J. Austral. Math. Soc. **70** (2001), 235–272

LOCALLY FULL HNN EXTENSIONS OF INVERSE SEMIGROUPS

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(Received 1 May 1998; revised 25 October 2000)

Communicated by D. Easdown

Abstract

We investigate a locally full HNN extension of an inverse semigroup. A normal form theorem is obtained and applied to the word problem. We construct a tree and show that a maximal subgroup of a locally full HNN extension acts on the tree without inversion. Bass-Serre theory is employed to obtain a group presentation of the maximal subgroup as a fundamental group of a certain graph of groups associated with the $\mathscr{D}$-structure of the original semigroup.

2000 *Mathematics subject classification*: primary 20M18, 20M05, 20E08, 20E06.
Keywords and phrases: HNN extensions, Inverse semigroups, Normal form, Word problem, Bass-Serre theory.

1. Introduction and preliminaries

1.1. Background The study on the embeddability of HNN extensions of semigroups began in [7]. The HNN extension of a semigroup was formalized in terms of presentations, and several types of embeddings of semigroups into HNN extensions and the relationship with amalgamated free products were studied in [15, 16]. Various inverse semigroups have an HNN extension structure. For example, free inverse semigroups, free inverse monoids and free Clifford semigroups can be presented as an HNN extension of a semilattice. Furthermore, any Bruck-Reilly extension of an inverse monoid is an HNN extension of an appropriate inverse monoid, and hence, any bisimple regular ω-semigroup and any simple regular ω-semigroup can be presented as an HNN extension ([18]).

There exist considerable applications of HNN extensions to algorithmic and structural problems in inverse semigroup theory as well as in group theory. The undecidability of Markov properties and several other properties of inverse semigroups are

proved using HNN extensions and amalgamated free products ([15, 16]). Furthermore, HNN extensions are applied to give an alternative proof of Reilly's theorem that every inverse semigroup can be embedded into a bisimple inverse monoid ([15]).

In [17], HNN extensions are employed to investigate an idempotent pure image of a free inverse semigroup and to prove the existence of an F-inverse cover over a free group for an inverse semigroup. There exist numerous similarities between an HNN extension of a semilattice and a free group. Therefore, an HNN extension of a semilattice is considered as a natural generalization of a free group. For example, an HNN extension of a semilattice is the universal object in a certain category on a fixed set of generators as a free group is the universal object of the class of groups generated by a fixed set. Contracted Schützenberger graphs of an HNN extension of a semilattice with respect to a certain subset form a forest. Recall that the Cayley graph of a free group with respect to its base is a tree. The Nielsen-Schreier subgroup theorem in group theory can be generalized to the class of full HNN extensions of semilattices with an identity.

These results suggest more applications of HNN extensions to algorithmic and structural problems in inverse semigroup theory and motivate us to study the structure of HNN extensions. An HNN extension is called *locally full* if the associated inverse monoids are full in the local monoids defined by their identities. This condition makes the structure of HNN extensions transparent. In fact, locally full HNN extensions resemble HNN extensions of groups and have many nice properties. Therefore, we study locally full HNN extensions in this paper.

Let us outline the paper. In this section, we review several fundamental results in inverse semigroup theory. In Section 2, we obtain the normal form theorem for a locally full HNN extension. In Section 3, we apply it to solve the word problem. In Section 4, we build a tree on which a maximal subgroup of a locally full HNN extension acts. In Section 5, the Bass-Serre theory is employed to obtain a group presentation for the maximal subgroup. In Section 6, we give several examples.

For basic results in inverse semigroup theory we refer the reader to [9, 13]. We use basic results and terminologies from group theory without mention. The reader is referred to [3, 10, 14] for definitions and results on combinatorial group theory.

1.2. Presentations We briefly review inverse semigroup presentations and HNN extensions. A semigroup S is called *inverse* if there exists a unique element x^{-1} satisfying $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$ for every x in S. It is equivalent to say that S is a regular semigroup (in the sense of von Neumann) whose idempotents form a semilattice. The set of idempotents of a semigroup S is denoted by $E(S)$. The class of inverse semigroups has a free object, that is, there exists a free inverse semigroup on any non-empty set. Let X be a non-empty set. Then the free semigroup on X is just the set of all non-empty words on X and is denoted by X^+. Now we

take a copy X^{-1} of X such that X^{-1} is disjoint from X. We have a one to one correspondence $^{-1} : X \to X^{-1}$ ($x \to x^{-1}$ for all $x \in X$). We extend the mapping $^{-1}$ to the correspondence $^{-1} : (X \cup X^{-1}) \to (X \cup X^{-1})$ by defining $(x^{-1})^{-1} = x$ for all $x^{-1} \in X^{-1}$. If $w \in (X \cup X^{-1})^+$, then we can write $w = x_1x_2 \cdots x_n$, where $x_i \in X \cup X^{-1}$ for every $i = 1, 2, \ldots, n$. We denote the element $x_n^{-1}x_{n-1}^{-1} \cdots x_2^{-1}x_1^{-1}$ by w^{-1}. Note that the operation $^{-1}$ is an involution. Consider the relation R_0 consisting of all pairs of the form $(ww^{-1}w, w)$ and $(ww^{-1}zz^{-1}, zz^{-1}ww^{-1})$, where w and z are elements of $(X \cup X^{-1})^+$. The *Wagner congruence* θ is the congruence generated by R_0. Then the semigroup $(X \cup X^{-1})^+/\theta$ is the *free inverse semigroup* on X. Denote the free inverse semigroup on X by $\mathrm{FIS}(X)$.

Every inverse semigroup is a homomorphic image of a suitable free inverse semigroup; if S is an inverse semigroup, there exists a certain non-empty set X such that $\mathrm{FIS}(X)/\eta = S$ for some congruence η on $\mathrm{FIS}(X)$. If η is generated by some set R of relations on $(X \cup X^{-1})^+$, then we say that *S is presented by the set X of generators and the set R of relations*. Then we write $S = \mathrm{Inv}(X \mid R)$. If $|X|$ is finite, S is called *finitely generated*. If both $|X|$ and $|R|$ are finite, then S is called *finitely presented*.

Assume that an inverse semigroup S has a presentation $\mathrm{Inv}(X \mid R)$. Take a set Y disjoint from $X \cup X^{-1}$ and the set R' of relations on $(Y \cup Y^{-1})^+$. Then the inverse semigroup presented by $\mathrm{Inv}(X, Y \mid R, R')$ is denoted by $\mathrm{Inv}(S, Y \mid R')$ for brevity.

The inverse semigroup $\mathrm{Inv}(X \mid R)$ is regarded as the freest inverse semigroup generated by the set X subject to the relation R because of the following well-known result, which is equivalent to von Dyck's theorem in group theory.

