p}^{\perp}$. Then (V, σ) decomposes as $(V, \sigma) = (V_p, \sigma_p) \oplus (V_c, \sigma_c)$, where (V_p, σ_p) is induced by π_p and (V_c, σ_c) is induced by π_c. Since π_c annihilates J, (V_c, σ_c) is coisometric, as we have just seen. The issue is to see that (V_p, σ_p) has no coisometric summands. But if (V', σ') is any summand of (V_p, σ_p), then (V', σ') is induced by a summand of π_p by Corollary 2.13. But no summand of π_p can annihilate J, since the restriction of π_p to J is nondegenerate. Hence no summand (V', σ') of (V, σ) can be coisometric, by the first assertion of the corollary. ■

Remark 2.16 If (V, σ) is induced by a representation $\pi \colon A \to B(\mathfrak{H})$, then for any ideal J_0 of A, we may decompose π into $\pi_1 \oplus \pi_2$ where $\pi_1 | J_0$ is nondegenerate and where π_2 annihilates J_0. This decomposition induces a corresponding decomposition of (V, σ) as

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$(V, \sigma) = (V_1, \sigma_1) \oplus (V_2, \sigma_2)$, where (V_i, σ_i) is induced by π_i. This fact will be useful for analyzing the representations of the relative Cuntz-Pimsner algebras $\mathcal{O}(J_0, E)$ introduced in [18, Definition 2.18].

3 An Abstract Wold Decomposition

In the presence of additional assumptions on E, Theorem 2.9 may be refined somewhat. See Theorem 3.5. While a proof could be generated from Theorem 2.9 and Corollary 2.15, another proof, that exhibits interesting connections with general operator algebra, will be given.

We adopt the language of Hilbert modules over general operator algebras from [10] and [15]. Let $\mathcal{A}$ be an arbitrary operator algebra. For us, this means that $\mathcal{A}$ is a norm closed algebra of operators on some Hilbert space and that $\mathcal{A}$ is either unital or approximately unital, meaning that $\mathcal{A}$ has a contractive approximate identity. Thanks to the wonderful theorem of Blecher, Ruan, and Sinclair [3], $\mathcal{A}$ may be given an abstract characterization independent of any Hilbert space on which $\mathcal{A}$ might sit.

A Hilbert module over $\mathcal{A}$ is simply a Hilbert space on which $\mathcal{A}$ is represented *completely contractively*. If $\mathfrak{H}$ is the Hilbert space and if $\rho \colon \mathcal{A} \to B(\mathfrak{H})$ is the representation, then the module action is given by the formula $a\xi := \rho(a)\xi$. We say that ρ is the representation associated with $\mathfrak{H}$ and that $\mathfrak{H}$ is the Hilbert module associated with ρ. Each perspective, Hilbert modules and (completely contractive) representations, has its advantages and we shall pass from one to the other as is convenient.

Observe that if $\mathfrak{H}$ is a Hilbert module over an operator algebra $\mathcal{A}$, then $\mathrm{End}(\mathfrak{H})$ will denote the collection of all operators S in $B(\mathfrak{H})$ such that S is a module map of $\mathfrak{H}$, *i.e.*, S satisfies $S(ah) = a(Sh)$, for all $a \in \mathcal{A}$ and $h \in \mathfrak{H}$. Of course, $\mathrm{End}(\mathfrak{H})$ is just the commutant of $\rho(\mathcal{A})$ where ρ is the representation determined by $\mathfrak{H}$.

Recall from [15] that a short exact sequence of Hilbert modules over $\mathcal{A}$,

$$0 \to \mathfrak{K} \xrightarrow{\Psi} \mathfrak{M} \xrightarrow{\Phi} \mathfrak{P} \to 0,$$

is called *isometric* in case Ψ is an isometric module map and Φ is a coisometric module map. Such a sequence is called *orthogonally split* when there is a contractive module map $\Phi' \colon \mathfrak{P} \to \mathfrak{M}$ such that $\Phi \circ \Phi' = I_{\mathfrak{P}}$. Equivalently, the sequence is orthogonally split when Φ^* is a module map. In this event, $\mathfrak{M}$ is isomorphic (*i.e.*, unitarily equivalent) to the (orthogonal) direct sum of $\mathfrak{K}$ and $\mathfrak{P}$. A Hilbert module $\mathfrak{K}$ over $\mathcal{A}$ is called *orthoinjective* in case whenever it appears as the first term in a short exact isometric sequence, the sequence is orthogonally split. To say the same thing differently, $\mathfrak{K}$ is orthoinjective iff whenever $\mathfrak{K}$ appears (isometrically) as a submodule of another Hilbert module $\mathfrak{M}$, $\mathfrak{K}$ is an (orthogonal) direct summand of $\mathfrak{M}$. Similarly, a Hilbert module $\mathfrak{P}$ over $\mathcal{A}$ is called *orthoprojective* in case whenever $\mathfrak{P}$ appears as the last term of a short exact isometric sequence, the sequence is orthogonally split. Equivalently, $\mathfrak{P}$ is orthoprojective if whenever $\mathfrak{P}$ appears (isometrically) as a compression of a Hilbert module $\mathfrak{M}$ to a co-invariant subspace, then $\mathfrak{P}$ is an orthogonal summand of $\mathfrak{M}$.

Orthoprojective Hilbert modules are a natural generalization of isometries and orthoinjective Hilbert modules are a natural generalization of coisometries. Indeed, if $\mathcal{A}$ is the disc

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algebra, then Hilbert modules over $\mathcal{A}$ are determined by contraction operators. A contraction T corresponds to an orthoprojective Hilbert module (resp. an orthoinjective Hilbert module) if and only if T is an isometry (resp. coisometry). This example is generalized in [18]. It is helpful to recall some of the details here. A *Shilov* Hilbert module $\mathfrak{S}$ over an arbitrary operator algebra $\mathcal{A}$ is one such that the associated representation ρ is obtained by taking a C^*-representation π of the C^*-envelope of $\mathcal{A}$, $\pi\colon C^*(\mathcal{A}) \to B(\mathfrak{H}_\pi)$, restricting π to $\mathcal{A}$, and then restricting $\pi(\mathcal{A})$ to an invariant subspace. That is, $\mathfrak{S}$ may be viewed as a subspace of $\mathfrak{H}_\pi$ that is invariant under $\pi(\mathcal{A})$ and then ρ is given by the formula

$$\rho(a) = \pi(a)|\mathfrak{S}.$$

The following lemma summarizes facts from [18, Proposition 4.2 and Corollary 4.6] that we will need.

Lemma 3.1 *Let $\mathfrak{S}$ be a Hilbert module over $\mathfrak{T}_+(E)$ and let ρ be the associated representation. Then the following assertions are equivalent:*

1. *$\mathfrak{S}$ is a Shilov module.*
2. *$\mathfrak{S}$ is orthoprojective.*
3. *ρ is the restriction to $\mathfrak{T}_+(E)$ of $\sigma \times V$ for a unique isometric covariant representation (V, σ) of E on $\mathfrak{S}$.*

The interest in these notions is further heightened when one recognizes, as was proved in [17], that a Hilbert module over a general operator algebra $\mathcal{A}$ is simultaneously orthoprojective and orthoinjective if and only if the associated representation is the restriction to $\mathcal{A}$ of a boundary representation of the C^*-envelope of $\mathcal{A}$, in the following sense: If B is the C^*-envelope of $\mathcal{A}$, then a C^*-representation π of B on a Hilbert space $\mathfrak{H}_\pi$ is called a *boundary representation of* B for $\mathcal{A}$ iff π is the only completely positive map from B to $B(\mathfrak{H}_\pi)$ whose restriction to $\mathcal{A}$ coincides with the restriction of π to $\mathcal{A}$. This notion was first defined by Arveson in [1], but with the extra assumption that π is irreducible. A Hilbert module over $\mathcal{A}$ that comes from a boundary representation of B in this way is called a *boundary* Hilbert module.

The following result generalizes the familiar fact that every contraction decomposes into a direct sum of a coisometry and a contraction that has no coisometric summands. If the operator happens to be an isometry to begin with, then the coisometric summand is a unitary operator—which necessarily comes from a boundary representation of the C^*-envelope of the disc algebra—and the other part is a pure isometry.

Theorem 3.2 *If $\mathfrak{M}$ is a Hilbert module over an operator algebra $\mathcal{A}$, then $\mathfrak{M}$ may be written as $\mathfrak{M} = \mathfrak{M}_p \oplus \mathfrak{M}_i$, where $\mathfrak{M}_i$ is orthoinjective and $\mathfrak{M}_p$ is pure in the sense that it contains no orthoinjective submodules. One or the other summands may, of course, be zero. Further, if $\mathfrak{M}$ is orthoprojective, then $\mathfrak{M}_p$ is orthoprojective and $\mathfrak{M}_i$ is a boundary Hilbert module.*

Proof First observe that if the Hilbert module $\mathfrak{M}$ is expressed as $\mathfrak{M} = \overline{\bigcup \mathfrak{M}_\alpha}$, where each $\mathfrak{M}_\alpha$ is an orthoinjective Hilbert module, then $\mathfrak{M}$ is orthoinjective. Indeed, consider a short exact isometric sequence $0 \to \mathfrak{M} \xrightarrow{\Phi} \mathfrak{K} \to \mathfrak{P} \to 0$. Then for every α, $0 \to \mathfrak{M}_\alpha \xrightarrow{\Phi|\mathfrak{M}_\alpha}$

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$\mathfrak{K} \to \mathfrak{K} \ominus \Phi(\mathfrak{M}_\alpha) \to 0$ is a short exact isometric sequence. Consequently $\mathfrak{K} \ominus \Phi(\mathfrak{M}_\alpha)$ is a submodule of $\mathfrak{K}$, *i.e.*, $\Phi(\mathfrak{M}_\alpha)$ is an orthogonal summand of $\mathfrak{K}$. But then $\mathfrak{K} \ominus \Phi(\mathfrak{M}) = \mathfrak{K} \ominus \vee \Phi(\mathfrak{M}_\alpha) = \bigcap_\alpha \mathfrak{K} \ominus \Phi(\mathfrak{M}_\alpha)$ is a submodule of $\mathfrak{K}$, showing that $\Phi(\mathfrak{M})$ is an orthogonal summand of $\mathfrak{K}$. Thus $\mathfrak{M}$ is orthoinjective. So, to prove the theorem, all one needs to do is to set $\mathfrak{M}_i := \bigvee\{\mathfrak{K} \subseteq \mathfrak{M} \mid \mathfrak{K} \text{ is orthoinjective}\}$. Then the assertion shows that $\mathfrak{M}_i$ is orthoinjective and, of course, no submodule of $\mathfrak{M}_p := \mathfrak{M}_i^\perp$ is orthoinjective. The fact that $\mathfrak{M}_p$ is orthoprojective, if $\mathfrak{M}$ is follows from [15, Proposition 3.5], which asserts that the direct sum of two Hilbert modules over an operator algebra is orthoprojective if and only if each summand is. Likewise, if $\mathfrak{M}$ is orthoprojective so that $\mathfrak{M}_i$ is also orthoprojective, then since it is also orthoinjective, it is a boundary Hilbert module by the main result of [17]. ■

