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EIGENVALUES OF THE CURVATURE OPERATOR FOR CERTAIN HOMOGENEOUS MANIFOLDS

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Introduction. Given a Riemannian manifold M, the Riemann tensor R induces the curvature operator

$$\rho\colon \Lambda^2 T_pM \longrightarrow \Lambda^2 T_pM$$

on the exterior power $\Lambda^2 T_pM$ of the tangent space, defined by the formula

$$\langle \rho(X \wedge Y), U \wedge V \rangle = \langle R(X,Y)V, U \rangle$$

where the inner product is defined by $\langle X \wedge Y, U \wedge V \rangle = \langle X, U \rangle \langle Y, V \rangle - \langle X, V \rangle \langle Y, U \rangle$. From the symmetries of R, it follows that ρ is self-adjoint and so has only real eigenvalues. R also induces the sectional curvature function K on 2-planes in T_pM by $K(\pi) = \langle \rho(X \wedge Y), X \wedge Y \rangle$ where $\{X, Y\}$ is an orthonormal basis of the 2-plane π. Conditions on the sign of ρ or of K are well known to be important in geometry. The main aim of this note is to explore, in the homogeneous case, some intermediate curvature conditions, some of which have recently proved to be useful.

Given a nonzero element ω in $\Lambda^2 T_pM$, there is a number $2r$ such that $\omega^r \neq 0$ but $\omega^{r+1} = 0$. This is also the minimal $2r$ such that $\omega = \sum_{i=1}^{r} U_i \wedge V_i$, U_i, $V_i \in T_pM$. The number $2r$ is called the rank of ω. We will say that ρ is r-negative if $\langle \rho(\omega), \omega \rangle < 0$ for all 2-forms of rank $\leqq 2r$, with obvious similar definitions for r-positive, r-nonnegative, etc. Clearly the condition that ρ is 1-negative is equivalent to the condition that K is negative, while the condition that ρ is $[\frac{1}{2} \dim M]$ – negative is equivalent to the condition that ρ is negative. J. D. Moore [Mo] essentially used the condition that ρ is 2-positive. Sampson [Sa] introduced a curvature on the complexified tangent space of a real Riemannian manifold which he called "Hermitian curvature" and was interested in finding non symmetric spaces with negative Hermitian curvature. Since ρ 2-negative implies negative Hermitian curvature, one of our results will show that there exist many homogeneous examples.

In section 1, we modify an argument of Heintze [H] to show that the class of homogeneous manifolds admitting an invariant metric with $K < 0$ coincides with

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the class admitting an invariant metric with $\rho < 0$. This gives the aforementioned examples of negative Hermitian curvature. These results contrast with known results on the class of homogeneous manifolds admitting an invariant metric with $K > 0$ since the examples of Aloff-Wallach [A-W] do not admit any Riemannian metrics with $\rho > 0$, by results of Micaleff and Moore [M-M]. We also give an example where $K < 0$ but ρ is not 2-negative.

In section 2, we consider various additional conditions on the class of naturally reductive homogeneous spaces. For example, we define a strongly normal space to be a normal space for which $[\ ,\ ]_{\mathfrak{m}}$ defines a Lie algebra on the orthocomplement $\mathfrak{m}$ of the isotropy algebra. Strongly normal spaces have $\rho \geqq 0$ (but examples will show that the converse is false) and we are able to classify them (except in dimension 3, only symmetric spaces appear as irreducible components). This condition also occurs in work of Sagle [S_1][S_2] and we give some examples and results relating to his.

Section 1.

1.1 If M is a connected, simply connected homogeneous Riemannian manifold with $K \leqq 0$, then M is isometric to a solvable Lie group G with a left invariant metric, [H], [W]. Moreover, if $\mathfrak{g}$ is the Lie algebra of the solvable group G and $\langle\ ,\ \rangle$ is the inner product induced on $\mathfrak{g}$ by a left invariant metric on G with $K < 0$, then Heintze [H] proved.

(A) $\dim \mathfrak{g}' = \dim \mathfrak{g} - 1$ where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$

(B) there exists an element $A \in \mathfrak{g}$ with $\langle A, \mathfrak{g}'\rangle = 0$ such that if $D = \frac{1}{2}(\operatorname{ad} A + \operatorname{ad} A^*)$ and $S = \frac{1}{2}(\operatorname{ad} A - \operatorname{ad} A^*)$ then on $\mathfrak{g}'$, D and $F = D^2 + [D, S]$ are positive operators.

In the next subsections, we adapt the argument of Heintze to show that conditions (A) and (B) ensure the existence of left invariant metrics with $\rho < 0$.

1.2 Assume $\mathfrak{g}$ is a Lie algebra with inner product $\langle\ ,\ \rangle$, derived algebra $\mathfrak{g}'$ and $\mathfrak{g} = g' \oplus \mathbb{R}A$ (vector space sum) where $\langle \mathfrak{g}', A\rangle = 0$. With respect to the associated left invariant metric on G, covariant derivative is given by

(1.2.0) $2\langle \nabla_x Y, Z\rangle = \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle$

which implies

(1.2.1) $\nabla_A A = 0$

(1.2.2) $\nabla_A = S \equiv \dfrac{1}{2}(\operatorname{ad} A - \operatorname{ad} A^*)$ on $\mathfrak{g}'$

(1.2.3) For $X \in \mathfrak{g}'$, $\nabla_x A = -DX \equiv -\dfrac{1}{2}(\operatorname{ad} A + \operatorname{ad} A^*)(X)$

(1.2.4) If ∇' denotes the connection associated to $\langle\ ,\ \rangle|\mathfrak{g}'$, then for $X, Y \in \mathfrak{g}', \nabla_x Y = \nabla'_x Y + \alpha(X, Y)$ where $\alpha(X, Y) = \langle DX, Y\rangle A/|A|^2$

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Consider the operators

(1.2.5) $\Lambda^2 D : \Lambda^2 \mathfrak{g}' \longrightarrow \Lambda^2 \mathfrak{g}'$ defined by $\Lambda^2 D(X \wedge Y) = DX \wedge DY$

(1.2.6) $E: \Lambda^2 \mathfrak{g}' \longrightarrow \mathfrak{g}' \wedge \mathbb{R}A$ defined by

$$\langle E(X \wedge Y), Z \wedge A \rangle = \langle -\alpha(X, \nabla'_Y Z) + \alpha(Y, \nabla'_x Z) + \alpha([X, Y], Z), A \rangle$$

(1.2.7) $F \wedge I: \mathfrak{g}' \wedge \mathbb{R}A \longrightarrow \mathfrak{g}' \wedge \mathbb{R}A$ defined by $F \wedge I(X \wedge A) = F(X) \wedge A$ where $F = D^2 + [D, S]$

Lemma 1.2. *With respect to the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{R}A$, the curvature operator ρ has the matrix representation*

$$\begin{bmatrix} \rho' - \Lambda^2 D & E^* \\ E & -F \wedge I \end{bmatrix}$$

where ρ' is the curvature operator of the metric restricted to $\mathfrak{g}'$ and A has been normalized so that $\|A\| = 1$.