PROPOSITION 1.1. *Suppose that an inverse semigroup S is presented by* $\mathrm{Inv}(X \mid R)$ *and ϕ is a homomorphism of* $\mathrm{FIS}(X)$ *into an inverse semigroup T. If $\phi(w_1) = \phi(w_2)$ in T for all $w_1, w_2 \in (X \cup X^{-1})^+$ with $w_1 = w_2$ in R, then there exists a homomorphism $\psi : S \to T$ such that $\phi(x) = \psi(\varphi(x))$ for every $x \in X$, where φ is the natural homomorphism of* $\mathrm{FIS}(X)$ *into S.*

1.3. HNN extensions and embeddability Let S be an inverse semigroup, and let A_i and B_i ($i \in I$) be inverse subsemigroups of S. Suppose that $e_i \in A_i \subset e_iSe_i$, $f_i \in B_i \subset f_iSf_i$ for some idempotents e_i, f_i of S and that $\phi_i : A_i \to B_i$ is an isomorphism for every $i \in I$. Then the inverse semigroup S^* presented by

$$\mathrm{Inv}(S, t_i\ (i \in I) \mid t_i^{-1}at_i = \phi_i(a) \text{ for every } a \in A_i,\ t_i^{-1}t_i = f_i,\ t_it_i^{-1} = e_i \text{ for every } i \in I), \tag{1.1}$$

or equivalently,

$$\mathrm{Inv}(S, t_i\ (i \in I) \mid t_i^{-1}at_i = \phi_i(a) \text{ for every } a \in A_i',\ t_i^{-1}t_i = f_i,\ t_it_i^{-1} = e_i \text{ for every } i \in I), \tag{1.2}$$

https://doi.org/10.1017/S1446788700002639 Published online by Cambridge University Press

where A'_i is a set of generators of A_i, is called the HNN *extension of S associated with* $\phi_i : A_i \to B_i$ $(i \in I)$. Each element t_i in S^* is called a *stable letter*. In general, a class $\mathbf{C}$ of semigroups is said to have the *weak* HNN *property* if $\mathbf{C}$ satisfies the following condition.

Suppose that $S, A, B \in \mathbf{C}$, $e \in A \subset eSe$, $f \in B \subset fSf$ for some $e, f \in E(S)$. Let $\phi : A \to B$ be an isomorphism. Then there exist $T \in \mathbf{C}$ and an embedding $\psi : S \hookrightarrow T$ such that

$$t'\psi(a)t = \psi(\phi(a)) \quad \text{for all } a \in A, \tag{1.3}$$

and

$$t't = \psi(f) \quad \text{and} \quad tt' = \psi(e) \quad \text{for some } t \in T \text{ and its inverse } t'. \tag{1.4}$$

A class $\mathbf{C}$ of semigroups is said to have the *strong* HNN *property* if $\mathbf{C}$ satisfies the following condition.

Suppose that $S, A, B \in \mathbf{C}$ and $A \subset eSe$, $B \subset fSf$ for some $e, f \in E(S)$. Let $\phi : A \to B$ be an isomorphism. Then there exist $T \in \mathbf{C}$ and an embedding $\psi : S \hookrightarrow T$ satisfying the conditions (1.3), (1.4) and

$$t'\psi(S)t \cap \psi(S) = t'\psi(A)t = \psi(B) \tag{1.5}$$

PROPOSITION 1.2 ([15, 16]). *The class of inverse semigroups has the strong* HNN *property.*

By Proposition 1.2, an inverse semigroup S is always embedded into S^* presented by (1.1). We usually identify S with the corresponding inverse subsemigroup of the HNN extension under the natural isomorphism, and hence, (1.5) implies

$$t_i^{-1}St_i \cap S = t_i^{-1}A_it_i = B_i \quad \text{for every } i \in I \tag{1.6}$$

in the HNN extension.

We note that the class of inverse semigroups also has the strong amalgamation property ([5, 6]). The weak and strong HNN embeddability of several classes of semigroups and the relationship with the amalgamation property are investigated in [15, 16].

1.4. Locally full HNN extensions If A_i is full in e_iSe_i and B_i is full in f_iSf_i, that is, $E(A_i) = E(e_iSe_i)$ and $E(B_i) = E(f_iSf_i)$ for every $i \in I$, then the HNN extension S^* is called *locally full*. If M is an inverse monoid and A_i and B_i are full submonoids, that is, $E(A_i) = E(M)$ and $E(B_i) = E(M)$, then the HNN extension M^* is called *full*. Full HNN extensions of inverse monoids and locally full HNN extensions of semilattices are investigated in [15] and [17], respectively.

For example, the free Clifford semigroup on the set $\{t_1, t_2, \ldots, t_n\}$ is presented by

$$\begin{aligned}\mathrm{Inv}(E, t_1, t_2, \ldots, t_n \mid\ & t_i^{-1} f t_i = \phi_i(f) = f \text{ for every } f \in Ee_i, \\ & t_i^{-1} t_i = t_i t_i^{-1} = e_i \text{ for every } i = 1, 2, \ldots n),\end{aligned}$$

where E is the free semilattice on the set $X = \{e_1, e_2, \ldots, e_n\}$ and ϕ_i is the identity mapping of Ee_i for every $i = 1, 2, \ldots, n$. Hence, any free Clifford semigroup is a locally full HNN extension of a free semilattice.

The formation of a locally full HNN extension yields no new idempotents as we see in Corollary 3.1. Hence, the partial order structure of a locally full HNN extension is not so complicated as a more general HNN extension. We can generalize Britton's lemma to the class of locally full HNN extensions. The following is obtained in [17].

PROPOSITION 1.3. *Any* HNN *extension of a semilattice E with stable letters t_i $(i \in I)$ can be presented as a locally full* HNN *extension of a semilattice E_0 with $|I|$ stable letters, where E_0 is an extension of E.*

A similar result holds for HNN extensions of inverse semigroups.

PROPOSITION 1.4. *Any* HNN *extension of an inverse semigroup S with stable letters t_i $(i \in I)$ can be presented as a locally full* HNN *extension of T_0 with $|I|$ stable letters, where T_0 is an inverse semigroup extension of S.*

PROOF. Let S be an inverse semigroup. Suppose that A_i and B_i are inverse subsemigroups of S such that $e_i \in A_i \subset e_i S e_i$ and $f_i \in B_i \subset f_i S f_i$ for some $e_i, f_i \in E(S)$ for every $i \in I$. Let ϕ_i be an isomorphism of A_i onto B_i for every $i \in I$. We show that the HNN extension S^*, presented by (1.1), is a locally full HNN extension of a certain inverse semigroup with the same cardinal number of stable letters. Let T_0 be an inverse subsemigroup of S^* generated by S and $E(S^*)$. Let C_i be the inverse subsemigroup of S^* generated by A_i and $E(S^*)e_i$ for each $i \in I$. Let D_i be the inverse subsemigroup of S^* generated by B_i and $E(S^*)f_i$ for each $i \in I$. We note that C_i and D_i are inverse subsemigroups of S^* such that $e_i \in C_i \subset e_i S^* e_i$ and $f_i \in D_i \subset f_i S^* f_i$ for each $i \in I$. We also note that $E(C_i) = E(e_i S^* e_i)$ and $E(D_i) = E(f_i S^* f_i)$ for each $i \in I$. We define a mapping ϕ_i^* of C_i into D_i by $\phi_i^*(s) = t_i^{-1} s t_i$ for $s \in C_i$. Since $t_i^{-1} A_i t_i \subset B_i$ and $t_i^{-1} E(S^*) e_i t_i \subset E(S^*) f_i$ for each $i \in I$, we have $t_i^{-1} C_i t_i \subset D_i$ for each $i \in I$. Hence, ϕ_i^* is well defined. Similarly, we can show that $t_i D_i t_i^{-1} \subset C_i$ for each $i \in I$. It is clear that ϕ_i^* is an isomorphism of C_i onto D_i for each $i \in I$ and that the restriction $\phi_i^*|_{A_i}$ of ϕ_i^* to A_i is equal to ϕ_i. Let T be the inverse semigroup presented by

$$\begin{aligned}\mathrm{Inv}(T_0, p_i\ (i \in I) \mid\ & p_i^{-1} c p_i = \phi_i^*(c) \text{ for every } c \in C_i, \\ & p_i^{-1} p_i = f_i,\ p_i p_i^{-1} = e_i \text{ for every } i \in I).\end{aligned}$$