Corollary 3.3 *Every submodule of a pure Hilbert module over an operator algebra $\mathcal{A}$ is pure.*

Proof An orthoinjective summand of a submodule of a Hilbert module $\mathfrak{M}$ is an orthoinjective summand of $\mathfrak{M}$, too. ■

To apply these results to the setting of tensor algebras, we use a special case of Rieffel's imprimitivity theorem [25, Theorem 6.29]. For the reader's benefit, we recall the key ideas behind it. Suppose that F is a Hilbert C^*-module over a C^*-algebra A and that B is a C^*-subalgebra of $\mathcal{L}(F)$. Then if π is a representation of A on a Hilbert space $\mathfrak{H}$, we may induce it to a representation π^F of $\mathcal{L}(F)$ and restrict π^F to B to obtain a representation of B. We will denote the restriction also by π^F. The question arises: Given a representation ρ of B on the Hilbert space $\mathfrak{H}_\rho$, when is it (unitarily equivalent to) π^F for some representation π of A? Rieffel's imprimitivity theorem asserts that this is the case if and only if there is a (nondegenerate) representation σ of $\mathcal{K}(F)$ on $\mathfrak{H}_\rho$ such that

$$\sigma(bk) = \rho(b)\sigma(k) \tag{3.1}$$

for all $b \in B$ and $k \in \mathcal{K}(F)$. The necessity of the condition is clear. For the sufficiency, simply observe that if such a σ exists, then as we indicated prior to Lemma 2.1, $\sigma^{\tilde{F}}$ represents $\mathcal{K}(\tilde{F}) \simeq A$ on $\tilde{F} \otimes_\sigma \mathfrak{H}_\rho$ and $\sigma \simeq (\sigma^{\tilde{F}})^F|\mathcal{K}(F)$. It follows that $\rho \simeq (\sigma^{\tilde{F}})^F|B$.

For our application of this result, first recall that when $\varphi(A) \subseteq \mathcal{K}(E)$, the entire algebra $\mathcal{K}\big(\mathcal{F}(E)\big)$ is contained in $\mathcal{T}(E)$ [19, Theorem 3.13]. So a representation of $\mathcal{T}(E)$ yields a (possibly degenerate) representation of $\mathcal{K}\big(\mathcal{F}(E)\big)$ by restriction.

Lemma 3.4 *Assume $\varphi(A) \subseteq \mathcal{K}(E)$. Then a representation ρ of $\mathcal{T}(E)$ is of the form $\pi^{\mathcal{F}(E)}$ (restricted to $\mathcal{T}(E)$) for some representation π of A if and only if the restriction of ρ to $\mathcal{K}\big(\mathcal{F}(E)\big)$ is nondegenerate.*

Proof If $\rho = \pi^{\mathcal{F}(E)}$ restricted to $\mathcal{T}(E)$, then as $\mathcal{K}\big(\mathcal{F}(E)\big) \subseteq \mathcal{T}(E)$ by the remarks preceding the statement of the lemma, $\rho|\mathcal{K}\big(\mathcal{F}(E)\big)$ is nondegenerate. For the opposite implication, simply note that in the preceding discussion, the representation ρ *restricts* to $\mathcal{K}\big(\mathcal{F}(E)\big)$ to give a representation σ of $\mathcal{K}\big(\mathcal{F}(E)\big)$, which is nondegenerate by hypothesis. Evidently, then, equation (3.1) is satisfied where $F = \mathcal{F}(E)$ and $B = \mathcal{T}(E)$. ■

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Our refinement of the Wold decomposition theorem, Theorem 2.9, is an immediate corollary of this lemma.

Theorem 3.5 *Suppose the correspondence E is strict in the sense that the essential subspace $\varphi(A)E$ is a summand of E, i.e., it is of the form QE for a projection in $\mathcal{L}(E)$, suppose, too, that φ is injective and that $\varphi(A) \subseteq \mathcal{K}(E)$. If $\mathfrak{S}$ is a Shilov module for $\mathcal{T}_+(E)$ and if $\mathfrak{S} = \mathfrak{S}_p \oplus \mathfrak{S}_i$ is the abstract Wold decomposition of $\mathfrak{S}$ as in Theorem 3.2, then $\mathfrak{S}_p$ is a Hilbert module that is induced from a Hilbert module over A and $\mathfrak{S}_i$ is a boundary Hilbert module obtained by restricting a C^*-representation of $\mathcal{O}(E)$ to $\mathcal{T}_+(E)$. Conversely, every Hilbert module over $\mathcal{T}_+(E)$ that is induced is pure.*

Proof As we mentioned at the outset of this section, the first part of this lemma is easily proved on the basis of Theorem 2.9 and Corollary 2.15. We give an alternate proof, based on Lemma 3.4, that takes advantage of the hypothesis that $\varphi(A) \subseteq \mathcal{K}(E)$. We write ρ for the representation of $\mathcal{T}_+(E)$ associated with $\mathfrak{S}$. Since $\mathfrak{S}$ is Shilov, we may apply Lemma 3.1 to write ρ as $\rho = \sigma \times V$ where (V, σ) is an isometric covariant representation of E on $\mathfrak{S}$. Since $\mathfrak{S}_p \subseteq \mathfrak{S}$, $\mathfrak{S}_p$ is Shilov and we can assume for the time being that $\mathfrak{S}_p = \mathfrak{S}$; *i.e.*, we may drop the "p". Since (V, σ) is isometric, ρ extends to a C^*-representation of $\mathcal{T}(E)$ on $\mathfrak{S}$. Write $\mathfrak{N} = \Big(\mathcal{K}\big(\mathcal{F}(E)\big)\mathfrak{S}\Big)^{\perp}$ and recall that since $\varphi(A) \subseteq \mathcal{K}(E)$, $\mathcal{K}\big(\mathcal{F}(E)\big)$ is an ideal in $\mathcal{T}(E)$. Thus $\mathcal{T}(E)\mathfrak{N} \subseteq \mathfrak{N}$. Also $\mathcal{K}\big(\mathcal{F}(E)\big)\mathfrak{N} = 0$. Hence $\mathfrak{N}$ is a module for $\mathcal{O}(E)$ (which is $\mathcal{T}(E)/\mathcal{K}\big(\mathcal{F}(E)\big)$ by [19, Theorem 3.13]). In particular, $\mathfrak{N}$ is orthoinjective by Theorem 6.4 of [18] and the main theorem of [17]. By our assumption, $\mathfrak{N} = \{0\}$. Thus $\mathfrak{S} = \big[\mathcal{K}\big(\mathcal{F}(E)\big)^r akS\big]$. It follows, then, from the imprimitivity theorem, Lemma 3.4, that there is a representation π of A on a Hilbert space $\mathfrak{H}$ such that $\mathfrak{S}$ is unitarily equivalent to $\mathcal{F}(E) \otimes_\pi \mathfrak{H}$. That is, $\mathfrak{S}$ is induced.

To see that $\mathfrak{S}_i$ is a boundary Hilbert module, simply note that since $\mathfrak{S}$ is Shilov, $\mathfrak{S}$ is also orthoprojective. Thus $\mathfrak{S}_i$ is a boundary Hilbert module by the second half of Theorem 3.2. The fact that $\mathfrak{S}_i$ comes from a representation of $\mathcal{O}(E)$ follows from Corollary 6.6 of [18].

For the converse assertion, simply note that if $\mathfrak{S} = \mathcal{F}(E) \otimes_\pi \mathfrak{H}$ is an induced Hilbert module for $\mathcal{T}_+(E)$, and if its abstract Wold decomposition is $\mathfrak{S} = \mathfrak{S}_p \oplus \mathfrak{S}_i$, then $\mathfrak{S}_p$ and $\mathfrak{S}_i$ reduce $\sigma^{\mathcal{F}(E)}\big(\mathcal{T}(E)\big)$ and, in particular, they reduce $\sigma^{\mathcal{F}(E)}\Big(\mathcal{K}\big(\mathcal{F}(E)\big)\Big)$, since $\mathcal{K}\big(\mathcal{F}(E)\big) \subseteq \mathcal{T}(E)$. By Theorem 6.23 of [25], or by Proposition 2.11 above, both the Hilbert modules $\mathfrak{S}_p$ and $\mathfrak{S}_i$ are induced. However, as we have noted before, $\mathfrak{S}_i$ is a boundary Hilbert module, arising, in particular, from restricting a representation of $\mathcal{O}(E) = \mathcal{T}(E)/\mathcal{K}\big(\mathcal{F}(E)\big)$ to $\mathcal{T}_+(E)$. Thus, $\mathfrak{S}_i$ viewed as a Hilbert module over $\mathcal{T}(E)$ kills the compacts. By Theorem 6.23 of [25], again, $\mathfrak{S}_i$ (which is induced) must be induced by the zero representation of A, *i.e.*, $\mathfrak{S}_i$ must be zero. ∎

Remark 3.6 We now have two uses of the word "pure" in the context of Hilbert modules over tensor algebras: The first, from Corollary 2.15, refers to induced Hilbert modules that contain no coisometric summands and the second, from Theorem 3.2, refers to Hilbert modules that contain no orthoinjective summands. Under the hypotheses of Theorem 3.5, these two notions are the same. However, while in general every orthoinjective (Shilov) module corresponds to a coisometric, isometric covariant representation, we do not know when the converse holds.

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4 Beurling's Theorem

The proof of the Beurling-Lax-Halmos theorem essentially boils down to showing this: If S is a shift on a Hilbert space $\mathfrak{H}$ and if $\mathfrak{M}$ is an invariant subspace, then the dimension of the wandering subspace $\mathfrak{M}_0$ $(= \mathfrak{M} \ominus S\mathfrak{M})$ associated to $\mathfrak{M}$ is less than or equal to the dimension of the wandering subspace $\mathfrak{H}_0$ $(= \mathfrak{H} \ominus S\mathfrak{H})$ associated to $\mathfrak{H}$. This observation and our analysis of induced isometric covariant representations, in particular, Proposition 2.11, suggests that when studying a submodule $\mathfrak{M}$ of an induced Shilov module over $\mathcal{T}_+(E)$, induced, say by a representation π_0 of A, one should try to compare π_0 with the representation π of A obtained by restricting $\pi_0^{\mathcal{F}(E)} \circ \varphi_\infty$ to the wandering subspace $\mathfrak{M}_0$ determined by $\mathfrak{M}$. As we shall see in Remark 4.6 below, simple examples show that without additional hypotheses, π_0 and π may be incomparable in the strongest sense: each contains subrepresentations that are disjoint from the other. Our objective in this section is to identify natural hypotheses that disallow this sort of disjointness and to show how to use them to prove a natural generalization of Beurling's theorem for tensor algebras.