Proof. (1.2.4) just says that α is the second fundamental form of the hypersurface G' in G. The standard Gauss equation ([KN II], prop. 4.1) says for X, Y, Z, $W \in \mathfrak{g}'$

$$\begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle R'(X, Y)Z, W \rangle + \langle \alpha(X, Z), \alpha(Y, W) \rangle \\ &\quad - \langle \alpha(Y, Z), \alpha(X, W) \rangle \\ &= \langle R'(X, Y)Z, W \rangle + \langle DX, Z \rangle \langle DY, W \rangle \\ &\quad - \langle DY, Z \rangle \langle DX, W \rangle \\ &= \langle R'(X, Y)Z, W \rangle - \langle DX \wedge DY, W \wedge Z \rangle \end{aligned}$$

Note that (1.2.3) says the connection $\nabla^\perp$ on the normal bundle $T(G')^\perp$ is given by $\nabla^\perp_x A = -DX$ (see [KN II], prop. 3.4). Let $\tilde{\nabla}$ be the connection on $T(G') + T(G')^\perp$ obtained by combining the connections ∇' and $\nabla^\perp$. Then

$$\begin{aligned} (\tilde{\nabla}_x \alpha)(Y, Z) &= \nabla^\perp_x(\alpha(Y, Z)) - \alpha(\nabla'_x Y, Z) - \alpha(Y, \nabla'_x Z) \\ &= -\langle DY, Z \rangle DX - \alpha(\nabla'_x Y, Z) - \alpha(Y, \nabla'_x Z). \end{aligned}$$

The Codazzi equation ([KN II], prop. 4.3) says

$$\begin{aligned} \langle R(X, Y)Z, A \rangle &= \langle (\tilde{\nabla}_x \alpha)(Y, Z) - (\tilde{\nabla}_Y \alpha)(X, Z), A \rangle \\ &= \langle -\alpha(Y, \nabla'_x Z) + \alpha(X, \nabla'_Y Z) - \alpha([X, Y], Z), A \rangle \end{aligned}$$

This proves $\rho(X \wedge Y) = (\rho' - \Lambda^2 D)(X \wedge Y) + E(X \wedge Y)$.

Since ρ is self adjoint, we need only compute

$$\begin{aligned}\langle \rho(Z \wedge A), Y \wedge A\rangle &= \langle R(Z,A)A, Y\rangle \\ &= -\langle \nabla_A A, \nabla_Z Y\rangle + \langle \nabla_Z A, \nabla_A Y\rangle - \langle \nabla_{[Z,A]}A, Y\rangle \\ &= \langle -DZ, SY\rangle + \langle \nabla_{(S+D)Z}A, Y\rangle \\ &= \langle SDZ, Y\rangle - \langle D(S+D)Z, Y\rangle \\ &= -\langle FZ, Y\rangle\end{aligned}$$

1.3 Let $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{R}A$ be as before with $\|A\| = 1$. For $s > 0$, let A_s be the linear transformation of $\mathfrak{g}$ defined by

$$(1.3.1) \qquad A_s X = sX \quad \text{for} \quad X \in \mathfrak{g}', \quad A_s(A) = A.$$

Now let $[\ ,\ ]_s$ be the Lie product defined on the set $\mathfrak{g}$ so that

$$(1.3.2) \qquad A_s\colon (\mathfrak{g}, [\ ,\ ]_s) \longrightarrow (\mathfrak{g}, [\ ,\ ])$$

is a Lie algebra isomorphism. As noted by Heintze, $[\ ,\ ]_0 = \lim\limits_{s\to 0} [\ ,\ ]_s$ also defines a Lie algebra structure on $\mathfrak{g}$. Note that the derived algebra of each $(\mathfrak{g}, [\ ,\ ]_s)$, $0 \leqq s \leqq 1$, agrees with $\mathfrak{g}'$ as a set (but the derived algebra of $(\mathfrak{g}, [\ ,\ ]_0)$ is abelian). Also note that the adjoint action of A does not change. We keep the same inner product $\langle\ ,\ \rangle$ on each $(\mathfrak{g}, [\ ,\ ]_s)$ and let ∇_s, ∇'_s be the induced covariant derivatives on $\mathfrak{g}$, $\mathfrak{g}'$, ρ_s, ρ'_s the induced curvature operators and D_s, E_s, F_s the operators of section 1.2.

The standard formula (1.2.0) for covariant derivatives in left-invariant metrics now shows that $\nabla'_s = s\nabla'$ so $\rho'_s = s^2\rho'$. Also $D_s = D$, $F_s = F$, and $E_s = sE$ so

$$(1.3.3) \qquad \rho_s = s^2\begin{bmatrix}\rho' & 0\\ 0 & 0\end{bmatrix} + s\begin{bmatrix}0 & E^*\\ E & 0\end{bmatrix} - \begin{bmatrix}\Lambda^2 D & 0\\ 0 & F\wedge I\end{bmatrix}$$

PROPOSITION 1.3. *If $\mathfrak{g}$ as above is a Lie algebra for which the associated left invariant metric on G has $K < 0$, then G has a invariant metric with $\rho < 0$.*

Proof. We use the construction above. Since $K < 0$, part (B) of Heintze's result implies that

$$\begin{bmatrix}\Lambda^2 D & 0\\ 0 & F\wedge I\end{bmatrix}$$

is a positive operator. By (1.3.3), for all small enough s, ρ_s is negative definite. For such s, the inner product $\langle\ ,\ \rangle_s$ on $(\mathfrak{g}, [\ ,\ ])$ so that $A_s\colon (\mathfrak{g}, \langle\ ,\ \rangle) \longrightarrow (\mathfrak{g}, \langle\ ,\ \rangle_s)$ is an orthogonal isomorphism has the desired property.

Remark 1.3. It follows from [H], lemma 2, that if $(\mathfrak{g}, \langle\ ,\ \rangle)$ is symmetric with $K < 0$, then $[D, S] = 0$. Hence, the construction above shows that there exist many nonsymmetric homogeneous manifolds with $\rho < 0$. In fact, this will always be the result of the construction if the original metric is nonsymmetric since D and S do not change with s. This gives many examples with negative Hermitian curvature, as defined by Sampson [Sa] in his work on harmonic maps.

Combining Proposition 1.3 and the results summarized in 1.1, one obtains

COROLLARY 1.3. *Let M be a homogeneous manifold. Then the following conditions are equivalent.*

(i) M admits an invariant Riemannian metric with negative curvature.

(ii) M admits an invariant Riemannian metric with $\rho < 0$.

Remark. This contrasts with the case of positive curvature since the Aloff–Wallach examples of positively curved homogeneous manifolds [A-W] do not admit Riemannian metrics with $\rho > 0$ since their 3-homology is nontrivial, (see [M-M]).

1.4 This subsection is devoted to give an example of a left invariant metric of negative curvature whose curvature operator is not 2-negative.

Consider the nilpotent Lie algebra $\mathfrak{n}$ which has basis $\{e_1, e_2, e_3\}$ and bracket relations

(1.4.1) $[e_1, e_j] = 0, [e_2, e_3] = e_1$

and let $\delta\colon \mathfrak{n} \longrightarrow \mathfrak{n}$ be the derivation given by

(1.4.2) $\delta(e_j) = \lambda_j e_j \quad \text{for} \quad j = 1, 2, 3 \quad \text{with} \quad \lambda_1 = \lambda_2 + \lambda_3$

Consider the solvable Lie algebra $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}A$ where $\mathrm{ad}_{\mathfrak{n}}\, A = \delta$. On $\mathfrak{g}$, take the inner product $\langle\ ,\ \rangle$ making $\{e_1, e_2, e_3, A\}$ orthonormal.

Keeping the notation of 1.2, we compute the curvature operator. Since ad A is self-adjoint (1.2.1–3) imply

(1.4.3) $\nabla_A = 0, \nabla_x A = -\delta(X) = -D(X), F = D^2 = \delta^2$

Then (1.4.2) gives

(1.4.4) $\alpha(e_i, e_j) = \delta_{ij}\lambda_i A.$

For ∇', we get from (1.2.0) that

(1.4.5)
$$\nabla'_{e_1} e_2 = \nabla'_{e_2} e_1 = -\frac{1}{2}\, e_3,$$
$$\nabla'_{e_1} e_3 = \nabla'_{e_3} e_1 = \frac{1}{2}\, e_2,$$
$$\nabla'_{e_2} e_3 = \frac{1}{2}\, e_1, \nabla'_{e_3} e_2 = -\frac{1}{2}\, e_1$$
$$\nabla'_{e_j} e_j = 0$$

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Now we compute (somewhat tediously)

$$(1.4.6)\quad \rho'(e_1 \wedge e_j) = \frac{1}{4}\, e_1 \wedge e_j, \rho'(e_2 \wedge e_3) = -\frac{3}{4}\, e_2 \wedge e_3$$

$$(1.4.7)\quad \begin{aligned} E(e_1 \wedge e_2) &= \frac{-\lambda_1 + \lambda_2}{2}\, e_3 \wedge A \\ E(e_1 \wedge e_3) &= \frac{\lambda_1 - \lambda_3}{2}\, e_2 \wedge A \\ E(e_2 \wedge e_3) &= \frac{\lambda_2 + \lambda_3}{2}\, e_1 \wedge A \end{aligned}$$

Lemma 1.2 now gives ρ. One finds

$$(1.4.8)\quad \begin{aligned} \langle \rho(e_1 \wedge e_3 + e_2 \wedge A), e_1 \wedge e_3 + e_2 \wedge A\rangle &= \langle (\rho' - \Lambda^2 D)(e_1 \wedge e_3), e_1 \wedge e_3\rangle \\ &\quad + 2\langle E(e_1 \wedge e_3), e_2 \wedge A\rangle - \langle (F \wedge I)(e_2 \wedge A), e_2 \wedge A\rangle \\ &= \left(\frac{1}{4} - \lambda_1\lambda_3\right) + (\lambda_1 - \lambda_3) - \lambda_2^2 \end{aligned}$$

which is positive in particular for $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2} + \epsilon$, $\lambda_3 = \frac{1}{2} - \epsilon$, $0 < \epsilon < \frac{1}{2}$.