Then T is a locally full HNN extension of T_0. We show that T is isomorphic to S^*. We define a mapping Φ of $S \cup \{t_i \mid i \in I\}$ into T by $\Phi(s) = s$ for $s \in S$ and $\Phi(t_i) = p_i$ for each $i \in I$. Since the defining relations of S^* are satisfied by T under Φ, the mapping Φ can be extended to a homomorphism by Proposition 1.1. We next define a mapping Ψ of $T_0 \cup \{p_i \mid i \in I\}$ into S^* by $\Psi(s) = s$ for $s \in S$, $\Psi(e) = e$ for $e \in E(S^*)$ and $\Psi(p_i) = t_i$ for each $i \in I$. It is easy to check that the defining relations are satisfied by S^* under Ψ. Hence, Ψ can be extended to a homomorphism by Proposition 1.1. Apparently $\Psi \circ \Phi = id_{S^*}$ and so Φ is injective. We next show that Φ is surjective. To prove this we show that every e in $E(S^*)$ can be written as a product of p_i's and elements of S in T. Take any $e \in E(S^*)$. Since e is in S^*, the element e can be written as a product of several stable letters t_i's and their inverses and elements of S. Since $t_i t_i^{-1}, t_i^{-1} t_i \in E(S)$, we may assume that e is written as

$$e = s_0 t_{i_1}^{\epsilon_1} s_1 t_{i_2}^{\epsilon_2} s_2 \cdots s_{n-1} t_{i_n}^{\epsilon_n} s_n,$$

where $s_i \in S$ for $i = 0, 1, 2, \ldots, n$ and $\epsilon_i = \pm 1$ for $i = 1, 2, \ldots, n$. Note that

$$\begin{aligned} e &= e^2 = e^{-1} e \\ &= s_n^{-1} t_{i_n}^{-\epsilon_n} s_{n-1}^{-1} t_{i_{n-1}}^{-\epsilon_{n-1}} s_{n-2}^{-1} \cdots s_1^{-1} t_{i_1}^{-\epsilon_1} s_0^{-1} s_0 t_{i_1}^{\epsilon_1} s_1 t_{i_2}^{\epsilon_2} s_2 \cdots s_{n-1} t_{i_n}^{\epsilon_n} s_n. \end{aligned}$$

Therefore, every element of e in $E(S^*)$ can be written as

$$s_n^{-1} t_{i_n}^{-\epsilon_n} s_{n-1}^{-1} t_{i_{n-1}}^{-\epsilon_{n-1}} s_{n-2}^{-1} \cdots s_1^{-1} t_{i_1}^{-\epsilon_1} s_0^{-1} s_0 t_{i_1}^{\epsilon_1} s_1 t_{i_2}^{\epsilon_2} s_2 \cdots s_{n-1} t_{i_n}^{\epsilon_n} s_n,$$

where $s_i \in S$ for $i = 0, 1, 2, \ldots, n$ and $\epsilon_i = \pm 1$ for $i = 1, 2, \ldots, n$. We claim that if e can be written in S^* as above then we have

$$e = s_n^{-1} p_{i_n}^{-\epsilon_n} s_{n-1}^{-1} p_{i_{n-1}}^{-\epsilon_{n-1}} s_{n-2}^{-1} \cdots s_1^{-1} p_{i_1}^{-\epsilon_1} s_0^{-1} s_0 p_{i_1}^{\epsilon_1} s_1 p_{i_2}^{\epsilon_2} s_2 \cdots s_{n-1} p_{i_n}^{\epsilon_n} s_n$$

in T by induction on n. If $n = 0$, then the claim is trivial. Suppose that the claim is true for non-negative integers less that n ($n \geq 1$). Assume that

$$e = s_n^{-1} t_{i_n}^{-\epsilon_n} s_{n-1}^{-1} t_{i_{n-1}}^{-\epsilon_{n-1}} s_{n-2}^{-1} \cdots s_1^{-1} t_{i_1}^{-\epsilon_1} s_0^{-1} s_0 t_{i_1}^{\epsilon_1} s_1 t_{i_2}^{\epsilon_2} s_2 \cdots s_{n-1} t_{i_n}^{\epsilon_n} s_n$$

in S^*. Let

$$f = s_{n-1}^{-1} t_{i_{n-1}}^{-\epsilon_{n-1}} s_{n-2}^{-1} \cdots s_1^{-1} t_{i_1}^{-\epsilon_1} s_0^{-1} s_0 t_{i_1}^{\epsilon_1} s_1 t_{i_2}^{\epsilon_2} s_2 \cdots t_{i_{n-1}}^{\epsilon_{n-1}} s_{n-1}.$$

By the inductive hypothesis,

$$f = s_{n-1}^{-1} p_{i_{n-1}}^{-\epsilon_{n-1}} s_{n-2}^{-1} \cdots s_1^{-1} p_{i_1}^{-\epsilon_1} s_0^{-1} s_0 p_{i_1}^{\epsilon_1} s_1 p_{i_2}^{\epsilon_2} s_2 \cdots p_{i_{n-1}}^{\epsilon_{n-1}} s_{n-1}$$

in T. Then we have

$$e = s_n^{-1} t_{i_n}^{-\epsilon_n} f t_{i_n}^{\epsilon_n} s_n = s_n^{-1} t_{i_n}^{-\epsilon_n} f t_{i_n}^{\epsilon_n} t_{i_n}^{-\epsilon_n} t_{i_n}^{\epsilon_n} s_n.$$