Proposition 4.1 *Let $\mathfrak{S}$ be a Shilov Hilbert module over $\mathcal{T}_+(E)$ that is induced by a representation $\pi\colon A \to B(\mathfrak{H})$, let (V,σ) be the associated isometric covariant representation of E, and let L be the endomorphism of $\sigma(A)'$ constructed from V. If $\mathfrak{M}$ be a submodule of $\mathfrak{S}$, with projection $P_{\mathfrak{M}}$; if $\mathfrak{M}_0$ is the corresponding wandering subspace, i.e., the range of $P_{\mathfrak{M}} - L(P_{\mathfrak{M}})$; and if π_0 be obtained by restricting $\sigma(\cdot)$ to $\mathfrak{M}_0$, then π_0 is unitarily equivalent to a subrepresentation of π if and only if there is a partial isometry in* $\mathrm{End}(\mathfrak{S})$ *with final space $\mathfrak{M}$.*

Proof Suppose π_0 is unitarily equivalent to a subrepresentation of π, and let $u_0\colon \mathfrak{H} \to \mathfrak{M}_0$ be a partial isometry such that $u_0^*\pi_0(\cdot)u_0 \leq \pi(\cdot)$. Recall that $\mathfrak{H}$ may be viewed as a summand of $\mathfrak{S}$ and that π is just σ restricted to this summand. Thus, if u is defined to be zero off $\mathfrak{H}$ and u_0 on $\mathfrak{H}$, then u is a partial isometry in the commutant of $\sigma(A)$ with initial space contained in $\mathfrak{H}$ and final space $\mathfrak{M}_0$. For $k \geq 0$, set $Q_k = P_k - P_{k+1} = L^k(I) - L^{k+1}(I)$, as above, and set $Q_k^{\mathfrak{M}} = L^k(P_{\mathfrak{M}}) - L^{k+1}(P_{\mathfrak{M}})$. Then by Proposition 2.11 and Corollary 2.4, $P_{\mathfrak{M}} = \sum_{k\geq 0}^{\oplus} Q_k^{\mathfrak{M}}$. Since u has initial space contained in the range of Q_0 and range equal to the range of $Q_0^{\mathfrak{M}}$, $L^k(u)$ has initial space contained in the range of Q_k and final space equal to the range of $Q_k^{\mathfrak{M}}$. Consequently, $U := \sum_{k\geq 0}^{\oplus} L^k(u)$ is a well-defined partial isometry with range $\mathfrak{M}$. Further, since $u \in \sigma(A)'$ and L is an endomorphism of $\sigma(A)'$, U lies in $\sigma(A)'$. It remains, then, to prove that U commutes with $V(\xi)$, $\xi \in E$. On the basis of Lemma 2.2, it is easy to see that $L^k(u)V(\xi) = V(\xi)L^{k-1}(u)$ for all $k \geq 1$. Indeed, for $h \in \mathfrak{S}$,

$$\begin{aligned}
L^k(u)V(\xi)h &= \tilde{V}_k(I_k \otimes u)\tilde{V}_k^* V_1(\xi \otimes h)\\
&= \tilde{V}_1(I_1 \otimes \tilde{V}_{k-1})(I_1 \otimes I_{k-1} \otimes u)(I_1 \otimes \tilde{V}_{k-1})^* \tilde{V}_1^* \tilde{V}_1(\xi \otimes h)\\
&= \tilde{V}_1\big(I_1 \otimes L^{k-1}(u)\big)(\xi \otimes h)\\
&= V(\xi)L^{k-1}(u)h.
\end{aligned}$$

Consequently, since $Q_0V(\xi) = 0$, so that $uV(\xi) = 0$, we see that

$$UV(\xi) = \sum_{k\geq 0}^{\oplus} L^k(u)V(\xi) = uV(\xi) + \sum_{k\geq 1}^{\oplus} L^k(u)V(\xi)$$
$$= 0 + \sum_{k\geq 1}^{\oplus} V(\xi)L^{k-1}(u) = V(\xi)U.$$

Conversely, if U is any partial isometry in $\operatorname{End}(\mathfrak{S})$ whose final space is $\mathfrak{M}$, then by Lemma 2.5,

$$UP_1U^* = U\tilde{V}_1\tilde{V}_1^*U^* = \tilde{V}_1(I\otimes U)(I\otimes U)^*\tilde{V}_1^*$$
$$= L(UU^*) = L(P_{\mathfrak{M}}).$$

Consequently, $UQ_0U^* = U(I-P_1)U^* = P_{\mathfrak{M}} - L(P_{\mathfrak{M}}) = Q_0^{\mathfrak{M}}$. This shows that the restriction of U to Q_0 is a partial isometry with initial space contained in $\mathfrak{H}$ and final space equal to the range of $Q_0^{\mathfrak{M}}$. Since this restriction commutes with σ (restricted to $\mathfrak{H}$), we see that the restriction implements an equivalence between π_0 and a subrepresentation of π. ∎

Following the traditional theory of shifts, we call a partial isometry in the commutant of a representation an *inner operator*. We want to establish the level of uniqueness that is available in the representation of an invariant subspace as the range of an inner operator. This requires a little preparation that is of interest in its own right. The following lemma was inspired by Theorem 2 of [26]; the proof is a minor adaptation of the proof given there.

Lemma 4.2 *Let $\mathfrak{S}$ be a Shilov Hilbert module over $\mathfrak{T}_+(E)$, afforded by an isometric covariant representation (V,σ) of E, and let $U \in \operatorname{End}(\mathfrak{S})$ be an inner operator. Then*

1. *the initial projection of U, U^*U, also lies in $\operatorname{End}(\mathfrak{S})$; and*
2. *for all $T \in \mathfrak{T}_+(E)$, $\sigma\times V(T)$ satisfies the equation*

$$U^*\big(\sigma\times V(T)\big)UU^* = \sigma\times V(T)U^*.$$

Proof Of course U^*U commutes with $\sigma(A)$, so to prove the first assertion, it suffices to prove that if $\mathfrak{M}$ is the range of U^*U, then $\mathfrak{M}$ reduces $\{V(\xi) \mid \xi\in E\}$. First observe that $\mathfrak{M} = U^*\mathfrak{S}$. Consequently, for $\xi\in E$, $V(\xi)^*\mathfrak{M} = V(\xi)^*U^*\mathfrak{H} = U^*V(\xi)^*\mathfrak{H} \subseteq U^*\mathfrak{H} = \mathfrak{M}$, proving that $\mathfrak{M}$ is invariant under $\{V(\xi)^* \mid \xi\in E\}$. On the other hand by Lemma 2.5, for $h\in\mathfrak{M}$ and $\xi\in E$,

$$\|UV(\xi)h\| = \|U\tilde{V}_1(\xi\otimes h)\| = \|\tilde{V}_1(I\otimes U)(\xi\otimes h)\|$$
$$= \|(I\otimes U)(\xi\otimes h)\| = (\xi\otimes Uh, \xi\otimes Uh)^{1/2}$$
$$= \big(\sigma(\langle\xi,\xi\rangle)Uh, Uh\big)^{1/2} = \big(U\sigma(\langle\xi,\xi\rangle)^{1/2}h, U\sigma(\langle\xi,\xi\rangle)^{1/2}h\big)^{1/2}$$
$$= \big(\sigma(\langle\xi,\xi\rangle)^{1/2}h, \sigma(\langle\xi,\xi\rangle)^{1/2}h\big)^{1/2} = (\xi\otimes h, \xi\otimes h)^{1/2}$$
$$= \|\tilde{V}_1(\xi\otimes h)\| = \|V(\xi)h\|,$$

which shows that $V(\xi)h \in \mathfrak{M}$. Thus $\mathfrak{M}$ is invariant under $\{V(\xi) \mid \xi \in E\}$, and the first assertion is proved.

The second assertion is immediate from the first (we need only check the equation on generators, $V(\xi)$): Since U and $U^*U \in \mathrm{End}(\mathfrak{S})$, it follows that $U^*V(\xi)UU^* = U^*UV(\xi)U^* = V(\xi)U^*UU^* = V(\xi)U^*$. ■

Corollary 4.3 *In the notation of Proposition 4.1, if U_1 and U_2 are two inner operators in* $\mathrm{End}(\mathfrak{S})$ *with the same range, then $U_2 = U_1W$, where W is a constant inner operator that maps the initial space of U_2 onto the initial space of U_1.*

Proof Recall that a constant inner operator is a partial isometry in the commutant of $\sigma \times V\big(\mathcal{T}(E)\big)$. Observe that $U_1^*U_2$ is a partial isometry in $\sigma(A)'$ that maps the initial space of U_2 to the initial space of U_1 and $U_1(U_1^*U_2) = U_2$. Therefore, it suffices to show that both $U_1^*U_2$ and $(U_1^*U_2)^* = U_2^*U_1$ lie in $\mathrm{End}(\mathfrak{S})$. By the second assertion of Lemma 4.2, for all $\xi \in E$, $V(\xi)U_1^*U_2 = U_1^*V(\xi)U_1U_1^*U_2$. Since $U_1U_1^*$ is the projection on the range of U_1, which is the projection onto the range of U_2, by hypothesis, $U_1U_1^*U_2 = U_2$. Consequently, $U_1^*V(\xi)U_1U_1^*U_2 = U_1^*V(\xi)U_2 = U_1^*U_2V(\xi)$. This shows that $U_1^*U_2$ lies in $\mathrm{End}(\mathfrak{S})$; the proof that $U_2^*U_1 \in \mathrm{End}(\mathfrak{S})$ is the same. ■

Recall that two representations π_1 and π_2 of a C^*-algebra A are called quasi-equivalent, and one writes $\pi_1 \approx \pi_2$, in case a multiple of π_1 is unitarily equivalent to a multiple of π_2 (see [7, 5.3]). We say that π_1 is *quasi-contained in* π_2, and write $\pi_1 \preceq \pi_2$, in case π_1 is quasi-equivalent to a subrepresentation of π_2. Using Proposition 5.3.1 of [7], it is not difficult to see that $\pi_1 \preceq \pi_2$ if and only if there is a family $\{V_i\}$ of partial isometries, with V_i mapping $\mathfrak{H}_{\pi_2}$ to $\mathfrak{H}_{\pi_1}$ and intertwining π_1 and π_2, such that $\sum_i V_iV_i^*$ is the identity operator on $\mathfrak{H}_{\pi_1}$.