The final task will be to show that this metric has negative curvature for some of these parameter values.

To compute sectional curvatures, we decompose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1$ is spanned by $\{A, e_1\}$ and $\mathfrak{g}_2$ is spanned by $\{e_2. e_3\}$. It follow from the expression of ρ and (1.4.3)–(1.4.7) that for X_i, $Y_i \in \mathfrak{g}_i$, $i = 1, 2$, $R(X_i, Y_i)$ maps $\mathfrak{g}_j$ into $\mathfrak{g}_j$ for $j = 1, 2$. Hence for $X = X_1 + X_2$, $Y = Y_1 + Y_2$, X_i, $Y_i \in \mathfrak{g}_i$, we get

$$(1.4.9)\quad \begin{aligned} R(X, Y, X, Y) &= R(X_1, Y_1, X_1, Y_1) + R(X_2, Y_2, X_2, Y_2) \\ &\quad + R(X_1, Y_2, X_1, Y_2) + R(X_2, Y_1, X_2, Y_1) \\ &\quad + 2R(X_1, Y_1, X_2, Y_2) + 2R(X_1, Y_2, X_2, Y_1) \end{aligned}$$

We next compute the above terms when $\epsilon = 0$, i.e. $\lambda_1 = 1, \lambda_2 = \lambda_3 = 1/2$. Set

$$(1.4.10)\quad \begin{aligned} X_1 &= \alpha A + \beta e_1, Y_1 = \alpha' A + \beta' e_1, \Delta_1 = \alpha\beta' - \beta\alpha' \\ X_2 &= \gamma e_2 + \delta e_3, Y_2 = \gamma' e_2 + \delta' e_3, \Delta_2 = \gamma\delta' - \delta\gamma' \end{aligned}$$

It easily follows that

$$(1.4.11)\quad \begin{aligned} R(X_1, Y_1, X_1, Y_1) &= \Delta_1^2 R(A, e_1, A, e_1) = -\Delta_1^2 \\ R(X_1, Y_1, X_2, Y_2) &= \Delta_1\Delta_2 R(A, e_1, e_2, e_3) = -\frac{1}{2}\, \Delta_1\Delta_2 \\ R(X_2, Y_2, X_2, Y_2) &= \Delta_2^2 R(e_2, e_3, e_2, e_3) = -\Delta_2^2 \end{aligned}$$

For the remaining terms, set $Z = xe_2 + ye_3$, $\tilde{Z} = ye_2 - xe_3$. Using (1.4.3)–(1.4.5), we get

(1.4.12) $\nabla_{e_1} Z = \frac{1}{2}\tilde{Z}, \nabla_{Y_2} Z = \frac{1}{2}(\langle Y_2, Z\rangle A + \langle Y_2, \tilde{Z}\rangle e_1)$

Since $\tilde{\tilde{Z}} = -Z$, (1.4.12) implies

$$\nabla_{Y_2}\tilde{Z} = \frac{1}{2}(\langle Y_2, \tilde{Z}\rangle A - \langle Y_2, Z\rangle e_1)$$

Thus

$$\begin{aligned} R(X_1, Y_2)Z &= \alpha R(A, Y_2)Z + \beta R(e_1, Y_2)Z \\ &= -\frac{\alpha}{2}\nabla_{Y_2} Z + \beta\nabla_{e_1}\nabla_{Y_2} Z - \frac{\beta}{2}\nabla_{Y_2}\tilde{Z} \\ &= -\frac{1}{4}(\alpha\langle Y_2, \tilde{Z}\rangle + \beta\langle Y_2, Z\rangle)e_1 - \frac{1}{4}(\alpha\langle Y_2, Z\rangle - \beta\langle Y_2, \tilde{Z}\rangle)A \end{aligned}$$

and

$$\begin{aligned} R(X_1, Y_2, X_2, Y_1) &= \frac{\beta'}{4}(\alpha\langle Y_2, \tilde{X}_2\rangle + \beta\langle Y_2, X_2\rangle) \\ &\quad + \frac{\alpha'}{4}(\alpha\langle Y_2, X_2\rangle - \beta\langle Y_2, \tilde{X}_2\rangle) \\ &= -\frac{1}{4}\Delta_1\Delta_2 + \frac{1}{4}\langle X_1, Y_1\rangle\langle X_2, Y_2\rangle \end{aligned}$$

from which it follows that

$$R(X_1, Y_2, X_1, Y_2) = -\frac{1}{4}\|X_1\|^2\|Y_1\|^2$$

$$R(X_2, Y_1, X_2, Y_1) = -\frac{1}{4}\|X_2\|^2\|Y_1\|^2.$$

From (1.4.9) we get

$$\begin{aligned} \text{(1.4.13)}\quad R(X, Y, X, Y) &= -\left(\Delta_1^2 + \frac{3}{2}\Delta_1\Delta_2 + \Delta_2^2\right) \\ &\quad - \frac{1}{4}(\|X_1\|^2\|Y_2\|^2 - 2\langle X_1, Y_1\rangle\langle X_2, Y_2\rangle + \|X_2\|^2\|Y_1\|^2) \end{aligned}$$

The first terms in (1.4.13) give a negative definite quadratic form in Δ_1 and Δ_2 and given a two plane, we can always find a basis $\{X, Y\}$ such that $\langle X_1, Y_1\rangle = 0$. This shows that the sectional curvature of the metric with parameter values

$\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 1/2$ is negative. Thus there exist $\epsilon > 0$ such that the same holds for the metric with parameter values $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2} + \epsilon$, $\lambda_3 = \frac{1}{2} - \epsilon$.

If G is a Lie group with Lie algebra $\mathfrak{g}$ as in Section 1.4 then

PROPOSITION 1.4. *There is a one parameter family of left invariant metrics on G all of which have negative curvature but the curvature operator has some positive eigenvalues.*

Remark. Proposition 1.4 is in sharp contrast to the case of positive curvature since by a result of Wallach [W] a Lie group with a left invariant metric of positive curvature is covered by S^3 with the canonical metric, and so in particular it has positive curvature operator.

2.1 Let M be a connected homogeneous Riemannian manifold. Throughout section 2, we will assume M is naturally reductive. By Kostant's theorem [K_1], [K_2], this means we can assume there exists a connected Lie group G acting by isometries on M such that

(NR1) G act almost effectively on M

(NR2) Let K be the isotropy subgroup at an arbitrary base point $b \in M$ and let $\mathfrak{k} \subset \mathfrak{g}$ be the corresponding Lie algebras. Then there is an Ad G invariant, symmetric, nondegenerate, bilinear form Q on $\mathfrak{g}$ which is nondegenerate on $\mathfrak{k}$ and positive definite on $\mathfrak{m} = \mathfrak{k}^\perp$. Furthermore $Q|\mathfrak{m}$ induces the Riemannian inner product on $T_bM \simeq \mathfrak{m}$.