Note that $f\, t_{i_n}^{\epsilon_n} t_{i_n}^{-\epsilon_n} \in E(S^*) t_{i_n}^{\epsilon_n} t_{i_n}^{-\epsilon_n}$. Then we have

$$\begin{aligned} t_{i_n}^{-\epsilon_n}(f\, t_{i_n}^{\epsilon_n} t_{i_n}^{-\epsilon_n}) t_{i_n}^{\epsilon_n} &= (\phi_i^*)^{\epsilon_n}(f\, t_{i_n}^{\epsilon_n} t_{i_n}^{-\epsilon_n}) && \text{(by the definition of } \phi_i^*) \\ &= p_{i_n}^{-\epsilon_n}(f\, t_{i_n}^{\epsilon_n} t_{i_n}^{-\epsilon_n}) p_{i_n}^{\epsilon_n} && \text{(by the relation } p_i^{-1} c p_i = \phi_i^*(c) \\ & && \text{for } c \in C_i \in T) \\ &= p_{i_n}^{-\epsilon_n} f p_{i_n}^{\epsilon_n} p_{i_n}^{-\epsilon_n} p_{i_n}^{\epsilon_n} && \text{(as } t_{i_n} t_{i_n}^{-1} = e_{i_n} = p_{i_n} p_{i_n}^{-1} \text{ and} \\ & && t_{i_n}^{-1} t_{i_n} = f_{i_n} = p_{i_n}^{-1} p_{i_n} \in T) \\ &= p_{i_n}^{-\epsilon_n} f p_{i_n}^{\epsilon_n}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} e &= s_n^{-1} t_{i_n}^{-\epsilon_n} f\, t_{i_n}^{\epsilon_n} t_{i_n}^{-\epsilon_n} t_{i_n}^{\epsilon_n} s_n = s_n^{-1} p_{i_n}^{-\epsilon_n} f p_{i_n}^{\epsilon_n} s_n \\ &= s_n^{-1} p_{i_n}^{-\epsilon_n} s_{n-1}^{-1} p_{i_{n-1}}^{-\epsilon_{n-1}} s_{n-2}^{-1} \cdots s_1^{-1} p_{i_1}^{-\epsilon_1} s_0^{-1} s_0 p_{i_1}^{\epsilon_1} s_1 p_{i_2}^{\epsilon_2} s_2 \cdots p_{i_{n-1}}^{\epsilon_{n-1}} s_{n-1} p_{i_n}^{\epsilon_n} s_n. \end{aligned}$$

Hence, the claim is true for every non-negative integer. It follows that every $e \in E(S^*)$ can be written as a product of p_i's and elements of S. Since T is generated by S, $E(S^*)$ and p_i's, the homomorphism Φ is surjective. Consequently Φ is an isomorphism of S^* onto T. □

2. Normal form theorem

Any element of an HNN extension of a group has a unique expression called a *normal form* (or *canonical form*). The method using such an expression is useful to study algorithmic and structural problems. In this section we obtain a normal form of an element of a locally full HNN extension of an inverse semigroup.

2.1. Normal form Let S be an inverse semigroup, and let H be an inverse subsemigroup of S. We define a relation $\sim_H$ on S.

For $x, y \in S$, we define

$$x \sim_H y \tag{2.1}$$

if one of the followings holds:

(E1) $x = y$.

(E2) $x^{-1}x, y^{-1}y \in E(H)$ and there exists $h \in H$ such that $xh = y$ with $x^{-1}x = hh^{-1}$.

PROPOSITION 2.1. *The relation $\sim_H$ is an equivalence relation on S.*

PROOF. Clearly, $x \sim_H x$ for every $x \in S$. Suppose that $x \sim_H y$. Then either (E1) or (E2) holds. In the case of (E1), $x = y$ and so clearly $y \sim_H x$. We assume (E2).

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Then, we have $x^{-1}x, y^{-1}y \in E(H)$ and $xh = y$ and $x^{-1}x = hh^{-1}$ for some $h \in H$. Then, $yh^{-1} = xhh^{-1} = xx^{-1}x = x$. We note that $h^{-1} \in H$. It is easy to see that $y^{-1}y = h^{-1}(h^{-1})^{-1}$. Hence, $y \sim_H x$.

Suppose now that $x \sim_H y$, $y \sim_H z$. We consider only the case that $x^{-1}x$, $y^{-1}y$, $z^{-1}z \in E(H)$ and $xh = y$, $yk = z$, $x^{-1}x = hh^{-1}$ and $y^{-1}y = kk^{-1}$ for some h, k in H. Then $z = x(hk)$ and $hk \in H$. It is easy to see that $x^{-1}x = (hk)(hk)^{-1}$. Hence, $x \sim_H z$. Consequently, we have shown that $\sim_H$ is an equivalence. □

The $\sim_H$ class of S containing an element x is denoted by $\overline{x}$. We denote the equivalence class of S containing x for Green's relation $\mathscr{H}$, $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{D}$ by $H_S(x)$, $R_S(x)$, $L_S(x)$ and $D_S(x)$, respectively.

LEMMA 2.2. *For an element h of H we have $\overline{h} = R_S(h) \cap H = R_H(h)$.*

PROOF. Suppose $x \in \overline{h}$. Then $x \sim_H h$, and hence, $x = h$ or $hk = x$ and $h^{-1}h = kk^{-1}$ for some $k \in H$. Since $h, k \in H$, we have $x = hk \in H$. Then $x\mathscr{R}h$ in S, since $x = hk$ and $h = xk^{-1}$. Hence, $x \in R_S(h) \cap H$. Take x from $R_S(h) \cap H$. Then $x\mathscr{R}h$ in S and $x \in H$. Hence, $xx^{-1} = hh^{-1}$ and so $x \in R_H(h)$. Take x from $R_H(h)$. Then, $x \in H$ and $x\mathscr{R}h$ in H and so $xx^{-1} = hh^{-1}$. Then, we have $h(h^{-1}x) = xx^{-1}x = x$. We note that $h^{-1}x \in H$ (as $h, x \in H$). It is easy to see that $(h^{-1}x)(h^{-1}x)^{-1} = h^{-1}h$. Hence, $x \in \overline{h}$. □

LEMMA 2.3. *For $x, y \in S$, $x \sim_H y$ if and only if*

(1) *$x = y$ and $x^{-1}x \notin E(H)$ or*
(2) *$xx^{-1} = yy^{-1}$, $x^{-1}y \in H$ and $x^{-1}x$, $y^{-1}y \in E(H)$.*

PROOF. 'Only if' part: Suppose $x \sim_H y$. If (E1) holds, then we have (1) or (2) according to whether or not $x^{-1}x$ is in $E(H)$. We now suppose that (E2) holds. Then $x^{-1}x, y^{-1}y \in E(H)$. There exists $h \in H$ such that $xh = y$ and $x^{-1}x = hh^{-1}$. Then we have $x^{-1}y = x^{-1}xh \in E(H)H \subset H$. It is easy to see that $yy^{-1} = xx^{-1}$. Hence, (2) holds.