Definition 4.4 A representation π of A is said to be *quasi-invariant* with respect to a correspondence E if π is quasi-equivalent to the representation $\pi^{\mathcal{F}(E)} \circ \varphi_\infty = \sigma$ of A.

We also write π_n for $\pi^E \circ \varphi^{(n)}$, $n \geq 0$. So, when we form the isometric covariant representation (V, σ) induced by π, then, clearly, σ $(= \pi^{\mathcal{F}(E)} \circ \varphi_\infty) = \sum^\oplus \pi_n$. Observe that since $\pi_0 = \pi$, we see that $\pi \leq \sigma$.

Lemma 4.5 *The representation π of A is quasi-invariant with respect to E if and only if every π_n is quasi-contained in π.*

Proof Suppose π is quasi-invariant with respect to E. If $\pi_n \npreceq \pi$ then there is a subrepresentation $\tau \leq \pi_n$ such that τ is disjoint from π. However, then, τ is contained in σ, which is quasi-equivalent to π, *and* τ is disjoint from π—which is clearly impossible. Conversely, if $\pi_n \precsim \pi$ for all n, then $\sigma = \sum \pi_n \precsim \pi \leq \sigma$. Thus $\pi \sim \sigma$. ■

Remark 4.6 The terminology, *quasi-invariant*, is suggested by ergodic theory and the theory of dynamical systems. Indeed, suppose the C^*-algebra A is $C(X)$, for a compact Hausdorff space X, suppose $E = C(X)$, also, and suppose that φ is given by a homeomorphism τ of X, so that $\varphi(f)\xi(x) = f \circ \tau(x)\xi(x)$. Then as is discussed in [18, Example 2.6], the module $E^{\otimes n}$ may be identified with $E = C(X)$, but the left action φ_n of $A = C(X)$

is given by the formula $\varphi_n(f)\xi(x) = f \circ \tau^n(x)\xi(x)$. If μ is a positive measure on X and π is the representation of $C(X)$ on $L^2(\mu)$ given by multiplication, then π_n is the representation of $C(X)$ on $L^2(\mu)$ given by multiplication by functions composed with τ^n; *i.e.*, $\pi_n(f)\xi(x) = f \circ \tau^n(x)\xi(x)$. It follows in this case that π_n is quasi-contained in π if and only if $\mu \circ \tau^{-n} \ll \mu$. From Lemma 4.5, we conclude that π is quasi-invariant with respect to $E = C(X)$ if and only if $\mu \equiv \sum_{n=0}^{\infty} \mu \circ \tau^{-n}$. Now recall that a measure μ is called quasi-invariant under τ in case $\mu \circ \tau^{-1} \ll \mu$. A moment's reflection reveals that this is, in fact, the same as saying that $\mu \equiv \sum_{n=0}^{\infty} \mu \circ \tau^{-n}$. Thus, we see that our notion of quasi-invariance extends the ergodic/dynamical-systems-theoretic notion.

This example helps to illustrate, too, the point made at the outset of this section that if (V, σ) is a covariant representation obtained by inducing a representation π_0 of A, so that $\sigma = \pi_0^{\mathcal{F}(E)} \circ \varphi_\infty$, and if π is obtained by restricting σ to a wandering subspace, then π and π_0 may be entirely unrelated. Indeed, in our example when $E = A = C(X)$, suppose π_0 is the representation determined by a point mass δ_x, and assume for the sake of this discussion that x is not a periodic point. Then π_{0n} is the representation determined by $\delta_{\tau^n(x)}$. The representation σ, then, is determined by counting measure along the forward orbit of x. The Hilbert space of (V, σ) can be taken to be $\ell^2(\mathbb{Z}_+)$, with the representation σ given by the formula $\sigma(f)\xi(n) = f\big(\tau^n(x)\big)\xi(n)$ and with V given by the formula $V(f)\xi(n) = f\big(\tau^n(x)\big)\xi(n-1)$, $\xi \in \ell^2(\mathbb{Z}_+)$, $f \in C(X)$. It follows, for example, that if $\mathfrak{M}$ is the subspace of $\ell^2(\mathbb{Z}_+)$ consisting of all functions ξ such that $\xi(0) = 0$, then the wandering subspace associated with $\mathfrak{M}$ is the set of all functions ξ such that ξ vanishes at all $n \neq 1$. The representation π associated with this subspace is the representation associated with $\delta_{\tau(x)}$; *i.e.*, $\pi = \pi_0 \circ \varphi = \pi_{01}$. Evidently, π_0 and π are disjoint.

The following result can be viewed as Beurling's theorem in our context. It extends Theorem 2.4 of [22] and Theorem 2.1 of [5].

Theorem 4.7 *Let π be a representation of A on a Hilbert space $\mathfrak{H}$ and let $\mathfrak{S} = \mathcal{F}(E) \otimes_\pi \mathfrak{H}$ be the Shilov Hilbert module over $\mathcal{T}_+(E)$ that is induced by π. If π is quasi-invariant with respect to E and if $\mathfrak{M} \subseteq \mathfrak{S}$ is a submodule, then there is a family of inner operators $\{V_i\}$ contained in* $\mathrm{End}(\mathfrak{S})$*, with orthogonal ranges, such that*

$$\mathfrak{M} = \sum\nolimits^{\oplus} V_i \mathfrak{S}. \tag{4.1}$$

Conversely, if every submodule $\mathfrak{M}$ of $\mathfrak{S}$ can be represented in this form, for a suitable family $\{V_i\}$ of inner operators with orthogonal ranges, then π is quasi-invariant with respect to E.

Proof Let (V, σ) be the isometric covariant representation of E associated with $\mathfrak{S}$, so that $\sigma = \pi^{\mathcal{F}(E)} \circ \varphi_\infty$, and let L be the endomorphism of $\sigma(A)'$ determined (V, σ). As in Proposition 4.1, let $P_{\mathfrak{M}}$ be the projection of $\mathfrak{S}$ onto $\mathfrak{M}$ and let $\mathfrak{M}_0$ be the range of $P_{\mathfrak{M}} - L(P_{\mathfrak{M}})$. If $\pi_{\mathfrak{M}_0}$ is the representation of A by restricting σ to $\mathfrak{M}_0$, then $\pi_{\mathfrak{M}_0}$ is a subrepresentation of σ, which, in turn, is quasi-equivalent to π. Thus $\pi_{\mathfrak{M}_0} \precsim \pi$. Hence there is a family $\{V_{0,i}\}$ such that $\sum_i V_{0,i} V_{0,i}^*$ is the projection onto $\mathfrak{M}_0$, the initial space of each $V_{0,i}$ is contained in $\mathfrak{H}$, and $V_{0,i}$ intertwines π and $\pi_{\mathfrak{M}_0}$. Hence $V_{0,i} \in \sigma(A)'$. If we set $V_i = \sum\nolimits^{\oplus}{}_{k=0}^{\infty} L^k\,(V_{0,i})$ as in the proof of Proposition 4.1, we conclude that $V_i \in \mathrm{End}(\mathfrak{S})$ and $\mathfrak{M} = \sum\nolimits^{\oplus} V_i \mathfrak{S}$ as

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required. The ranges of the V_i are orthogonal since this is the case for the $V_{0,i}$ (and L is normal).

For the converse assertion, observe that if every submodule of $\mathfrak{S}$ can be represented in the form of equation (4.1), then in particular, the spaces $P_n\mathfrak{S}$ $(= \sum_{k\geq n} E^{\otimes k} \otimes_\pi \mathfrak{H})$ have such a representation. Consequently, we may write $P_k\mathfrak{S} = \sum_i^{\oplus} V_i\mathfrak{S}$, for inner operators in $\mathrm{End}(\mathfrak{S})$ having orthogonal ranges. As we saw in the proof of Proposition 4.1, each V_i maps a subspace of $\mathfrak{H}$ (regarded as the 0-th summand of $\mathfrak{S}$) onto a subspace of $E^{\otimes k} \otimes_\pi \mathfrak{H}$ isometrically and in such a way that the restriction of V_i to $\mathfrak{H}$ intertwines π and π_n. Further, if we write $V_{0,i}$ for the restriction V_i to $\mathfrak{H}$, it follows easily from the fact that $\sum_i V_iV_i^*$ is the projection onto $P_k\mathfrak{S}$ that $\sum_i V_{0,i}V_{0,i}^*$ is the projection onto $E^{\otimes k} \otimes_\pi \mathfrak{H}$. We conclude, then, that $\pi_k \preceq \pi$ for all k. By Lemma 4.5, π is quasi-invariant. ∎

5 Quiver Algebras

In this section, we apply the analysis presented so far to study what happens when A is a finite dimensional C^*-algebra and E is finite dimensional over $\mathbb{C}$. Such a situation is usefully described in terms of quivers. Our goal, is to identify situations when the invariant subspace structure of induced Shilov Hilbert modules over the tensor algebra is nicely related to the ideal structure of the weakly closed algebra generated by the representation.

A (finite) *quiver* is simply a directed graph with n vertices $\{v_1, \ldots, v_n\}$ and C_{ij} arrows from v_j to v_i. Here, C_{ij} is a non-negative integer. We let A denote the C^*-direct sum of n copies of $\mathbb{C}$ indexed by the vertices. It will be convenient to view A as the algebra of all $n \times n$ diagonal matrices. The correspondence over A that we shall associate with the quiver, and denote by $E(C)$, is defined to be the direct sum of Hilbert spaces, $\sum_{1\leq i,j\leq n} \mathfrak{H}_{ij}$, where $\mathfrak{H}_{ij}$ is a Hilbert space of dimension C_{ij}. For each $\mathfrak{H}_{ij}$, with $C_{ij} > 0$, we fix an orthonormal basis $\{e_{ij}^{(k)} : 1 \leq k \leq C_{ij}\}$, so that $E(C) = \operatorname{span}\{e_{ij}^{(k)} : 1 \leq i, j \leq n, 1 \leq k \leq C_{ij}\}$. The Hilbert space $E(C)$ may be viewed as an A-A bimodule via the formulae:

$$e_{ij}^{(k)} e_{ll} = \delta_{jl} e_{ij}^{(k)}$$

$$\varphi(e_{ll}) e_{ij}^{(k)} = \delta_{i,l} e_{ij}^{(k)}.$$

Also, $E(C)$ has an A-valued inner product defined by the formula

$$\langle e_{ij}^{(k)}, e_{lm}^{(p)} \rangle = \delta_{k,p}\delta_{i,l}\delta_{j,m} e_{jj}.$$

This makes $E(C)$ into a C^*-correspondence over A.