The metric is called normal if we can choose G so that Q is positive definite on $\mathfrak{g}$. It is well known that a normal homogeneous metric has nonnegative sectional curvature. One of our aims is to find additional conditions on a normal homogeneous metric so as to guarantee that the curvature operator is nonnegative.

Note that assuming G is connected and K is closed, conditions (NR1) and (NR2) can be replaced by

(NR1)′ $\mathfrak{k}$ contains no proper nonzero ideals of $\mathfrak{g}$

(NR2)′ There is a symmetric, nondegenerate, bilinear form Q on $\mathfrak{g}$ which is nondegenerate on $\mathfrak{k}$ and positive definite on $\mathfrak{m} = \mathfrak{k}^\perp$ and for which each $\mathrm{ad}_{\mathfrak{g}}X$, $X \in \mathfrak{g}$, is skew symmetric. (Of course, if a Riemannian metric is given a priori on $M = G/K$, we would assume $Q|\mathfrak{m}$ induces this metric.) These conditions imply

(2.1.1) $\quad \mathfrak{g} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$

(where as usual $[\mathfrak{m}, \mathfrak{m}]$ is the linear span of bracket products from $\mathfrak{m}$) since otherwise the orthogonal complement of $\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ is an ideal in $\mathfrak{k}$. A pair $(\mathfrak{g}, \mathfrak{k})$ of Lie algebras $\mathfrak{g} \supset \mathfrak{k}$ satisfying (NR1)′ and (NR2)′ will be called a natural pair.

It is well known that the De Rham components of a simply connected naturally reductive space are again naturally reductive. In fact, the corresponding pair $(\mathfrak{g}, \mathfrak{k})$ can be decomposed as an orthogonal direct sum of natural pairs $(\mathfrak{g}_i, \mathfrak{k}_i)$ where each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$ and the form Q on $\mathfrak{g}$ is the sum of associated forms Q_i on $\mathfrak{g}_i$. Thus we will often restrict attention to pairs corresponding to irreducible spaces. Since the associated connection is the natural torsionless connection (canonical connection of the first find) results of Nomizu [N] (see also Sagle [S_1], Kostant [K_1, K_2]) show that the following condition is always necessary for irreducibility

(2.1.2) No proper nonzero subspace $\nu \subset \mathfrak{m}$ is invariant under the operators

$$X \longmapsto [Y, X]_{\mathfrak{m}} \quad \text{for all} \quad Y \in \mathfrak{g}$$

Actually, in case G is compact, Kostant shows (2.1.2) is equivalent to irreducibility.

We call a natural pair $(\mathfrak{g}, \mathfrak{k})$, with associated Q, weakly irreducible if (2.1.2) holds.

2.2 Let $(\mathfrak{g}, \mathfrak{k})$ be a natural pair with associated form Q, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. For $X \in \mathfrak{g}$, $X_{\mathfrak{k}}$, $X_{\mathfrak{m}}$ will denote $\mathfrak{k}$, $\mathfrak{m}$ components.

Definition. The pair $(\mathfrak{g}, \mathfrak{k})$ is called supernatural if there exists an associated Q so that

(2.2.3) $$J_{\mathfrak{k}}(X, Y, Z) \equiv [[X, Y]_{\mathfrak{k}}, Z] + [[Y, Z]_{\mathfrak{k}}, X] + [[Z, X]_{\mathfrak{k}}, Y]$$

vanishes identically for $X, Y, Z \in \mathfrak{m}$.

The pair $(\mathfrak{g}, \mathfrak{k})$ is called supernormal if it is both normal and supernatural for the same Q.

Note the condition $J_{\mathfrak{k}} \equiv 0$ on $\mathfrak{m}$ is equivalent to

(2.2.4) $\mathfrak{m}$ with the product $[\ ,\ ]_{\mathfrak{m}}$ is a Lie algebra.

From [KNII, p. 202], one finds that the curvature tensor for a naturally-reductive space is give by

(2.2.5) $$R(X, Y)Z = -[[X, Y]_{\mathfrak{k}}, Z] - \frac{1}{2}[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} + \frac{1}{4}[X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{4}[Y, [X, Z]_{\mathfrak{m}}]_{\mathfrak{m}}, X, Y, Z \in \mathfrak{m}.$$

Then it is clear from (2.2.4) that the curvature operator for a supernatural space is given by

(2.2.6) $$Q(\rho(X \wedge Y), W \wedge Z) = Q(R(X, Y)Z, W) = Q([X, Y]_{\mathfrak{k}}, [W, Z]_{\mathfrak{k}}) + Q([X, Y]_{\mathfrak{m}}, [W, Z]_{\mathfrak{m}}), X, Y, W, Z \in \mathfrak{m}$$

This immediately shows that

(2.2.7) the curvature operator for a supernormal space is nonnegative.

For a supernatural space (2.2.4) says $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is a Lie algebra. It is natural to ask whether the properties of this algebra are reflected in the geometry of M. As example 2 shows, the same Riemannian space can be represented in two different ways with the corresponding algebras $\mathfrak{m}$ of quite different types. Further, as we explain now, the type of the algebra $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is severely restricted.

In particular, Sagle [S_1] proves that

(2.2.8) for irreducible supernatural spaces, $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is either abelian or simple.

Actually, Sagle proves more in that he does not assume the existence of a Riemannian metric or bilinear form Q (he works directly with the natural torsionless connection; in our case, the Riemannian connection is of this type) nor does he assume $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is a Lie algebra. He does however assume the connection is non flat, i.e. has non-trivial holonomy. This however is not a restriction since an irreducible flat Riemannian space is 1-dimensional, i.e. dim $m = 1$, which means $\mathfrak{m}$ is abelian.

In our situation it is quite easy to prove a weaker result which suffices for our purposes.

Proposition 2.2. *Let $(\mathfrak{g}, \mathfrak{k})$ be a supernatural weakly irreducible pair. Then $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is either semisimple or abelian.*

Proof. The restriction $Q|\mathfrak{m}$ is a biinvariant positive definite inner product so $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is a compact algebra and $\mathfrak{m} = \mathfrak{z} \oplus \mathfrak{s}$ where $\mathfrak{z}$ is the center, $\mathfrak{s}$ is a semisimple ideal, and the decomposition is Q orthogonal. For $A \in \mathfrak{k}$, $X \in \mathfrak{z}$, $Y \in \mathfrak{m}$,

$$[[A, X], Y]_{\mathfrak{m}} = [A, [X, Y]_{\mathfrak{m}}] + [[A, Y], X]_{\mathfrak{m}} = 0$$

so $[\mathfrak{k}, \mathfrak{z}] \subset \mathfrak{z}$ and clearly $[\mathfrak{m}, \mathfrak{z}]_{\mathfrak{m}} = 0 \subset \mathfrak{z}$. By (2.1.2), either $\mathfrak{z} = 0$ or $\mathfrak{z} = \mathfrak{m}$ giving $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ semisimple or abelian, respectively.

Remark 2.2. If $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is abelian the map $\sigma(X) = X_{\mathfrak{k}} - X_{\mathfrak{m}}$ is an involutive automorphism with fixed point set $\mathfrak{k}$ and so $(\mathfrak{g}, \mathfrak{k}, \sigma)$ is an effective symmetric Lie algebra (KNII, p. 225) so any G-invariant Riemannian metric would be locally symmetric. The converse however is false, as is shown by example 2.3.

If further dim $\mathfrak{m} > 1$, $\mathfrak{g}$ is semisimple by [Prop. 7.5, KNII] hence $\mathfrak{k}$ is compactly embedded. We now check the last assertion. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and let K be the connected subgroup with Lie algebra $\mathfrak{k}$. If Z denotes the center of G let $G_1 = G/Z$ and $K_1 = K/K \cap Z$. We note that K_1 is closed in G_1 since it is the connected component of the fixed point set of an involutive automorphism. Now the homogeneous manifold G_1/K_1 admits a G_1-invariant Riemannian structure hence $\overline{\mathrm{Ad}(K_1)}$ is compact in $GL(\mathfrak{g})$. Being G_1 semisimple it follows that $\overline{\mathrm{Ad}(K_1)} = \mathrm{cl}\ \mathrm{AD}(K_1)$ where cl denotes the closure in $\mathrm{Int}(\mathfrak{g})$. Thus G_1 being centerless, K_1 is compact as asserted.