'If' part: Suppose (2) is true. Then $x^{-1}x, y^{-1}y \in E(H)$. Set $x^{-1}y = h \in H$. Then we have $xh = xx^{-1}y = yy^{-1}y = y$. It is easy to see that $x^{-1}x = hh^{-1}$. Hence, $x \sim_H y$. If (1) holds, clearly $x \sim_H y$. □

PROPOSITION 2.4. *If $H = E(S)$, then $\sim_H$ is the identity relation on S.*

PROOF. Suppose $x \sim_H y$ and the condition (E2) holds. Then there exists $h \in E(S)$ such that $xh = y$ and $x^{-1}x = hh^{-1}$. Since $h \in E(S)$, we have $x^{-1}x = hh^{-1} = h$. Then $y = xh = xx^{-1}x = x$. Hence, $x = y$ and so $\sim_H$ is the identity on S. □

PROPOSITION 2.5. *If $H = S$, then $\sim_H$ is identical to Green's $\mathscr{R}$-relation.*

PROOF. Obviously $\sim_H$ is included in $\mathscr{R}$. Suppose $x\mathscr{R}y$. Then, $xx^{-1} = yy^{-1}$ and so $x = xx^{-1}x = yy^{-1}x$. It is easy to see that $y^{-1}y = (y^{-1}x)(y^{-1}x)^{-1}$. Note that $y^{-1}x \in S = H$. Hence, (E2) holds, and so, $x \sim_H y$. It follows that $\sim_H$ is identical with $\mathscr{R}$. □

We begin to seek a unique expression of an element of a locally full HNN extension. Let S be an inverse semigroup, A_i and B_i $(i \in I)$ inverse subsemigroups of S. Suppose that $e_i \in A_i \subset e_iSe_i$, $f_i \in B_i \subset f_iSf_i$ for some idempotents e_i, f_i of S and that $\phi_i : A_i \to B_i$ is an isomorphism for every $i \in I$. We also assume that $E(A_i) = E(e_iSe_i)$ and $E(B_i) = E(f_iSf_i)$ for every $i \in I$. Let S^* be the inverse semigroup presented by

$$\text{(2.2)} \qquad \begin{aligned} \mathrm{Inv}(S, t_i\ (i \in I) \mid\ & t_i^{-1}at_i = \phi_i(a) \text{ for every } a \in A_i, \\ & t_i^{-1}t_i = f_i,\ t_it_i^{-1} = e_i \text{ for every } i \in I). \end{aligned}$$

Note that S^* is the locally full HNN extension of S associated with the isomorphisms ϕ_i $(i \in I)$. In the rest of the paper we assume that S, A_i, B_i and S^* are defined as above.

We choose and fix a set $\mathrm{C}(\sim_{A_i})$ of $\sim_{A_i}$ representatives of S for each $i \in I$. Similarly, we choose and fix a set $\mathrm{C}(\sim_{B_i})$ of $\sim_{B_j}$ representatives of S for each $j \in I$. If $r \in \mathrm{C}(\sim_{A_i})$ is in the $\sim_{A_i}$ class $R_S(e) \cap A_i$ (this is a $\sim_{A_i}$ class by Lemma 2.2) for some idempotent $e \in E(A_i)$, then we assume $r = e$. Similarly, if $s \in \mathrm{C}(\sim_{B_i})$ is in the $\sim_{B_j}$ class $R_S(f) \cap B_j$ for some idempotent $f \in E(B_j)$, then we assume $s = f$.

Let X be a sequence

$$(x_0, t_{i_1}^{\epsilon_1}, x_1, t_{i_2}^{\epsilon_2}, \ldots, t_{i_n}^{\epsilon_n}, x_n),$$

where $n \geq 0$, $x_k \in S$ for every $k = 0, 1, 2, \ldots, n$, t_{i_l} is a stable letter and $\epsilon_l = \pm 1$ for every $l = 1, 2, \ldots, n$. We consider the following conditions on the sequence X:

(C1) If $\epsilon_k = 1$, then $x_{k-1}^{-1}x_{k-1} \in E(A_{i_k})$. If $\epsilon_k = -1$, then $x_{k-1}^{-1}x_{k-1} \in E(B_{i_k})$, and $\phi_{i_k}^{\epsilon_k}(x_{k-1}^{-1}x_{k-1}) = x_kx_k^{-1}$ for every $k = 1, 2, \ldots, n$.

(C2) If $\epsilon_k = 1$, then x_{k-1} belongs to $\mathrm{C}(\sim_{A_{i_k}})$, and if $\epsilon_k = -1$, then x_{k-1} belongs to $\mathrm{C}(\sim_{B_{i_k}})$ for every $k = 1, 2, \cdots n$.

(C3) There exists no subsequence of the form (t_i^{-1}, e, t_i) $(e \in E(A_i))$ nor (t_i, e, t_i^{-1}) $(e \in E(B_i))$.

(C4) There exists no subsequence of the form (t_i^{-1}, a, t_i) $(a \in A_i)$ nor (t_i, b, t_i^{-1}) $(b \in B_i)$.

A sequence satisfying (C1), (C2) and (C3) ((C1) and (C4)) is called *normal* (*reduced*). Note that a normal sequence is always reduced. A *pinch* is a word of the form $t_i^{-1}at_i$

$(a \in A_i)$ or $t_i b t_i^{-1}$ $(b \in B_i)$, where t_i is a stable letter. Let x be an element of S^*. Suppose that x is written as

$$x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n. \tag{2.3}$$

If the sequence

$$(x_0, t_{i_1}^{\epsilon_1}, x_1, t_{i_2}^{\epsilon_2}, \cdots, t_{i_n}^{\epsilon_n}, x_n) \tag{2.4}$$

satisfies the condition (C1), then we say that x has a *form* (2.3). Moreover, if the sequence (2.4) is normal (reduced), then we say that x has a *normal form* (*reduced form*) (2.3). Note that a form is reduced if and only if it contains no pinch. We remark that a form in an inverse semigroup corresponds to the trace product in the corresponding inductive groupoid. The inductive groupoid approach is employed in [4].

LEMMA 2.6. *If an element x in S^* has a form $x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n$ then the following hold*:

(1) $xx^{-1} = x_0 x_0^{-1}$.
(2) $x^{-1}x = x_n^{-1} x_n$.

PROOF. We prove (1) using induction on n. If $n = 0$, then $x = x_0$ and the claim is trivially true. Next suppose that the claim is true for every non-negative integer less than n and that x has the form $x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n$. By the inductive hypothesis, we have

$$(x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_{n-1}}^{\epsilon_{n-1}} x_{n-1})(x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_{n-1}}^{\epsilon_{n-1}} x_{n-1})^{-1} = x_0 x_0^{-1}.$$

Therefore

$$\begin{aligned}
xx^{-1} &= (x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n)(x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n)^{-1} \\
&= (x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_{n-1}}^{\epsilon_{n-1}} x_{n-1}) t_{i_n}^{\epsilon_n} x_n x_n^{-1} t_{i_n}^{-\epsilon_n} (x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_{n-1}}^{\epsilon_{n-1}} x_{n-1})^{-1} \\
&= (x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_{n-1}}^{\epsilon_{n-1}} x_{n-1}) x_{n-1}^{-1} x_{n-1} (x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_{n-1}}^{\epsilon_{n-1}} x_{n-1})^{-1} \\
&\quad \text{(by (C1) and the relations } t_i^{-1} a t_i = \phi_i(a) \text{ for all } a \in A_i \\
&\quad \text{and } t_i b t_i^{-1} = \phi_i^{-1}(b) \text{ for all } b \in B_i) \\
&= (x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_{n-1}}^{\epsilon_{n-1}} x_{n-1})(x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_{n-1}}^{\epsilon_{n-1}} x_{n-1})^{-1} = x_0 x_0^{-1}.
\end{aligned}$$

This completes the induction. Similarly we can prove (2). □

2.2. Normal form theorem In this section we prove the normal form theorem for a locally full HNN extension.

THEOREM 2.7. *Let S be an inverse semigroup, and let A_i and B_i be inverse subsemigroups such that $e_i \in A_i \subset e_i S e_i$, $E(A_i) = E(e_i S e_i)$, $f_i \in B_i \subset f_i S f_i$ and*

$E(B_i) = E(f_i S f_i)$ for every $i \in I$. Let $\phi_i : A_i \to B_i$ be an isomorphism of A_i onto B_i for every $i \in I$. Let S^* be the locally full HNN *extension presented by* (2.2). *Every element of S^* has a unique normal form* (2.3).