It perhaps should be emphasized that the *Hilbert space structure* obtained by identifying $E(C)$ with $\sum_{1\leq i,j\leq n} \mathfrak{H}_{ij}$ plays no role in our discussion. The norm structure on $E(C)$ viewed as a Hilbert space is different from the norm structure on $E(C)$ viewed as a C^*-correspondence. It is the latter that is relevant for our considerations here.

If C, B are two $\mathbb{Z}_+$-valued $n \times n$ matrices, we have

Lemma 5.1 *The interior tensor product $E(C) \otimes_A E(B)$ is naturally isomorphic to $E(CB)$.*

Proof For every pair (i,l), with $1 \leq i,l \leq n$, the set $\{(k,r,j) \mid 1 \leq k \leq C_{ij}, 1 \leq r \leq B_{jl}, 1 \leq j \leq n\}$ has $\sum_{j=1}^{n} C_{ij}B_{jl}$ $(= (CB)_{il})$ elements. Hence we may write $E(CB)$ as the span of all $e_{i,l}^{(k,r,j)}$ for (k,r,j) in this set. Now define the map

$$e_{ij}^{(k)} \otimes e_{jl}^{(r)} \mapsto e_{il}^{(k,r,j)}$$

from $E(C) \otimes_A E(B)$ to $E(CB)$ and extend by linearity. It is straightforward to check that the extended map is a well defined A-A isomorphism that is unitary; *i.e.*, that preserves the A-valued inner products. ■

Corollary 5.2 *For each $n \geq 2$, $E(C)^{\otimes n} = E(C^n)$ and $\mathcal{F}\big(E(C)\big) = \sum_{n=0}^{\oplus\infty} E(C^n)$, where we identify $E(C^0)$ with A.*

Every representation π of A is determined by the multiplicities of the 1-dimensional representations δ_k, defined by $\delta_k(a) = a_{kk}$, $a \in A$. We write

$$m(\pi) = \big(m(\pi)_1, m(\pi)_2, \ldots, m(\pi)_n\big)$$

for the vector of multiplicities (*i.e.*, $m(\pi)_k$ is the multiplicity of δ_k in π); the $m(\pi)_k$ are non-negative integers or ∞. Given two representations of A, π', π'', it is easy to see that $\pi' \precsim \pi''$ if and only if $m(\pi'')_k = 0 \Rightarrow m(\pi')_k = 0$; *i.e.*, iff the support of $m(\pi')$ is contained in the support of $m(\pi'')$.

If π is a representation of A on a Hilbert space $\mathfrak{H}_\pi$, we may write $\mathfrak{H}_\pi = \sum_{k=1}^{n} \mathbb{C}^{m(\pi)_k}$, where $\mathbb{C}^\infty$ is interpreted as ℓ^2. When this is done, we have the following decomposition of $E(C) \otimes_\pi \mathfrak{H}_\pi$ that reflects the choices of the bases we have made:

$$\begin{aligned} E(C) \otimes_\pi \mathfrak{H}_\pi &= E(C) \otimes_\pi \sum \pi(e_{kk})\mathbb{C}^{m(\pi)_k} \\ &= \sum_k E(C)e_{kk} \otimes \mathbb{C}^{m(\pi)_k} \\ &= \sum_k \operatorname{span}\{e_{ik}^{(j)} \otimes e_p \mid 1 \leq p \leq m(\pi)_k, 1 \leq j \leq C_{ik}, 1 \leq i \leq n\} \\ &= \sum{}^{\oplus} \mathfrak{K}_i, \end{aligned}$$

where $\mathfrak{K}_i = \operatorname{span}\{e_{ik}^{(j)} \otimes e_p \mid 1 \leq p \leq m(\pi)_k, 1 \leq k \leq n, 1 \leq j \leq C_{ik}\}$; *i.e.*, $\dim \mathfrak{K}_i = \sum C_{ik} m(\pi)_k = \big(Cm(\pi)\big)_i$. Consequently, for each i, k, j, p as above and $a \in A$,

$$\pi^{E(C)}\big(\varphi(a)\big)(e_{ik}^{(j)} \otimes e_p) = a_{ii}(e_{ik}^{(j)} \otimes e_p).$$

Hence $m(\pi^{E(C)} \circ \varphi) = Cm(\pi)$.

Corollary 5.3 *A representation π of A is quasi-invariant if and only if for every $i \in \operatorname{supp} m(\pi)$ and $j \notin \operatorname{supp} m(\pi)$, $(C^l)_{ij} = 0$, for all $l \in \mathbb{N}$. Hence if C is irreducible then π is quasi-invariant if and only if for every $1 \leq k \leq n$, $m(\pi)_k \neq 0$.*

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Proof The fact that supp $Cm(\pi)$ is contained in supp $m(\pi)$ implies that for $i \in$ supp $m(\pi)$ and $j \notin$ supp $m(\pi)$, $0 = \big(Cm(\pi)\big)_j = \sum_k C_{jk} m(\pi)_k \geq C_{ji} m(\pi)_i$; hence $C_{ji} = 0$. Thus π is quasi-invariant if this holds for all C^n. If $S =$ supp $m(\pi)$ and if $S \neq \{1, \ldots, n\}$ then

$$S^c \times S \subseteq \{(i, j) \mid C_{ij} = 0\}.$$

So, if C is irreducible, this implies that $S = \{1, \ldots, n\}$. ■

Fix C and a representation π of A on a Hilbert space $\mathfrak{H}$ and let m denote the vector of multiplicities associated with π. We are interested in the Shilov module $\mathfrak{S}(C, m) := \mathcal{F}\big(E(C)\big) \otimes_\pi \mathfrak{H}$ for the algebra $\mathcal{T}_+\big(E(C)\big)$, and our objective is to understand the structure of $\operatorname{End}\big(\mathfrak{S}(C, m)\big)$ as a function of C and m. For this purpose, we write $\rho(C, m)$ or simply ρ for the representation associated to $\mathfrak{S}(C, m)$, and we shall write $\mathcal{A}(C, m)$ or simply $\mathcal{A}$ for the algebra $\rho(C, m)\Big(\mathcal{T}_+\big(E(C)\big)\Big)$. Of course, $\operatorname{End}\big(\mathfrak{S}(C, m)\big)$ is just $\mathcal{A}(C, m)'$. We first consider the case when $m = (1, 1, \ldots, 1)$, *i.e.*, the case when π is multiplicity free and of full support, so we denote the vector m by $\mathbf{1}$.

A crucial role in our analysis is played by a natural Hilbert space isomorphism, W, from $\mathfrak{S}(C, \mathbf{1})$ to $\mathfrak{S}(C^t, \mathbf{1})$. We emphasize that W is not a module map, but it does reflect the structure we have been developing. In order to define W we should note first that $\mathfrak{H} = \mathfrak{H}_\pi$ is simply $\mathbb{C}^n$ and we fix in $\mathbb{C}^n$ its usual orthonormal basis $\{e_i\}_{i=1}^n$. Writing $E = E(C)$, we see that $\mathfrak{S}(C, \mathbf{1}) = \mathcal{F}\big(E(C)\big) \otimes_\pi \mathfrak{H} = \sum_{k=0}^{\oplus\infty} E^{\otimes k} \otimes_\pi \mathfrak{H}$ and, in the notation established above,

$$E^{\otimes k} \otimes_\pi \mathfrak{H} = \operatorname{span}\{e_{i_1 i_2}^{(l_1)} \otimes e_{i_2 i_3}^{(l_2)} \otimes \cdots \otimes e_{i_k i_{k+1}}^{(l_k)} \otimes e_{i_{k+1}} : 1 \leq l_j \leq C_{i_j i_{j+1}}\}.$$

To shorten the notation we shall write a typical element in this spanning set as $e_{(i_1, \ldots, i_{k+1})}^{(l_1, \ldots, l_k)}$ or simply $e_{\underline{i}}^{\underline{l}}$. Also, for any n, we write θ for the permutation of an n-tuple of integers that reverses the order. Thus, for $\underline{i} = (i_1, \ldots, i_{k+1})$, we write $\theta(\underline{i}) = (i_{k+1}, \ldots, i_1)$ and similarly $\theta(\underline{l}) = (l_k, \ldots, l_1)$. We then define $W : \mathfrak{S}(C, \mathbf{1}) \to \mathfrak{S}(C^t, \mathbf{1})$ by setting

$$W e_{\underline{i}}^{\underline{l}} = e_{\theta(\underline{i})}^{\theta(\underline{l})}, \tag{5.1}$$

and extending W by linearity. It is easy to check that W is indeed a well defined Hilbert space isomorphism mapping $\mathfrak{S}(C, \mathbf{1})$ onto $\mathfrak{S}(C^t, \mathbf{1})$.

Proposition 5.4 *The map W relates the algebras $\mathcal{A}(C, \mathbf{1})$ and $\mathcal{A}(C^t, \mathbf{1})$ via the equation*

$$\mathcal{A}(C, \mathbf{1})' = W^* \overline{\mathcal{A}(C^t, \mathbf{1})}^w W,$$

where $\overline{\mathcal{A}(C^t, \mathbf{1})}^w$ denotes closure of $\mathcal{A}(C^t, \mathbf{1})$ in the weak operator topology.

Proof To prove that $W^* \mathcal{A}(C^t, \mathbf{1}) W \subseteq \mathcal{A}(C, \mathbf{1})'$, it suffices to show that for $e_{ij}^{(k)} \in E(C^t)$, $1 \leq k \leq C_{ji}$, and for $e_{ii} \in A$, we have

$$W^* \rho(C^t, \mathbf{1})(e_{ij}^{(k)}) W, \quad W^* \rho(C^t, \mathbf{1})(e_{ii}) W \in \mathcal{A}(C, \mathbf{1})',$$

where $\rho(C^t,\mathbf{1})(e_{ij}^{(k)})$ denotes $\rho(C^t,\mathbf{1})(T_{e_{ij}^{(k)}}) \in \mathcal{A}(C^t,\mathbf{1})$. For this we show that they both commute with $\rho(C,\mathbf{1})(e_{q,p}^{(r)})$ and with $\rho(C,\mathbf{1})(e_{qq})$. For simplicity we write ρ, ρ^t for $\rho(C,\mathbf{1})$ and $\rho(C^t,\mathbf{1})$.

$$\begin{aligned}
W^*\rho^t(e_{ij}^{(k)})W\rho(e_{q,p}^{(r)})e_{\underline{i}}^{\underline{l}} &= W^*\rho^t(e_{ij}^{(k)})We_{(q,\underline{i})}^{(r,\underline{l})}\cdot\delta_{p,i_1}\\
&= W^*\rho^t(e_{ij}^{(k)})e_{\theta(q,\underline{i})}^{\theta(r,\underline{l})}\cdot\delta_{p,i_1}\\
&= W^*e_{(i,j,i_n,\dots,i_2,p,q)}^{(k,l_n,\dots,l_1,r)}\delta_{p,i_1}\delta_{j,i_{n+1}}\\
&= \delta_{p,i_1}\delta_{j,i_{n+1}}e_{(q,p,\dots,i_n,j,i)}^{(r,l_1,\dots,k)},
\end{aligned}$$

$$\begin{aligned}
\rho(e_{q,p}^{(r)})W^*\rho^t(e_{ij}^{(k)})We_{\underline{i}}^{\underline{l}} &= \rho(e_{q,p}^{(r)})W^*\rho^t(e_{ij}^{(k)})e_{\theta(\underline{i})}^{\theta(\underline{l})}\\
&= \rho(e_{q,p}^{(r)})W^*e_{(i,j,i_n,\dots,i_1)}^{(k,l_n,\dots,l_1)}\delta_{j,i_{n+1}}\\
&= \rho(e_{q,p}^{(r)})e_{(i_1,\dots,i_n,j,i)}^{(l_1,\dots,k)}\delta_{j,i_{n+1}}\\
&= \delta_{j,i_{n+1}}\delta_{q,i_1}e_{(q,p,i_2,\dots,j,i)}^{(r,l_1,\dots,k)}.
\end{aligned}$$

The other equalities are proved similarly. This shows that $W^*\overline{\mathcal{A}(C^t,\mathbf{1})}^w W \subseteq \mathcal{A}(C,\mathbf{1})'$.