2.3 The following example illustrates the latitude available in choosing $\mathfrak{m}$ and is also important in our classification.

Let $\mathfrak{g}_i$, $i = 1, 2$, be isomorphic Lie algebras with brackets $[\ ,\ ]_i$ and inner products $\langle\ ,\ \rangle_i$ for which $\mathrm{ad}_{\mathfrak{g}_i}$ acts skewsymmetrically. Let $\phi: \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ be an isomorphism. Denote elements of $\mathfrak{g}_1$ by A, B, etc and of $\mathfrak{g}_2$ by X, Y, etc. Let ϕ^t be the transpose defined by $\langle \phi(A), Y\rangle_2 = \langle A, \phi^t(Y)\rangle_1$. On the vector space sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, define a product by

$$A \cdot B = [A, B]_1$$

$$A \cdot X = -X \cdot A = [\phi(A), X]_2$$

$$X \cdot Y = \phi^t[X, Y]_2 + [X, Y]_2$$

and consider the inner product $Q = \langle\ ,\ \rangle = \langle\ ,\ \rangle_1 + \langle\ ,\ \rangle_2$. It is easy to check that left multiplication in $\mathfrak{g}$ is Q-skewsymmetric and less easy, but routine, to check that $\mathfrak{g}$ is a Lie algebra with this product. With respect to Q, $\mathfrak{g}_2$ is the orthogonal complement to $\mathfrak{g}_1$.

Suppose $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{g}_1$. Then $\mathfrak{a}$ is an ideal of $\mathfrak{g}_1$ and from $X \cdot A = -[\phi(A), X]_2 = -\phi[A, \phi^{-1}(X)]_1$ for $X \in \mathfrak{g}_2, A \in \mathfrak{a}$, we see $[\mathfrak{a}, \mathfrak{g}_1]_1 = 0$, i.e. $\mathfrak{a} \subset$ center $(\mathfrak{g}_1)$. Thus $(\mathfrak{g}, \mathfrak{g}_1)$, satisfies $(\mathrm{NR1})'$ iff center $(\mathfrak{g}_1) = 0$. Since the existence of the form $\langle\ ,\ \rangle_i$ implies $\mathfrak{g}_i$ is compact, $(\mathrm{NR1})'$ is equivalent to $\mathfrak{g}_1$ semisimple. Assuming this, $(\mathfrak{g}, \mathfrak{g}_1)$ is clearly a natural pair which is weakly irreducible (2.1.2) iff $\mathfrak{g}_2$ (hence $\mathfrak{g}_1$) is simple. Further, $(\mathfrak{g}, \mathfrak{g}_1)$ with Q is clearly supernormal. Here $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}}) = (\mathfrak{g}_2, [\ ,\ ]_2)$.

Now assume $\mathfrak{g}_1$ simple. Then, up to multiple, $\langle\ ,\ \rangle_i$ is unique. Thus scaling $\langle\ ,\ \rangle_2$ by a constant multiple if necessary, we can assume ϕ preserves inner product and $\phi^t = \phi^{-1}$.

One can now verify that $\mathfrak{g}$ is isomorphic to the standard product algebra $\mathfrak{g}_1 \times \mathfrak{g}_2$ by $f: \mathfrak{g} \longrightarrow \mathfrak{g}_1 \times \mathfrak{g}_2$ defined by

$$f(A) = (A, \phi(A))$$

$$f(X) = \frac{1}{2}\left((1 - \sqrt{5})\phi^{-1}X,\ (1 + \sqrt{5})X\right)$$

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Let $\theta\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ be defined by

$$\theta(A+X) = (A+\phi^{-1}(X)) - X$$

One verifies θ is an involutive automorphism of $\mathfrak{g}$ with fixed point set $\mathfrak{g}_1$. It is easy to see that center $\mathfrak{g} = 0$ so $\mathfrak{g}$ is a compact semisimple Lie algebra and if G is any connected Lie group with Lie algebra $\mathfrak{g}$, then G is compact. By Mostow [M], if K is the connected subgroup of G corresponding to the simple algebra $\mathfrak{g}_1$, then K is closed hence $M = G/K$ is Riemannian locally symmetric ($\nabla R \equiv 0$) for any G invariant metric, see [He].

We continue with the assumption $\mathfrak{g}_i$ simple. The orthogonal complement of $\mathfrak{g}_1$ with respect to the Killing form Kill of $\mathfrak{g}$ will be the -1 eigenspace of θ which is $\mathfrak{p} = \{-\frac{1}{2}\phi^{-1}X + X\colon X \in \mathfrak{g}_2\}$. This gives the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k} = \mathfrak{g}_1$. Thus we have on the pair $(\mathfrak{g}, \mathfrak{g}_1)$, two forms Q and – Kill making the pair weakly irreducible supernormal and giving orthogonal complements $\mathfrak{g}_2$ and $\mathfrak{p}$ respectively. However, the Lie algebra structures induced on these complements are quite different ($\mathfrak{g}_2$ is simple and $\mathfrak{p}$ is abelian) even though, up to constant multiple, the Riemannian metrics induced on $M = G/K$ are the same locally symmetric metric. In fact, this is locally, just the biinvariant metric on the Lie group associated to $\mathfrak{g}_i$.

2.4 PROPOSITION 2.4. *Let $(\mathfrak{g}, \mathfrak{k})$ be a supernatural weakly irreducible pair with $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ semisimple. Then either $\mathfrak{k}$ is abelian or $\mathfrak{k}$ is semisimple. In the latter case, $(\mathfrak{g}, \mathfrak{k})$ is constructed as in example 2.3 and $\mathfrak{k} \simeq (\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is actually simple.*

Proof. We denote elements of $\mathfrak{k}$ by A, B, C, etc. and elements of $\mathfrak{m}$ by X, Y, or Z. For $A \in \mathfrak{k}$, ad A: $\mathfrak{m} \longrightarrow \mathfrak{m}$ is a derivation of $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$. Since $\mathfrak{m}$ is semisimple there is an element $\phi(A) \in \mathfrak{m}$ such that

$$(2.4.1)\quad [A, X] = [\phi(A), X]_{\mathfrak{m}}$$

Since $\mathfrak{m}$ is centerless, $\phi(A)$ is uniquely defined by (2.4.1). Now

$$\begin{aligned}[\phi[A,B], X]_{\mathfrak{m}} &= [[A,B],X] = [A,[B,X]] - [B,[A,X]] \\ &= [\phi(A), [\phi(B), X]_{\mathfrak{m}}]_{\mathfrak{m}} - [\phi(B), [\phi(A), X]_{\mathfrak{m}}]_{\mathfrak{m}} \\ &= [[\phi(A), \phi(B)]_{\mathfrak{m}}, X]_{\mathfrak{m}}\end{aligned}$$

so ϕ: $\mathfrak{k} \longrightarrow (\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is a Lie homomorphism. It is easy to see (by 2.4.1) Ker ϕ is an ideal of $\mathfrak{g}$ so by effectiveness, ϕ is injective. Let $\mathfrak{k}' \subset \mathfrak{m}$ be the image of ϕ and write $\mathfrak{m} = \mathfrak{k}' \oplus \mathfrak{p}$ (orthogonally). Note the relations

$$(2.4.2)\quad \begin{aligned} &[A, \phi(B)] = [\phi(A), \phi(B)]_{\mathfrak{m}} = \phi[A,B] \\ &Q([X,Y]_{\mathfrak{k}}, A) = Q(X, [Y, A]) = Q(X, [Y, \phi(A)]) = Q([X,Y]_{\mathfrak{m}}, \phi(A)) \end{aligned}$$

Using (2.4.1), (2.4.2), and invariance of Q, it is easy to prove successively the following