The proof is divided into four lemmas. Lemma 2.8 shows that every element has a form and also provides an algorithm that finds a form. Lemma 2.9 shows that we can construct a reduced form from a form and provides an algorithm for doing so. Lemma 2.10 shows that we can obtain a normal form from a reduced form. Lemma 2.11 guarantees the uniqueness of the normal form.

LEMMA 2.8. *Every element of S^* has a form. Moreover, if the isomorphisms ϕ_i and ϕ_i^{-1} are effectively computable for every $i \in I$, then we can effectively find a form for any element of S^*.*

PROOF. Take an element $x \in S^*$. Note that S^* is generated by S and the stable letters t_i. Since $t_i t_i^{-1} = e_i$ and $t_i^{-1} t_i = f_i$, the element x can be written as

$$x = y_0 t_{i_1}^{\delta_1} y_1 t_{i_2}^{\delta_2} \cdots t_{i_m}^{\delta_m} y_m,$$

where $y_j \in S$ for $j = 0, 1, \ldots, m$ and $\delta_j = \pm 1$ for $j = 1, 2, \ldots, m$. Since we have $t_i t_i^{-1} = e_i$ and $t_i^{-1} t_i = f_i$, we may assume that $y_0^{-1} y_0 \le t_{i_1}^{\delta_1} t_{i_1}^{-\delta_1}$, $y_m y_m^{-1} \le t_{i_m}^{-\delta_m} t_{i_m}^{\delta_m}$, $y_k y_k^{-1} \le t_{i_k}^{-\delta_k} t_{i_k}^{\delta_k}$ for $k = 1, 2, \ldots, m$, and $y_k^{-1} y_k \le t_{i_{k+1}}^{\delta_{k+1}} t_{i_{k+1}}^{-\delta_{k+1}}$ for $k = 0, 1, \ldots, m-1$. Then

$$y_{k-1}^{-1} y_{k-1} \in E(A_{i_k}) \quad \text{if } \delta_k = 1 \tag{2.5}$$

and

$$y_{k-1}^{-1} y_{k-1} \in E(B_{i_k}) \quad \text{if } \delta_k = -1 \tag{2.6}$$

for $k = 1, 2, \ldots, m$. Using induction on m, we show that if x is written as $y_0 t_{i_1}^{\delta_1} y_1 t_{i_2}^{\delta_2} \cdots t_{i_m}^{\delta_m} y_m$ satisfying (2.5) and (2.6), then the element x has a form $x_0 t_{k_1}^{\delta_1} x_1 t_{k_2}^{\delta_2} \cdots t_{k_m}^{\delta_m} x_m$. If $m = 0$, the result is trivially true. Suppose that the result is true for any non-negative integer less than m $(1 \le m)$, and that x is written as $y_0 t_{i_1}^{\delta_1} y_1 t_{i_2}^{\delta_2} \cdots t_{i_m}^{\delta_m} y_m$ satisfying (2.5) and (2.6). We set

$$z = y_0 t_{i_1}^{\delta_1} y_1 t_{i_2}^{\delta_2} \cdots y_{m-2} t_{i_{m-1}}^{\delta_{m-1}} y_{m-1} \phi_{i_m}^{-\delta_m}(y_m y_m^{-1}).$$

Note that $x = z t^{\delta_m} y_m$ since $\phi^{-\delta_m}(y_m y_m^{-1}) t^{\delta_m} = t^{\delta_m} y_m y_m^{-1}$. By inductive hypothesis, the element z has a form $z_0 t_{i_1}^{\delta_1} z_1 t_{i_2}^{\delta_2} \cdots z_{m-2} t_{i_{m-1}}^{\delta_{m-1}} z_{m-1}$. It is easily verified that

$$z_0 t_{i_1}^{\delta_1} z_1 t_{i_2}^{\delta_2} \cdots z_{m-2} t_{i_{m-1}}^{\delta_{m-1}} z_{m-1} t_{i_m}^{\delta_m} \phi^{\delta_m}(z_{m-1}^{-1} z_{m-1}) y_m$$

is a form for x. It is also easy to see that there exists an algorithm that finds a form for an element if the isomorphisms ϕ_i and ϕ_i^{-1} $(i \in I)$ are effectively computable. □

LEMMA 2.9. *Every element of S^* has a reduced form. Moreover, if the isomorphisms ϕ_i and ϕ_i^{-1} are effectively computable and the membership problems for A_i and B_i in S are solvable for every $i \in I$, then we can effectively find a reduced form for a given element x of S^* provided that a form for x is given.*

PROOF. Take $x \in S^*$. Suppose that x has a form $x = x_0 t_{i_1}^{\delta_1} x_1 t_{i_2}^{\delta_2} \cdots t_{i_n}^{\delta_n} x_n$. If the form contains a pinch, that is, a subword of the form $t_{i_k}^{\delta_k} x_k t_{i_{k+1}}^{\delta_{k+1}}$, where $i_k = i_{k+1}$, $x_k \in A_{i_k}$, $\delta_k = -1$ and $\delta_{k+1} = 1$ (or $i_k = i_{k+1}$, $x_k \in B_{i_k}$, $\delta_k = 1$ and $\delta_{k+1} = -1$), then we replace it by $\phi_{i_k}^{-\delta_k}(x_k)$. Recall that $\phi_{i_k}^{-\delta_k}(x_k) = t_{i_k}^{\delta_k} x_k t_{i_{k+1}}^{\delta_{k+1}}$ in S^*. Then we have

$$x = x_0 t_{i_1}^{\delta_1} x_1 t_{i_2}^{\delta_2} \cdots t_{i_{k-1}}^{\delta_{k-1}} x_{k-1} \phi_{i_k}^{-\delta_k}(x_k) x_{k+1} t_{i_{k+2}}^{\delta_{k+2}} \cdots t_{i_n}^{\delta_n} x_n.$$

It is easy to see that this is a form of x. We continue this process until all pinches disappear, and then the resulting form is reduced. Obviously we can construct an algorithm to find a reduce form using the argument above if the isomorphisms ϕ_i and ϕ_i^{-1} ($i \in I$) are effectively computable and the membership problems for A_i and B_i in S are solvable for every $i \in I$. □

LEMMA 2.10. *Every element of S^* has a normal form. Furthermore, if x has a reduced form $x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n$, then a normal form for x is written as $u_0 t_{i_1}^{\epsilon_1} u_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} u_n$ for some $u_0, u_1, \ldots u_n \in S$.*

PROOF. Take an element x of S^*. By Lemma 2.8 and Lemma 2.9, x has a reduced form $x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n$. If $\epsilon_1 = 1$, then we rewrite x_0 as $x_0 = u_0 c_0$, where u_0 is a $\sim_{A_{i_1}}$ representative for x_0 and $c_0 \in A_{i_1}$ with $u_0^{-1} u_0 = c_0 c_0^{-1}$. If $\epsilon_1 = -1$, then we rewrite x_0 as $x_0 = u_0 c_0$, where u_0 is a $\sim_{B_{i_1}}$ representative for x_0 and $c_0 \in B_{i_1}$ with $u_0^{-1} u_0 = c_0 c_0^{-1}$. Then we have

$$x = u_0 c_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n = u_0 t_{i_1}^{\epsilon_1} \phi_{i_1}^{\epsilon_1}(c_0) x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n.$$