The proof of the reverse containment is more involved. We write Q_k for the projection of $\mathfrak{S}(C,\mathbf{1}) = \mathcal{F}\big(E(C)\big)\otimes\mathfrak{H}$ onto $E(C)^{\otimes k}\otimes\mathfrak{H}$ and we write Q_k^t for the projection of $\mathfrak{S}(C^t,\mathbf{1}) = \mathcal{F}\big(E(C^t)\big)\otimes\mathfrak{H}$ onto $E(C^t)^{\otimes k}\otimes\mathfrak{H}$, and for $k<0$, we set both Q_k and Q_k^t equal to 0. Clearly, then, $WQ_kW^* = Q_k^t$ and for $\xi\in E$,

$$W\rho(\xi)W^*Q_k^t = Q_{k+1}^tW\rho(\xi)W^*, \quad k\in\mathbb{Z},$$

where, again, we write $\rho(\xi)$ for $\rho(T_\xi)$. Let $U_t = \sum_{n=0}^\infty e^{-int}Q_n^t$, and for any operator R on $\mathfrak{S}(C^t,\mathbf{1})$, let $\alpha_t(R) = U_tRU_t^*$. Then on $\rho^t\Big(\mathcal{T}\big(E(C^t)\big)\Big)$, $\{\alpha_t\}$ is the gauge automorphism group. Also, we set $\Phi_j(R) = \int e^{-ijt}\alpha_t(R)\,dt$, for $j\in\mathbb{Z}$. Thus Φ_j is a bounded operator on $B\big(\mathfrak{S}(C^t,\mathbf{1})\big)$. A straightforward computation shows that for $R\in W\mathcal{A}(C,\mathbf{1})'W^*$, $\Phi_j(R)$ is given by the formula $\Phi_j(R) = \sum_k Q_k^tRQ_{k+j}^t$. Therefore for $\xi\in E$, we have

$$\begin{aligned}
W\rho(\xi)W^*\Phi_j(R) &= \sum_k W\rho(\xi)W^*Q_k^tRQ_{k+j}^t\\
&= \sum_k Q_{k+1}^tW\rho(\xi)W^*RQ_{k+j}^t\\
&= \sum_k Q_{k+1}^tRW\rho(\xi)W^*Q_{k+j}^t\\
&= \sum_k Q_{k+1}^tRQ_{k+j+1}^tW\rho(\xi)W^*\\
&= \Phi_j(R)W\rho(\xi)W^*,
\end{aligned}$$

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which shows that $\Phi_j(R) \in W\mathcal{A}(C,\mathbf{1})'W^*$.

Now, for $k \geq 1$ and $R \in W\mathcal{A}(C,\mathbf{1})'W^*$, set

$$\Sigma_k(R) = \sum_{|j|<k} \left(1 - \frac{|j|}{k}\right) \Phi_j(R) \in W\mathcal{A}(C,\mathbf{1})'W^*.$$

Then

$$\Sigma_k(R) \xrightarrow{\text{WOT}} R$$

because $\Sigma_k(R) = \int \alpha_t(R) K_k(t)\, dt$, where K_k is Fejer's kernel.

It is left to show that, for every j, $\Phi_j(R) \in \mathcal{A}(C^t,\mathbf{1})$. We now write σ^t for $\rho^t \circ \varphi_\infty$. Then σ^t is a representation of A and, by what was already proved, $\sigma^t(A) = \rho^t\big(\varphi_\infty(A)\big) \subseteq \mathcal{A}(C^t,\mathbf{1}) \subseteq \big(W\mathcal{A}(C,\mathbf{1})W^*\big)'$. It suffices to show that, for all i, j, k,

$$\sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk}) \in \mathcal{A}(C^t,\mathbf{1}).$$

We note that

$$Q_l\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big) = Q_l\rho\big(E(C)^{\otimes l}\big)\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)$$

(as $\rho\big(E(C)^{\otimes l}\big)\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big) \supseteq \rho\big(E(C)^{\otimes l}\big)(A \otimes \mathfrak{H}) = E(C)^{\otimes l} \otimes \mathfrak{H} = Q_l\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)$) and, thus,

$$Q_l^t\Big(\mathcal{F}\big(E(C^t)\big) \otimes \mathfrak{H}\Big) = Q_l^t W\rho\big(E(C)^{\otimes l}\big)W^*\Big(\mathcal{F}\big(E(C^t)\big) \otimes \mathfrak{H}\Big).$$

Hence

$$\mathcal{F}\big(E(C^t)\big) \otimes \mathfrak{H} = \bigvee_l Q_l^t W\rho\big(E(C)^{\otimes l}\big)W^*\Big(\mathcal{F}\big(E(C^t)\big) \otimes \mathfrak{H}\Big). \tag{5.2}$$

For $j > 0$ and $R \in \big(W\mathcal{A}(C,\mathbf{1})W^*\big)'$, $\Phi_j(R)Q_0^t = 0$. Thus

$$\begin{aligned} 0 &= W\rho\big(E(C)^{\otimes l}\big)W^*\Phi_j(R)Q_0^t \\ &= \Phi_j(R)W\rho\big(E(C)^{\otimes l}\big)W^*Q_0^t \\ &= \Phi_j(R)Q_l^t W\rho\big(E(C)^{\otimes l}\big)W^*. \end{aligned}$$

By equation (5.2), $\Phi_j(R) = 0$ whenever $j > 0$. So we can assume that $j \leq 0$ and write $p = |j|$. Fixing i and k, with $1 \leq i, k \leq n$, we consider $\sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk})e_k$, where $e_k \in \mathfrak{H} = A \otimes \mathfrak{H} \subseteq \mathcal{F}\big(E(C^t)\big) \otimes \mathfrak{H}$. Clearly it is an element of $E(C^t)^{\otimes p} \otimes \mathfrak{H}$ and, thus, can be written as a sum $\sum a(\underline{i}, \underline{l})e_{\underline{i}}^{\underline{l}}$ where each $\underline{i}$ is of length $p+1$. Also it is clear that $i_1 = i$, since it is in the range of $\sigma^t(e_{ii})$. Since $W\sigma(e_{kk})W^*e_k = e_k$, we have

$$\begin{aligned} \sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk})e_k &= \sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk})W\sigma(e_{kk})W^*e_k \\ &= W\sigma(e_{kk})W^*\sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk})e_k \\ &= W\sigma(e_{kk})\Big(\sum a(\underline{i}, \underline{l})e_{\theta(\underline{i})}^{\theta(\underline{l})}\Big). \end{aligned}$$

Thus $i_{p+1} = k$. Now, each $e_{\underline{i}}^{\underline{l}}$ is of the form $\xi_{\underline{i}}^{\underline{l}} \otimes e_k$, for $\xi_{\underline{i}}^{\underline{l}} \in E^{\otimes p}$. We claim, therefore, that

$$\sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk}) = \sum a(\underline{i}, \underline{l})\rho^t(\xi_{\underline{i}}^{\underline{l}}) \tag{5.3}$$

(note that this is a finite sum). Write the right hand side of equation (5.3) as S. We know that $\sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk})e_k = Se_k$ and both sides vanish when applied to e_q, $q \neq k$ (as $i_{p+1} = k$); hence $\sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk})Q_0^t = SQ_0^t$. For $q \geq 0$, $Q_q^t\Big(\mathcal{F}\big(E(C^t)\big) \otimes \mathfrak{H}\Big) = W\rho\big(E(C)^{\otimes q}\big)W^*Q_0^t\Big(\mathcal{F}\big(E(C^t)\big) \otimes \mathfrak{H}\Big)$ and, for $\xi \in E^{\otimes q}$, $h \in Q_0^t\Big(\mathcal{F}\big(E(C^t)\big) \otimes \mathfrak{H}\Big)$ we have

$$\begin{aligned}\sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk})W\rho(\xi)W^*h &= W\rho(\xi)W^*\sigma_t(e_{ii})\Phi_j(R)\sigma^t(e_{kk})h\\ &= W\rho(\xi)W^*Sh\\ &= SW\rho(\xi)W^*h.\end{aligned}$$

(We used the fact that $S \in \mathcal{A}(C^t, \mathbf{1}) \subseteq \big(W\mathcal{A}(C, \mathbf{1})W^*\big)'$.) This proves equation (5.3) and it follows that, for each i, j, k, $\sigma^t(e_{ii})\Phi_j(R)\sigma^t(e_{kk}) \in \mathcal{A}(C^t, \mathbf{1})$. This completes the proof. ■

Proposition 5.4 extends a result of Popescu [22, Corollary 1.3] and Davidson and Pitts [5, Theorem 1.2]. Their analyses fit into ours if one takes $n = 1$, $m = \mathbf{1}$ and C to be a 1×1 matrix, which necessarily must be C^t.

In order to deal with the case of a general m we need only note that as a consequence of Corollary 2.13, we have

Lemma 5.5 *If π is expressed as the direct sum of two representations $\pi = \pi_1 \oplus \pi_2$ and if the multiplicity of π_i is m_i, $i = 1, 2$, then $\rho(C, m_1) \oplus \rho(C, m_2) = \rho(C, m_1 + m_2)$.*

In the following, σ^t and $\mathfrak{H}$ are as in Proposition 5.4.