(2.4.3) $[\mathfrak{k}', \mathfrak{k}']_{\mathfrak{m}} \subset \mathfrak{k}'$

$[\mathfrak{k}, \mathfrak{k}'] \subset \mathfrak{k}'$

$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$

$[\mathfrak{k}', \mathfrak{p}]_{\mathfrak{k}} = 0$ and $[\mathfrak{k}', \mathfrak{p}]_{\mathfrak{m}} \subset \mathfrak{p}$

Now take $X \in \mathfrak{k}'$, Y, $Z \in \mathfrak{p}$. Then $0 = J_{\mathfrak{k}}(X, Y, Z) = [[Y, Z]_{\mathfrak{k}}, X]$. Thus $\phi[[Y, Z]_{\mathfrak{k}}, \phi^{-1}X] = [\phi[Y, Z]_{\mathfrak{k}}, X]_{\mathfrak{m}} = [[Y, Z]_{\mathfrak{k}}, X] = 0$. Since ϕ is injective, we get

(2.4.4) $[[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}}, \mathfrak{k}'] = 0$ and $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}} \subset \mathfrak{z}(\mathfrak{k})$

where $\mathfrak{z}(\mathfrak{k})$ is the center of $\mathfrak{k}$. In (2.4.2), assume $A \in \mathfrak{z}(\mathfrak{k})$ and let $X = \phi(B)$, $Y = \phi(C)$ where B, $C \in \mathfrak{k}$, giving

$$Q([X, Y]_{\mathfrak{k}}, A) = Q(\phi(B), [\phi(C), \phi(A)]_{\mathfrak{m}}) = Q(\phi(B), \phi[C, A]) = 0$$

Thus

(2.4.5) $Q(\mathfrak{z}(\mathfrak{k}), [\mathfrak{k}', \mathfrak{k}']) = 0$

Thus if $A \in \mathfrak{z}(\mathfrak{k})$ and $Q(A, [\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}}) = 0$, then

$$Q(A, [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{k}}) = Q(A, [\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}}) + Q(A, [\mathfrak{p}, \mathfrak{k}']_{\mathfrak{k}}) + Q(A, [\mathfrak{k}', \mathfrak{k}']_{\mathfrak{k}}) = 0$$

But then $\mathbb{R}A$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$ and by effectiveness, $A = 0$. Thus

(2.4.6) $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}} = \mathfrak{z}(\mathfrak{k})$

Since Q is positive definite on $\mathfrak{m}$, $\mathfrak{k}'$ is a compact algebra and so therefore is $\mathfrak{k}$. Thus we have $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus [\mathfrak{k}, \mathfrak{k}]$, $\mathfrak{k}' = \mathfrak{z}(\mathfrak{k}) \oplus [\mathfrak{k}', \mathfrak{k}']_{\mathfrak{m}}$, where both decompositions are Q-orthogonal and $\mathfrak{k}'$ is treated as a subalgebra of $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$. Clearly, ϕ maps $\mathfrak{z}(\mathfrak{k})$ onto $\mathfrak{z}(\mathfrak{k}')$ and $[\mathfrak{k}, \mathfrak{k}]$ onto $[\mathfrak{k}', \mathfrak{k}']_{\mathfrak{m}}$. Combining (2.4.6) and (2.4.2), we get

(2.4.7) $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{m}} \subset \mathfrak{p} \oplus \mathfrak{z}(\mathfrak{k}') \equiv \nu$

Now

$$[\mathfrak{p}, \nu]_{\mathfrak{m}} = [\mathfrak{p}, \mathfrak{p}]_{\mathfrak{m}} + [\mathfrak{p}, \mathfrak{z}(\mathfrak{k}')]_{\mathfrak{m}} \subset \nu + \mathfrak{p} = \nu$$

$$[\mathfrak{k}', \nu]_{\mathfrak{m}} = [\mathfrak{k}', \mathfrak{p}]_{\mathfrak{m}} \subset \mathfrak{p} \subset \nu$$

$$[\mathfrak{k}, \nu]_{\mathfrak{m}} = [\mathfrak{k}, \mathfrak{p}] + [\mathfrak{k}, \mathfrak{z}(\mathfrak{k}')]_{\mathfrak{m}} \subset \mathfrak{p} \subset \nu$$

where in the last line we use $Q([\mathfrak{k},\mathfrak{z}(\mathfrak{k}')],\mathfrak{k}') = Q(\mathfrak{k},[\mathfrak{z}(\mathfrak{k}'),\mathfrak{k}']) = 0$. By (2.1.2), $\nu = \mathfrak{m}$ or $\nu = 0$. In the first case, $\mathfrak{z}(\mathfrak{k}') = \mathfrak{k}'$ and hence $\mathfrak{k}'$ and $\mathfrak{k}$ are abelian. In the second, $\mathfrak{p} = \mathfrak{z}(\mathfrak{k}') = 0$ so in particular, $\mathfrak{k} \simeq \mathfrak{k}'$ are semisimple and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}'$ as a vector space. By (2.4.2)

$$Q([X,Y],A) = Q([X,Y]_{\mathfrak{m}},\phi(A)) = Q(\phi^t[X,Y]_{\mathfrak{m}},A)$$

so the product on $\mathfrak{g}$ agrees with that defined in 2.3 with $\mathfrak{g}_1 = \mathfrak{k}$, $\mathfrak{g}_2 = \mathfrak{k}'$.

2.5 We first present another example important for our classification

Example 2.5. Let $\mathfrak{g} = \operatorname{span}\{A,W,X,Y\}$, $\mathfrak{k} = \mathbb{R}A$ and define a skew-symmetric bilinear product and symmetric bilinear form Q so that

$$[A,W] = 0, [A,X] = \alpha Y, [A,Y] = -\alpha X$$

$$[W,X] = \beta Y, [W,Y] = -\beta X, [X,Y] = W + A.$$

(2.5.1) A, W, X, Y are orthogonal.

$$Q(X,X) = Q(Y,Y) = 1$$

$$Q(W,W) = r^2, Q(A,A) = \epsilon s^2, \epsilon = \pm 1, r,s > 0$$

One easily checks $(\mathfrak{g},\mathfrak{k})$ is supernatural with associated invariant form Q provided $\alpha = \epsilon s^2$ and $\beta = r^2$. From (2.2.6), one finds

$$\rho(X \wedge Y) = \left(\epsilon s^2 + \frac{1}{4}\, r^2\right) X \wedge Y$$

$$\rho\left(X \wedge \frac{W}{r}\right) = \frac{1}{4}\, r^2 X \wedge \frac{W}{r}$$

$$\rho\left(Y \wedge \frac{W}{r}\right) = \frac{1}{4}\, r^2 Y \wedge \frac{W}{r}$$

For $\epsilon = 1$, $(\mathfrak{g},\mathfrak{k})$ is supernormal and $\rho > 0$. However, for $\epsilon = -1$, $(\mathfrak{g},\mathfrak{k})$ is only supernatural and we have a family of naturally reductive metrics with ρ changing type from positive to mixed sign.

In all cases $[\mathfrak{g},\mathfrak{g}] = \operatorname{span}\{X, Y, W{+}A\}$ is isomorphic to $\mathfrak{su}(2)$ unless $r^2{+}\epsilon s^2 = 0$. Also for $r^2 + \epsilon s^2 \neq 0$, the center of $\mathfrak{g}$ is $\mathbb{R}(r^2 A - \epsilon s^2 W)$ and $\mathfrak{g}$ is isomorphic to $\mathfrak{u}(2)$. For $\epsilon = 1$, we get precisely the family of 3-dimensional Berger spheres ([B]) which are symmetric only for the standard sphere ($s = 0$). In all cases, $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}}) \simeq \mathfrak{su}(2)$.

PROPOSITION 2.5. *Let $(\mathfrak{g},\mathfrak{k})$ be a supernatural weakly irreducible pair with $\mathfrak{k}$ abelian and non zero. If $\mathfrak{m}$ is semisimple, $(\mathfrak{g},\mathfrak{k})$ is of the type of example 2.5. If*

$\mathfrak{m}$ *is abelian,* $(\mathfrak{g}, \mathfrak{k})$ *is an effective symmetric Lie algebra pair (cf. Remark 2.2) with* $\dim \mathfrak{g} = 3$, $\dim \mathfrak{m} = 2$.