It is easy to see that the resulting sequence is a reduced form for x. Then we have $\phi^{\epsilon_1}(c_0) x_1 = u_1 c_1$ where c_1 is a $\sim_{A_{i_2}}$ or $\sim_{B_{i_2}}$ representative according to $\epsilon_2 = \pm 1$. We rewrite the form by applying a similar argument to the subword $\phi_{i_1}^{\epsilon_1}(c_0) x_1 t_{i_2}^{\epsilon_2}$ and continue the same process. We eventually get a form $u_0 t_{i_1}^{\epsilon_1} u_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} u_n$ for x. Clearly it is a normal form for x. □

LEMMA 2.11. *A normal form for any element of S^* is unique.*

PROOF. We use the Artin-van der Waerden method. Let N be the set of normal sequences. Let $\mathrm{Sym}(\mathsf{N})$ be the symmetric inverse semigroup on N. We assume that an element of $\mathrm{Sym}(\mathsf{N})$ acts from the right. We define a mapping Φ of $S \cup \{t_i, t_i^{-1} \mid i \in I\}$

into Sym(N) as follows. For any $X \in \mathsf{N}$, if $X = (x_0, t_{i_1}^{\epsilon_1}, x_1, t_{i_2}^{\epsilon_2}, \dots, t_{i_n}^{\epsilon_n}, x_n)$, then we define $\mathbf{z}(X)$ to be the last component x_n. For $s \in S$, we define $\Phi(s)$. We set

$$\mathrm{Dom}(\Phi(s)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Sss^{-1}\} \tag{2.7}$$

and for $X = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_n}^{\epsilon_n}, x_n) \in \mathrm{Dom}(\Phi(s))$

$$X\Phi(s) = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_n}^{\epsilon_n}, x_n s). \tag{2.8}$$

Clearly $X\Phi(s)$ satisfies the condition (C1), (C2) and (C3), and hence, it belongs to N. It is easy to see that $\mathrm{Ran}(\Phi(s)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Ss^{-1}s\}$ and the inverse mapping of $\Phi(s)$ is $\Phi(s^{-1})$ for every $s \in S$. Therefore, $\Phi(s)$ belongs to Sym(N). It is also easy to see that for $s, r \in S$ we have

$$\Phi(sr) = \Phi(s)\Phi(r),$$

that is, Φ is a homomorphism on S.

We now define $\Phi(t_i)$ and $\Phi(t_i^{-1})$ for every $i \in I$. We set

$$\mathrm{Dom}(\Phi(t_i)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Se_i\}$$

and

$$\mathrm{Ran}(\Phi(t_i)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Sf_i\},$$

where $t_i t_i^{-1} = e_i$ and $t_i^{-1} t_i = f_i$. Note that if $X \in \mathrm{Dom}(\Phi(t_i))$, then $\mathbf{z}(X)^{-1}\mathbf{z}(X) \in E(A_i)$ since $E(A_i) = E(e_i S e_i)$. Take $X = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_n}^{\epsilon_n}, x_n)$ from $\mathrm{Dom}(\Phi(t_i))$. We define $\Phi(t_i)$ by

$$X\Phi(t_i) = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}\phi_i(x_n)) \quad \text{if } \mathbf{z}(X) = x_n \in A_{i_n},\ i = i_n \text{ and } \epsilon_n = -1, \tag{2.9}$$

$$X\Phi(t_i) = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_n, t_i, \phi_i(a)) \quad \text{otherwise}, \tag{2.10}$$

where $x_n = y_n a$ and y_n is a $\sim_{A_i}$ representative, $y_n^{-1} y_n = aa^{-1}$ and $a \in A_i$. Similarly, we define $\phi(t_i^{-1})$ as follows. We set

$$\mathrm{Dom}(\Phi(t_i^{-1})) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Sf_i\}$$

and

$$\mathrm{Ran}(\Phi(t_i^{-1})) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Se_i\}.$$

Note that if $X \in \mathrm{Dom}(\Phi(t_i^{-1}))$ then $\mathbf{z}(X)^{-1}\mathbf{z}(X) \in E(B_i)$ since $E(B_i) = E(f_i S f_i)$. Take $X = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_n}^{\epsilon_n}, x_n)$ from $\mathrm{Dom}(\Phi(t_i^{-1}))$. We define $\Phi(t_i^{-1})$ by

$$X\Phi(t_i^{-1}) = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}\phi_i^{-1}(x_n)), \quad \text{if } \mathbf{z}(X) = x_n \in B_{i_n},\ i = i_n \text{ and } \epsilon_n = 1, \tag{2.11}$$

$$X\Phi(t_i^{-1}) = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_n, t_i^{-1}, \phi_i^{-1}(b)) \quad \text{otherwise}, \tag{2.12}$$

https://doi.org/10.1017/S1446788700002639 Published online by Cambridge University Press

where $x_n = y_n b$ and y_n is a $\sim_{B_i}$ representative, $y_n^{-1} y_n = bb^{-1}$ and $b \in B_i$.

If all defining relations for S^* are preserved under Φ then the mapping Φ can be extended to a homomorphism $\Phi : S^* \to \mathrm{Sym}(\mathsf{N})$ by Proposition 1.1. To show all relations are preserved, we prove

$$\Phi(t_i)\Phi(t_i^{-1}) = \Phi(e_i) \quad \text{for every } i \in I, \tag{2.13}$$

$$\Phi(t_i^{-1})\Phi(t_i) = \Phi(f_i) \quad \text{for every } i \in I, \tag{2.14}$$

$$\Phi(t_i^{-1})\Phi(a)\Phi(t_i) = \Phi(\phi_i(a)) \quad \text{for every } a \in A_i. \tag{2.15}$$

Since Φ is a homomorphism on S, it preserves the relations for S.

We prove the relation (2.13). Since e_i is an idempotent, we have

$$\mathrm{Dom}(\Phi(e_i)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Se_i\}$$

by (2.7). Then we have

$$X\Phi(e_i) = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_n}^{\epsilon_n}, x_n e_i) = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_n}^{\epsilon_n}, x_n) = X$$

for any $X = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_n}^{\epsilon_n}, x_n) \in \mathrm{Dom}(\Phi(e_i))$ by (2.8). Therefore, $\Phi(e_i)$ is the identity mapping on $\mathrm{Dom}(\Phi(e_i))$. On the other hand, we have

$$\mathrm{Ran}(\Phi(t_i)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Sf_i\} = \mathrm{Dom}(\Phi(t_i^{-1})).$$

It follows that

$$\begin{aligned}\mathrm{Dom}(\Phi(t_i)\Phi(t_i^{-1})) &= (\mathrm{Ran}(\Phi(t_i)) \cap \mathrm{Dom}(\Phi(t_i^{-1})))\Phi(t_i)^{-1} \\ &= \mathrm{Ran}(\Phi(t_i))\Phi(t_i)^{-1} = \mathrm{Dom}(\Phi(t_i)) \\ &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Se_i\} = \mathrm{Dom}(\Phi(e_i)).\end{aligned}$$

Take an element X from $\mathrm{Dom}(\Phi(e_i)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Se_i\}$. Suppose that $X = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_n}^{\epsilon_n}, x_n)$, where $\mathbf{z}(X) = x_n \in Se_i$. We compute $X\Phi(t_i)\Phi(t_i^{-1})$. In computing $X\Phi(t_i)$ for X and t_i, there are two possible cases; (2.9) and (2.10).