Write $\tau(a) = W^*\sigma^t(a)W$, $a \in A$. Then

$$\tau(e_{ii})e_{\underline{i}}^{\underline{l}} = W^*\sigma^t(e_{ii})e_{\theta(\underline{i})}^{\theta(\underline{l})} = \delta_{i_n,i}W^*e_{\theta(\underline{i})}^{\theta(\underline{l})} = \delta_{i_n,i}e_{\underline{i}}^{\underline{l}}.$$

Hence $\tau(e_{ii})(\xi \otimes h) = \xi \otimes P_ih$ where $\xi \in \mathcal{F}\big(E(C)\big)$, $h \in \mathfrak{H}$ and P_i is the projection onto the subspace of $\mathfrak{H}$ spanned by e_i. From Proposition 5.4 we know that $\tau(e_{ii}) \in \mathcal{A}(C, \mathbf{1})'$. Suppose now that $m(\pi)$ is a vector of 1's and 0's and set $\Gamma = \{1 \leq i \leq n : m(\pi)_i = 1\}$. We can clearly view $\mathfrak{H}_\pi$ as a subspace of $\mathbb{C}^n$ ($= \mathfrak{H}_{\pi_0}$ where $m(\pi_0) = \mathbf{1}$) and $\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}_\pi$ is a subspace of $\mathcal{F}\big(E(C)\big) \otimes \mathbb{C}^n$. In fact, we can identify $\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}_\pi$ with $\sum_{i\in\Gamma} \tau(e_{ii})\Big(\mathcal{F}\big(E(C)\big) \otimes \mathbb{C}^n\Big)$. We write $\mathfrak{H}$ for $\mathbb{C}^n$ and $P(\Gamma)$ for $\sum \tau(e_{ii})$. Note that $P(\Gamma) \in \mathcal{A}(C, \mathbf{1})'$.

Lemma 5.6 *Viewing $\mathcal{A}(C, m) \subseteq B\bigg(P(\Gamma)\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)\bigg)$ we have*

$$\mathcal{A}(C, m)' = P(\Gamma)\mathcal{A}(C, \mathbf{1})'P(\Gamma).$$

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Proof From the considerations above it follows that $\mathcal{A}(C,m) = \mathcal{A}(C,\mathbf{1})P(\Gamma)$ and $P(\Gamma) \in \mathcal{A}(C,\mathbf{1})$. So the lemma follows. ■

Now we consider a general multiplicity vector m. Let $N = \max_i m_i$ and let $\Gamma_i = \{j : m_j \geq i\}$, $1 \leq i \leq N$. Given π with $m(\pi) = m$ we may write $\rho(C,m) = \sum_{i=1}^{N} \rho(C,\tilde{m}_i)$ where $(\tilde{m}_i)_j = \chi_{\Gamma_i}(j)$, thanks to Lemma 5.5. Hence $\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}_\pi$ can be identified with the direct sum,

$$\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}_\pi = \sum_{}^{N}{}^{\oplus} P(\Gamma_i)\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)$$

and

$$\rho(C,m) = \sum_{i=1}^{N} \rho(C,\mathbf{1})|P(\Gamma_i)\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big).$$

(Here, as above, $\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}$ is the representation space of $\rho(C,\mathbf{1})$). We can also write $\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)^{(N)}$ for the direct sum of N copies of $\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}$ and $P(m)$ for the projection of $\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)^{(N)}$ onto $\sum_{i=1}^{\oplus N} P(\Gamma_i)\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)$. Thus, $P(m) = \sum^{\oplus} P(\Gamma_i)$.

Theorem 5.7 *For every representation π of A on $\mathfrak{H}_\pi$ we can identify $\mathcal{F}\big(E(C)\big) \otimes_\pi \mathfrak{H}_\pi$ with $P\big(m(\pi)\big)\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)^{(N)}$, where N is $\max m(\pi)_i$, and then*

$$\operatorname{End}\Big(\mathcal{F}\big(E(C)\big) \otimes_\pi \mathfrak{H}_\pi\Big) = P\big(m(\pi)\big)\overline{M_N\big(W^*\mathcal{A}(C^t,\mathbf{1})W\big)}^{w} P\big(m(\pi)\big)$$

where W is the unitary operator defined by equation (5.1) and where $\overline{M_N\big(W^*\mathcal{A}(C^t,\mathbf{1})W\big)}^{w}$ is the algebra of $N \times N$ matrices over $\overline{W^*\mathcal{A}(C^t,\mathbf{1})W}^{w}$.

Proof Straightforward. ■

Our objective now is to identify useful, general conditions under which σ-weakly continuous linear functionals on $\mathcal{A}(E(C),m)$ are vector functionals.

Definition 5.8 We shall say that the quiver, or its incidence matrix C, satisfies *the entrance condition* in case there is a $k > 0$ such that for all $1 \leq i \leq n$, $\sum_{j=1}^{n} (C^k)_{ij} \geq 2$.

The reason for the terminology is that the condition means that there is a k such that each vertex is the range of at least two distinct paths of length k; *i.e.*, at least two distinct paths of length k *enter* each vertex. Observe that any quiver or matrix that is primitive in the sense that there is a k such that for all i and j, $(C^k)_{ij} > 0$ satisfies the entrance condition. (Primitive quivers are also called aperiodic.)

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Lemma 5.9 *Suppose that C satisfies the entrance condition, and suppose π is a representation of A with $m(\pi) = \mathbf{1}$. If $\mathfrak{S}$ is the $\mathcal{T}_+(E)$-module $\mathcal{F}\big(E(C)\big) \otimes_\pi \mathfrak{H}_\pi$, then there are infinitely many, pairwise orthogonal submodules $\{\mathfrak{S}_i\}_{i=1}^\infty$ of $\mathfrak{S}$ such that each is isomorphic to $\mathfrak{S}$; i.e., there are infinitely many inner isometries V_i such that $\mathfrak{S}_i = V_i \mathfrak{S}$.*

Proof First note that it suffices to prove that there exist two inner isometries, V_1 and W_1, with orthogonal ranges, say $\mathfrak{S}_1$, $\mathfrak{S}_1'$. Indeed, suppose we can prove this. Then the sequence of inner isometries $\{V_1^k W_1\}_{k=1}^\infty$ has pairwise orthogonal ranges. In order to construct the desired V_1 and W_1 (or $\mathfrak{S}_1$, $\mathfrak{S}_1'$), it will suffice to construct in $E^{\otimes k} \otimes_\pi \mathfrak{H}$ two orthogonal subspaces, $\mathfrak{M}_0$ and $\mathfrak{N}_0$, such that the representations of A, $\sigma|\mathfrak{M}_0$ and $\sigma|\mathfrak{N}_0$, will both be equivalent to π. (The index k in $E^{\otimes k} \otimes_\pi \mathfrak{H}$ is the k that appears in the entrance condition, *i.e.*, the hypothesis that $\sum_{j=1}^n (C^k)_{ij} \geq 2$.) The fact that $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are contained in $E^{\otimes k} \otimes_\pi \mathfrak{H}$ guarantees that $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are wandering subspaces. The isometries V_1 and W_1 are then constructed as in Proposition 4.1 and $\mathfrak{S}_1$, $\mathfrak{S}_1'$ are defined by the equations $\mathfrak{S}_1 = \overline{\rho\big(\mathcal{T}_+(E)\big)\mathfrak{M}_0}$, $\mathfrak{S}_1' = \overline{\rho\big(\mathcal{T}_+(E)\big)\mathfrak{N}_0}$.

To produce $\mathfrak{M}_0$ and $\mathfrak{N}_0$, note that the entrance condition, $\sum_{j=1}^n (C^k)_{ij} \geq 2$ for all i, implies that $m_k := m(E^{\otimes k} \otimes_\pi \mathfrak{H}) = C^k \mathbf{1} \geq 2\mathbf{1}$. Hence, in particular, $m_k \geq m \oplus m$. Thus, there are two orthogonal subspaces $\mathfrak{M}_0$ and $\mathfrak{N}_0$ of $E^{\otimes k} \otimes_\pi \mathfrak{H}$ that reduce σ such that the multiplicity of each is $\mathbf{1}$; *i.e.*, $\sigma|\mathfrak{M}_0$ and $\sigma|\mathfrak{N}_0$ are both equivalent to π. ∎

With this lemma available, we are in a position to apply the argument of [5, Theorem 2.10] to prove that every σ-weakly continuous linear functional on $\mathcal{A}(C, \mathbf{1})$ is a vector functional when C satisfies the entrance condition.

Theorem 5.10 *Suppose $m(\pi) = \mathbf{1}$ and that C satisfies the entrance condition. Then every σ-weakly continuous linear functional f of norm < 1 on $\mathcal{A}(C, \mathbf{1})$ is implemented by a pair of vectors of norm < 1; i.e., there are vectors ξ, η in $\mathcal{F}(E) \otimes_\pi \mathfrak{H}_\pi$, with $\|\xi\|, \|\eta\| < 1$, such that*

$$f(T) = \langle T\xi, \eta \rangle, \quad T \in \mathcal{A}(C, \mathbf{1}).$$

Proof The proof is as in [5, Theorem 2.10]. Since f is σ-weakly continuous and of norm < 1, there are vectors ξ_k, η_k, $\|\xi_k\|, \|\eta_k\| < 1$, and numbers $s_k \geq 0$ with $\sum s_k < 1$ such that $f(T) = \sum_{k=1}^\infty s_k \langle T\xi_k, \eta_k \rangle$, $T \in \mathcal{A}(C, \mathbf{1})$. Let $\{V_i\}$ be a sequence of inner isometries, with orthogonal ranges (such a sequence exists, by Lemma 5.9), and set

$$\xi = \sum_{k=1}^\infty s_k^{\frac{1}{2}} V_k \xi_k, \quad \eta = \sum s_k^{\frac{1}{2}} V_k \eta_k.$$

Then $\|\xi\|, \|\eta\| < 1$, and $f(T) = \sum s_k \langle T V_k \xi_k, V_k \xi_k \rangle = \langle T\xi, \eta \rangle$, $T \in \mathcal{A}(C, \mathbf{1})$. ∎

This result enables us to use the argument from [4, Theorem 2.1] to show that there is a bijective correspondence between the right ideals in the algebra $\overline{\mathcal{A}(C, \mathbf{1})}^w$ and $\operatorname{Lat} \mathcal{A}(C, \mathbf{1})'$.