Proof. First let $(\mathfrak{g}, \mathfrak{k})$ just be a natural weakly irreducible pair with $\mathfrak{k}$ abelian and nonzero. Let $Q = \langle \ , \ \rangle$ be an associated form. Set $\mathfrak{m}_0 = \{X \epsilon \mathfrak{m}\colon [\mathfrak{k}, X] = 0\}$ and $\mathfrak{p} = \mathfrak{m}_0^{\perp} \cap \mathfrak{m}$. Since $(\mathfrak{g}, \mathfrak{k})$ is effective and $\mathfrak{k} \neq 0$, we have $\mathfrak{p} \neq 0$ and clearly $\mathfrak{p}$ is ad $\mathfrak{k}$ invariant. Since $\{\operatorname{ad} A|\mathfrak{p}\colon A \in \mathfrak{k}\}$ is an abelian set of skew symmetric operators on $\mathfrak{p}$, we can decompose $\mathfrak{p}^{\mathbf{C}} = \oplus\Sigma\mathfrak{p}_{\lambda}^{\mathbf{C}}$, $\lambda \in \Lambda$, where each λ is a nontrivial (real) linear functional on $\mathfrak{k}$ and

(2.5.2) $\quad \mathfrak{p}_{\lambda}^{\mathbf{C}} = \{Z \in \mathfrak{p}^{\mathbf{C}}\colon [A, Z] = i\lambda(A)Z \text{ for } A \in \mathfrak{k}\}$

One checks easily that

$$\text{(2.5.3)} \quad \begin{aligned} &\overline{\mathfrak{p}_{\lambda}^{\mathbf{C}}} = \mathfrak{p}_{-\lambda}^{\mathbf{C}} \\ &[\mathfrak{p}_{\lambda}^{\mathbf{C}}, \mathfrak{p}_{\mu}^{\mathbf{C}}]_{\mathfrak{p}^{\mathbf{C}}} \subset \mathfrak{p}_{\lambda+u}^{\mathbf{C}} \end{aligned}$$

and that

$$\text{(2.5.4)} \quad \begin{aligned} &Z = Y + iX \in \mathfrak{p}_{\lambda}^{\mathbf{C}} \Rightarrow \\ &\langle X, Y\rangle = 0, \langle X, X\rangle = \langle Y, Y\rangle \text{ and} \\ &[A, X] = \lambda(A)Y, [A, Y] = \lambda(A)X \text{ for } A \in \mathfrak{k}. \end{aligned}$$

We can now find an orthonormal basis $\{X_i, Y_i\colon i = 1, 2, \ldots\}$ of $\mathfrak{p}$ and nontrivial linear functionals α_i on $\mathfrak{k}$ such that

(2.5.5) $\quad [A, X_i] = \alpha_i(A)Y_i, [A, Y_i] = -\alpha_i(A)X_i \quad \text{for} \quad A \in \mathfrak{k}.$

Of course, $\Lambda = \{\pm\alpha_i\colon i = 1, 2, \ldots\}$ but we are not assuming the α_i distinct. Let $\mathfrak{m}_i = \mathbb{R}X_i \oplus \mathbb{R}Y_i$. If $X \in \mathfrak{m}_0$, $Y \in \mathfrak{m}$ or $X \in \mathfrak{m}_i$, $Y \in \mathfrak{m}_j$ with $i \neq j$, skew-symmetry of ad X implies $[X, Y]_{\mathfrak{k}} = 0$ so

$$\text{(2.5.6)} \quad \begin{aligned} &[\mathfrak{m}_0, \mathfrak{m}]_t = 0 \\ &[m_i, m_j]_{\mathfrak{k}} = 0, \ i \neq j \end{aligned}$$

From (2.5.6) follows

$$\text{(2.5.7)} \quad \begin{aligned} &[\mathfrak{m}_0, \mathfrak{m}_0] \subset \mathfrak{m}_0 \\ &[\mathfrak{m}_0, \mathfrak{p}] = \subset \mathfrak{p} \end{aligned}$$

Then $J_{\mathfrak{k}}(X, Y, Z) = 0$ (cf. 2.2.3) for $X \in \mathfrak{m}_i$, $Y \in \mathfrak{m}_j$, $Z \in \mathfrak{m}_k$ provided either at least one of i, j, k is 0 or i, j, k are distinct. Thus $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ is a Lie algebra iff

(2.5.8) $\quad \alpha_j([X_i, Y_i]_{\mathfrak{k}}) = 0 \quad \text{for} \quad i \neq j.$

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Assume this from now on. Suppose $A \in \mathfrak{k}$ and $\alpha_i(A) = 0$ for all i. Then A is in the center of $\mathfrak{g}$ and, by effectiveness, $A = 0$. Again for $A \in \mathfrak{k}$, $\langle [X_i, Y_i], A\rangle = \alpha_i(A)$; since α_i is nontrivial we have $[X_i, Y_i]_{\mathfrak{k}} \neq 0$. With (2.5.8), this implies

$$\text{(2.5.9)} \quad \alpha_i([X_i, Y_i]_{\mathfrak{k}}) \neq 0$$

A consequence of (2.5.8) and (2.5.9) is that

$$\text{(2.5.10)} \quad \begin{aligned} \alpha_i &\neq \alpha_j \quad \text{for} \quad i \neq j \\ \alpha_i + \alpha_j &\neq \alpha_k \quad \text{for any} \quad i, j, k. \end{aligned}$$

In particular, each $\mathfrak{p}_\lambda^{\mathbf{C}}$ is one dimensional and $\mathfrak{m}_j^{\mathbf{C}} = \mathfrak{p}_{\alpha_j}^{\mathbf{C}} + \mathfrak{p}_{-\alpha_j}^{\mathbf{C}}$. From (2.5.7), $[\mathfrak{m}_0, \mathfrak{p}_\lambda^{\mathbf{C}}] \subset \mathfrak{p}_\lambda^{\mathbf{C}}$ so

$$\text{(2.5.11)} \quad [\mathfrak{m}_0, \mathfrak{m}_j] \subset \mathfrak{m}_j$$

Which implies with (2.5.6) that

$$\text{(2.5.12)} \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{p} \quad \text{for} \quad i \neq j,\ i,\ j \geqq 1$$

Now (2.5.3) implies $[\mathfrak{m}_i, \mathfrak{m}_j]_{\mathfrak{p}} \subset \mathfrak{m}_k \oplus \mathfrak{m}_l$ where $\alpha_k = \alpha_i + \alpha_j$ and $\pm\alpha_l = \alpha_i - \alpha_j$. Then (2.5.10) and (2.5.12) give

$$\text{(2.5.13)} \quad \begin{aligned} &[\mathfrak{m}_i, \mathfrak{m}_j] = 0 \quad \text{for} \quad i \neq j, i,\ j \geqq 1 \\ &[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{k} \oplus \mathfrak{m}_0 \end{aligned}$$

For $i \neq j$, (2.5.13) implies $0 = [\mathfrak{m}_j, [\mathfrak{m}_i, \mathfrak{m}_i]]$ and (2.5.8) implies $0 = [\mathfrak{m}_j, [\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{k}}]$ so

$$\text{(2.5.14)} \quad [\mathfrak{m}_j, [\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{m}_0}] = 0 \quad \text{for} \quad i \neq j, i,\ j \geqq 1.$$

Let $\mathfrak{n}_1 = \mathfrak{m}_1 \oplus [\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{m}_0}$. Note $[\mathfrak{m}_0, [\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{m}_0}] = [\mathfrak{m}_0, [\mathfrak{m}_1, \mathfrak{m}_1]]_{\mathfrak{m}_0} \subset [\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{m}_0}$. Combined with the above relations, we see $\mathfrak{n}_1$ is a subspace invariant under all operators $X \longmapsto [Y, X]_{\mathfrak{m}}$, $Y \in \mathfrak{g}$. Since we assume weak irreducibility, we have $\mathfrak{n}_1 = \mathfrak{m}$, i.e. $\mathfrak{p} = \mathfrak{m}_1$ and $\mathfrak{m}_0 = \mathbb{R}[X_1, Y_1]_{\mathfrak{m}_0}$. By (2.1.1), we have $\mathfrak{k} = \mathbb{R}[X_1, Y_1]_{\mathfrak{k}}$. If $\mathfrak{m}$ is semisimple, we must have $[X_1, Y_1]_{\mathfrak{m}_0} \neq 0$. Let $W = [X_1, Y_1]_{\mathfrak{m}_0}$, $A = [X_1, Y_1]_{\mathfrak{k}}$. By (2.5.11) and skew-symmetry of ad Z, we have $[W, X] = \beta Y$, $[W, Y] = \beta X$. This is example (2.5). On the other hand, $\mathfrak{m}$ abelian implies $\mathfrak{m}_0 = 0$ and dim $\mathfrak{g} = 3$.

2.6 The previous results now immediately give the following classification.

PROPOSITION 2.6. *Let* $(\mathfrak{g}, \mathfrak{k})$ *be a supernatural weakly irreducible pair with* $\dim \mathfrak{g} > 1$. *Then one of the following holds.*

(a) $\mathfrak{k} = 0$ *and* $\mathfrak{g}$ *is a simple Lie algebra of compact type.*

(b) $(\mathfrak{g}, \mathfrak{k})$ *is an irreducible effective orthogonal symmetric Lie algebra (cf. [He]) with* $\mathfrak{k} \neq 0$.

(c) $\mathfrak{k}$ *is one dimensional,* $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}}) \simeq \mathfrak{su}(2)$, *and* $(\mathfrak{g}, \mathfrak{k})$ *is of the type of example* 2.5.

Conversely, each of the above is a supernatural weakly irreducible pair.

Although we have now classified all the Lie algebra pairs $(\mathfrak{g}, \mathfrak{k})$, to classify the corresponding forms Q is a bit more delicate. In case (a), Q must be a multiple of the Killing form of $\mathfrak{g}$ and the corresponding Riemannian metric is locally symmetric and always supernormal. In case (b), there is more latitude in choosing Q and the orthocomplement $\mathfrak{m}$ (see Example 2.3) but again the corresponding Riemannian metric is always locally symmetric. In case (c), all possible Q are already described in example 2.5 by the proof of Prop. 2.5. Here we get a family of 3-dimensional Riemannian spaces in general not symmetric, which includes the Berger spheres.

Remark 2.6.1. As mentioned, normal metrics have sectional curvatures $\geqq 0$. The following example shows that there are however normal metrics where ρ has some negative eigenvalues; these are of course not supernormal. This example is the counterpart, in a sense, of Proposition 1.5.

Let $\mathfrak{g} = \mathfrak{so}(n)$ for $n \geqq 4$ and let $\mathfrak{k} = \mathfrak{so}(n-1)$ considered as the subalgebra of $\mathfrak{g}$ consisting of matrices with trivial last column. Let $\langle X, Y\rangle_0 = -\frac{1}{2}\operatorname{Tra} XY$, which is a negative multiple of the Killing form of $\mathfrak{g}$. Let $\mathfrak{p}$ be the orthocomplement of $\mathfrak{k}$. Finally, define $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by $\langle\ ,\ \rangle_0|_{\mathfrak{p}\times\mathfrak{p}} + \beta\langle\ ,\ \rangle_0|_{\mathfrak{k}\times\mathfrak{k}}$. By the results of [D'A-Z] (Sect. 4 and Th. 9), this defines a left-invariant metric on $G = SO(n)$ which is normal homogeneous (with respect to the group $G \times K$) for $0 < \beta \leqq 1$. Let $B_{ij} = -B_{ji} = E_{ij} - E_{ji} \in \mathfrak{g}$ where E_{ij} is the matrix with entry 1 in the i^{th} row, j^{th} column and 0 entries elsewhere. One finds for the curvature operator ρ that

$$\rho(B_{in} \wedge B_{in-1}) = \frac{\beta}{4} B_{in} \wedge B_{in-1} + \frac{1}{4} \sum_{\substack{r \neq i \\ r < n-1}} B_{rn} \wedge B_{rn-1}.$$

Then ρ has an eigenvector $(n-3)\, B_{1n} \wedge B_{1n-2} - \sum_{1<r<n-1} B_{rn} \wedge B_{rn-1}$, with eigenvalue $\frac{\beta-1}{4}$.

Remark 2.6.2. All supernormal metrics have $\rho \geqq 0$. So do all Riemannian symmetric spaces of compact type. Of course, the problem of classifying all Riemannian metrics (or the underlying manifolds) with $\rho > 0$ or $\rho \geqq 0$ is studied extensively (see Nishikawa, Chow-Yang, Moore, Hamilton, Mori, etc.). However, it is useful to note that even within the class of normal metrics there are examples with $\rho > 0$ which are neither symmetric nor supernormal. These are the Berger spheres of dimension $2n+1$ for $n > 1$ which can be represented as normal pairs (G, K) with $G = SU(n+1) \times \mathbb{R}$ (see 2.5 for the case $n = 1$). The necessary computations follow rather directly from the exposition given in [Ch].

2.7 In a series of papers, see [S_1] and [S_2] for example, A. Sagle considers what we call natural pairs (and various generalizations) and considers properties of the algebra $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ (without assuming it is a Lie algebra). In [S_2] Sagle assumes that there exist real constants b_1, b_3 and linear resp. bilinear, forms f, resp. f_0, on $\mathfrak{m}$ such that

$$\begin{aligned}(2.7.1)\quad \operatorname{ad}[X, Y]_{\mathfrak{k}}|\mathfrak{m} = {} & f_0(X, Y)I + f(Y)R(X) - f(X)R(Y) \\ & + b_1[R(X), R(Y)] + b_3 R([X, Y]_{\mathfrak{m}}) \quad \text{for} \quad X, Y \in \mathfrak{m}\end{aligned}$$

where $R(X)Z = [Z, X]_{\mathfrak{m}}$ (see [S_2, pp. 15–19]). A further condition he considers is that $f_0 \equiv f \equiv 0$ in (2.7.1).

Suppose $f_0 \equiv f \equiv 0$. Then for $X, Y \in \mathfrak{m}$, one computes

$$(2.7.2)\quad [[X, Y]_{\mathfrak{k}}, X] = (b_3 - b_1)[X, [X, Y]_{\mathfrak{m}}]_{\mathfrak{m}}.$$

In [Ch.], Chavel introduced operators B_x, T_x on $\mathfrak{m}$ by

$$(2.7.3)\quad T_x = -R(X), B_x(Y) = [[X, Y]_{\mathfrak{k}}, X]$$

and called the corresponding naturally reductive metric quasisymmetric if T_x and B_x commute. But (2.7.2) implies $B_x = (b_3 - b_1)R_X^2$. Further, [D'A] proves that quasi-symmetric (in this sense) implies locally symmetric. Thus we have proven the following.

PROPOSITION 2.7. *Suppose $M = G/K$ is naturally reductive with a natural pair $(\mathfrak{g}, \mathfrak{k})$ which satisfies (2.7.1) with $f_0 \equiv f \equiv 0$. Then M is locally symmetric.*

In [S_2], there is a Theorem (p. 18) which implies that, assuming (2.7.1) holds, a natural pair with $(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})$ a simple Lie, Malcev, or flat algebra must also satisfy $f_0 \equiv f \equiv 0$, and hence by Prop. 2.7 it is locally symmetric. We conclude that example 2.5 does not satisfy the conditions of Sagle's theorem. In fact a direct check shows that there does not exist any f_0, f, b_1, and b_3 so that (2.7.1) hold in example 2.5. In any event, (2.7.1) is perhaps too restrictive a condition, at least in the presence of additional hypothesis which force $f_0 \equiv f \equiv 0$.

Added in proof. The authors thank Prof. Thomas Wolter for pointing out (1) our Proposition 1.3 is in Azencott-Wilson, Trans. A.M.S., 215 (p. 357), (2) our example in section 1.4 with parameters $(1, \frac{1}{2}, \frac{1}{2},)$ is complex hyperbolic space, and (3) several misprints.

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