We start by adopting the following notation. We write $\mathcal{T}_0$ for the ideal in $\mathcal{T}_+(E)$ generated by $\{T_\xi \mid \xi \in E\}$. We write $\mathcal{A}$ for $\mathcal{A}(C, \mathbf{1})$ and $\mathcal{B} = \overline{\mathcal{A}}^w$. By $\operatorname{Id}_r(\mathcal{B})$ we denote all the right ideals in $\mathcal{B}$ that are closed in the weak operator topology and by $\operatorname{Lat} \mathcal{A}'$ the lattice of all

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closed $\mathcal{A}'$-invariant subspaces. If S is a family of vectors in a Hilbert space $\mathfrak{K}$, we shall write $[S]$ for the closed linear span of S in $\mathfrak{K}$. For $\mathcal{J} \in \mathrm{Id}_r(\mathcal{B})$ and $\mathfrak{M} \in \mathrm{Lat}\,\mathcal{A}'$ we set $\mu(\mathcal{J}) = \Big[\mathcal{J}\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)\Big]$ and $\iota(\mathfrak{M}) = \Big\{T \in \mathcal{B} : T\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big) \subseteq \mathfrak{M}\Big\}$.

Lemma 5.11

1. $\mu\big(\mathrm{Id}_r(\mathcal{B})\big) \subseteq \mathrm{Lat}\,\mathcal{A}'$.
2. $\iota(\mathrm{Lat}\,\mathcal{A}') \subseteq \mathrm{Id}_r(\mathcal{B})$.

Proof Obvious. ∎

In order to show that μ and ι are inverse maps, we need the following lemma.

Lemma 5.12 *Let $m(\pi) = \mathbf{1}$ and fix $\xi \in \mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}_\pi$. Then there is an inner operator $U \in \mathcal{A}(C,\mathbf{1})'$ whose range is $[\mathcal{A}(C,\mathbf{1})\xi]$.*

Proof First assume that ξ is a wandering vector; *i.e.*, assume $\rho(\mathcal{T}_0)\xi \perp \xi$. If we set $\xi' = \sum \frac{\sigma(e_{kk})\xi}{\|\sigma(e_{kk})\xi\|}$, where the sum ranges over all k such that $\sigma(e_{kk})\xi \neq 0$, then it is easy to check that $[\sigma(A)\xi] = [\sigma(A)\xi']$. Hence ξ' is also a wandering vector and $[\mathcal{A}(C,\mathbf{1})\xi] = [\mathcal{A}(C,\mathbf{1})\xi']$. We may therefore replace ξ by ξ' and assume that, for every k, $\|\sigma(e_{kk})\xi\|$ is either 0 or 1. Then we may define a partial isometry $U_0 \colon \mathfrak{H} \to [\sigma(A)\xi]$ by setting $U_0 e_k = \sigma(e_{kk})\xi$, for every k and defining U_0 to be zero on the orthogonal complement of span $\mathfrak{M}_0$ of the e_k's. Then $U_0 \in \sigma(A)'$ and its range is $[\sigma(A)\xi]$. Since $\mathfrak{M}_0$ and $[\sigma(A)\xi]$ are wandering subspaces, we can then use Proposition 4.1 to obtain U as required. Now suppose ξ is an arbitrary vector in $\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}$. Write $\mathfrak{N} = [\mathcal{A}(C,\mathbf{1})\xi]$ and $\mathfrak{N}_0 = \mathfrak{N} \ominus [\rho(\mathcal{T}_0)\mathfrak{N}]$. By Proposition 2.11, we know that $\mathfrak{N}_0$ is different from zero and that if we set $\eta = P_{\mathfrak{N}_0}\xi$, then $\eta \neq 0$. Further, η is a wandering vector and, clearly, $[\mathcal{A}(C,\mathbf{1})\eta] \subseteq [\mathcal{A}(C,\mathbf{1})\xi]$ $(= \mathfrak{N})$. If we show that $[\mathcal{A}(C,\mathbf{1})\xi] \subseteq [\mathcal{A}(C,\mathbf{1})\eta]$, we will be done. In fact, it will suffice to show that $\mathfrak{N}_0 \subseteq [\mathcal{A}(C,\mathbf{1})\eta]$ (since $\mathfrak{N} = [\mathcal{A}(C,\mathbf{1})\mathfrak{N}_0]$). So fix $h \in \mathfrak{N}_0$. Since $h \in \mathfrak{N} = [\mathcal{A}(C,\mathbf{1})\xi]$ there is a sequence $\{T_n\} \subseteq \mathcal{A}(C,\mathbf{1})$ such that $T_n\xi \to h$. Thus, also, $P_{\mathfrak{N}_0}T_n\xi \to P_{\mathfrak{N}_0}h = h$. For every n, $T_n = T_n' + T_n''$ where $T_n' \in \sigma(A)$ and $T_n'' \in \rho(\mathcal{T}_0)$. But $P_{\mathfrak{N}_0}T_n''\xi = 0$ since $T_n''\xi \in [\rho(\mathcal{T}_0)\mathfrak{N}] \subseteq \mathfrak{N}_0^\perp$. Thus $P_{\mathfrak{N}_0}T_n'\xi \to h$. As $P_{\mathfrak{N}_0}$ commutes with $\sigma(A)$ we conclude that $T_n'\eta = T_n'P_{\mathfrak{N}_0}\xi = P_{\mathfrak{N}_0}T_n'\xi \to h$. This shows that $h \in [\mathcal{A}(C,\mathbf{1})\eta]$. Since this holds for all $h \in \mathfrak{N}_0$, we are done. ∎

Theorem 5.13 *If C satisfies the entrance condition, then $\mu = \iota^{-1}$. Hence, in this case there is a bijective, order preserving correspondence between* $\mathrm{Id}_r(\mathcal{B})$ *and* $\mathrm{Lat}\,\mathcal{A}'$.

Proof The inclusions $\mu\big(\iota(\mathfrak{M})\big) \subseteq \mathfrak{M}$ ($\mathfrak{M} \in \mathrm{Lat}\,\mathcal{A}'$) and $\mathcal{J} \subseteq \iota\big(\mu(\mathcal{J})\big)$ ($\mathcal{J} \in \mathrm{Id}_r(\mathcal{B})$) follow immediately from the definitions. We shall prove the reverse inclusions.

For the inclusion $\mathfrak{M} \subseteq \mu\big(\iota(\mathfrak{M})\big)$, observe that we have $\mathcal{A}' = W^*\mathcal{B}(C^t)W$, where $\mathcal{B}(C^t) = \overline{\mathcal{A}(C^t,\mathbf{1})}^w$, by Proposition 5.4. Since $\mathcal{A}'\mathfrak{M} \subseteq \mathfrak{M}$ we have $\mathcal{B}(C^t)W\mathfrak{M} \subseteq W\mathfrak{M}$. We write $\mathfrak{N}$ for $W\mathfrak{M} \subseteq \mathcal{F}\big(E(C^t)\big) \otimes \mathfrak{H}$. By Proposition 2.11,

$$\mathfrak{N} = \mathfrak{N}_0 \oplus [\rho^t(E)\mathfrak{N}_0] \oplus [\rho^t(E^{\otimes 2})\mathfrak{N}_0] \oplus \cdots .$$

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Hence

$$\mathfrak{M} = \mathfrak{M}_1 \oplus [\mathcal{R}_1\mathfrak{M}_1] \oplus [\mathcal{R}_2\mathfrak{M}_1] \oplus \cdots$$

where $\mathfrak{M}_1 = W^*\mathfrak{N}_0$ and $\mathcal{R}_k = W^*\rho^t\big(E(C^t)^{\otimes k}\big)W \subseteq \mathcal{A}'$. Fix $\xi \in \mathfrak{M}_1$. Applying Lemma 5.12 to $W\xi$ ($\in \mathfrak{N}_0$) we find an inner operator $U \in \mathcal{A}(C^t, \mathbf{1})' = W\mathcal{B}W^*$ whose range is $[\mathcal{A}(C^t, \mathbf{1})W\xi]$. Write U_0 for W^*UW ($\in \mathcal{B}$). Then U_0 is an inner operator whose range is $W^*[\mathcal{A}(C^t, \mathbf{1})W\xi] = [\mathcal{A}'\xi]$. Since $[\mathcal{A}'\xi] \subseteq \mathfrak{M}$ (as $\xi \in \mathfrak{M}_1$ and $\mathcal{A}'\mathfrak{M} \subseteq \mathfrak{M}$) and $[\mathcal{A}'\xi] = U_0\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)$, we see that $U_0 \in \iota(\mathfrak{M})$. But then $\xi \in [\mathcal{A}'\xi] = U_0\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big) \subseteq \Big[\iota(\mathfrak{M})\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)\Big] = \mu\big(\iota(\mathfrak{M})\big)$. This shows that $\mathfrak{M}_1 \subseteq \mu\big(\iota(\mathfrak{M})\big)$. However, since $\mu\big(\iota(\mathfrak{M})\big)$ is in $\operatorname{Lat}\mathcal{A}'$, $\mathfrak{M} = W^*\mathfrak{N} = W^*[\mathcal{A}(C^t, \mathbf{1})\mathfrak{N}_0] = [W^*\mathcal{A}(C^t, \mathbf{1})W\mathfrak{M}_1] = [\mathcal{A}'\mathfrak{M}_1] \subseteq \big[\mathcal{A}'\mu\big(\iota(\mathfrak{M})\big)\big] \subseteq \mu\big(\iota(\mathfrak{M})\big)$.

For the inclusion, $\mathcal{J} \supseteq \iota\big(\mu(\mathcal{J})\big)$, let $\mathcal{J} \in \mathrm{Id}_r(\mathcal{B})$ and fix $\xi \in \mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}$. Using Lemma 5.12, we may find an inner operator $U \in \mathcal{A}(C, \mathbf{1})'$ whose range is $[\mathcal{A}\xi]$. We then have $[\mathcal{J}\xi] = [\mathcal{J}\mathcal{A}\xi] = \Big[\mathcal{J}U\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)\Big] = U\Big[\mathcal{J}\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)\Big] = U\mu(\mathcal{J})$ and, similarly, $\big[\iota\big(\mu(\mathcal{J})\big)\xi\big] = \big[\iota\big(\mu(\mathcal{J})\big)\mathcal{A}\xi\big] = \Big[\iota\mu(\mathcal{J})U\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)\Big] = U\Big[\iota\mu(\mathcal{J})\Big(\mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}\Big)\Big] \subseteq U\mu(\mathcal{J})$. Since $\iota\big(\mu(\mathcal{J})\big) \supseteq \mathcal{J}$, $[\iota\mu(\mathcal{J})\xi] \supseteq [\mathcal{J}\xi] = U\mu(\mathcal{J})$. Hence $[\iota\mu(\mathcal{J})\xi] = U\mu(\mathcal{J}) = [\mathcal{J}\xi]$. This holds for every $\xi \in \mathcal{F}\big(E(C)\big) \otimes \mathfrak{H}$. The argument used in [4, Theorem 2.1], using Theorem 5.10, now applies to show that $\mathcal{J} = \iota\big(\mu(\mathcal{J})\big)$. ∎

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