Suppose that (2.9) is true. Then we have $x_n \in A_i$, $i = i_n$ and $\epsilon_n = -1$. Remark that in this case, it cannot happen that $x_{n-1} \in B_i$, $i = i_{n-1}$ and $\epsilon_{n-1} = 1$ since $X \in \mathsf{N}$. We have

$$X\Phi(t_i) = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}\phi_i(x_n)).$$

By the remark above, $t_{i_{n-1}}^{\epsilon_{n-1}} x_{n-1}\phi_i(x_n) t_i^{-1}$ is not a pinch, and so, we have

$$\begin{aligned}X\Phi(t_i)\Phi(t_i^{-1}) &= (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}\phi_i(x_n))\Phi(t_i^{-1}) \\ &= (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, y_{n-1}, t_i^{-1}, \phi_i^{-1}(b)),\end{aligned}$$

where $x_{n-1}\phi_i(x_n) = y_{n-1}b$, y_{n-1} is a $\sim_{B_i}$ representative and $b \in B_i$ with $y_{n-1}^{-1} y_{n-1} = bb^{-1}$. Then $y_{n-1} = x_{n-1}\phi_i(x_n)b^{-1}$. Note that

$$\begin{aligned}(x_{n-1}\phi_i(x_n))^{-1}(x_{n-1}\phi_i(x_n)) &= \phi_i(x_n^{-1})x_{n-1}^{-1}x_{n-1}\phi_i(x_n) \\ &= \phi_i(x_n^{-1})\phi_i(x_n x_n^{-1})\phi_i(x_n) = \phi_i(x_n^{-1}x_n).\end{aligned}$$

Since $x_{n-1}\phi_i(x_n) = y_{n-1}b$, we have

$$(x_{n-1}\phi_i(x_n))^{-1}(x_{n-1}\phi_i(x_n)) = (y_{n-1}b)^{-1}(y_{n-1}b) = b^{-1}y_{n-1}^{-1}y_{n-1}b = b^{-1}b.$$

It follows that $b^{-1}b = \phi_i(x_n^{-1}x_n)$. Then we have

$$\begin{aligned}(\phi_i(x_n)b^{-1})(\phi_i(x_n)b^{-1})^{-1} &= \phi_i(x_n)b^{-1}b\phi_i(x_n^{-1}) \\ &= \phi_i(x_n)\phi_i(x_n^{-1}x_n)\phi_i(x_n^{-1}) = \phi_i(x_nx_n^{-1}) = x_{n-1}^{-1}x_{n-1}.\end{aligned}$$

Note that $x_{n-1}^{-1}x_{n-1} \in E(B_i)$, $y_{n-1}^{-1}y_{n-1} = bb^{-1} \in E(B_i)$ and $\phi_i(x_n)b^{-1} \in B_i$. Hence, (E2) holds for x_{n-1} and y_{n-1}, and so, $x_{n-1} \sim_{B_i} y_{n-1}$. Since both x_{n-1} and y_{n-1} are $\sim_{B_i}$ representative, we have $x_{n-1} = y_{n-1}$. By (C1), we have $\phi_i^{-1}(x_{n-1}^{-1}x_{n-1}) = x_nx_n^{-1}$. Then $\phi_i(x_n) = x_{n-1}^{-1}x_{n-1}\phi_i(x_n) = y_{n-1}^{-1}y_{n-1}b = b$, and so, $\phi_i^{-1}(b) = x_n$. Consequently, we have $X\Phi(t_i)\Phi(t_i^{-1}) = X$.

Next suppose that (2.10) holds. Then we have

$$X\Phi(t_i) = (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_n, t_i, \phi_i(a)),$$

where $x_n = y_na$ and y_n is a $\sim_{A_i}$ representative with $y_n^{-1}y_n = aa^{-1}$ and $a \in A_i$. Remark that $t_i\phi_i(a)t_i^{-1}$ is a pinch. Hence, we have

$$\begin{aligned}X\Phi(t_i)\Phi(t_i^{-1}) &= (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_n, t_i, \phi_i(a))\Phi(t_i^{-1}) \\ &= (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_n\phi_i^{-1}(\phi_i(a))) \\ &= (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_na) \\ &= (x_0, t_{i_1}^{\epsilon_1}, \dots, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, x_n) = X.\end{aligned}$$

Consequently, we have $X\Phi(t_i)\Phi(t_i^{-1}) = X$ in both (2.9) and (2.10). It follows that

$$\Phi(t_i)\Phi(t_i^{-1}) = id_{\mathrm{Dom}(\Phi(t_i))} = id_{\mathrm{Dom}(\Phi(e_i))} = \Phi(e_i).$$

We have proved (2.13). Similarly, we can prove (2.14). This implies that $\Phi(t_i^{-1}) = \Phi(t_i)^{-1}$ in $\mathrm{Sym}(\mathsf{N})$ for every $i \in I$.

Next we prove the relation (2.15). Take a from A_i, where $i \in I$. Note that

$$\mathrm{Ran}(\Phi(a)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Sa^{-1}a\}$$

and

$$\mathrm{Dom}(\Phi(t_i)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Se_i\}.$$

Since $a \in A_i \subset e_iSe_i$, we have $a^{-1}a \leq e_i$ and $aa^{-1} \leq e_i$. Then we have

$$\begin{aligned}\mathrm{Ran}(\Phi(a)) \cap \mathrm{Dom}(\Phi(t_i)) &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Sa^{-1}a \cap Se_i\} \\ &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Sa^{-1}a\} = \mathrm{Ran}(\Phi(a)).\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Dom}(\Phi(a)\Phi(t_i)) &= (\mathrm{Ran}(\Phi(a)) \cap \mathrm{Dom}(\Phi(t_i)))\Phi(a)^{-1} = (\mathrm{Ran}(\Phi(a)))\Phi(a)^{-1} \\ &= \mathrm{Dom}(\Phi(a)) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Saa^{-1}\}.\end{aligned}$$

Since

$$\mathrm{Ran}(\Phi(t_i^{-1})) = \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Se_i\},$$

then

$$\begin{aligned}\mathrm{Ran}(\Phi(t_i^{-1})) \cap \mathrm{Dom}(\Phi(a)\Phi(t_i)) &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Saa^{-1} \cap Se_i\} \\ &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Saa^{-1}\}\end{aligned}$$

since $aa^{-1} \leq e_i$. Recall that $\Phi(t_i^{-1}) = \Phi(t_i)^{-1}$ in $\mathrm{Sym}(\mathsf{N})$. Then it is easy to see that

$$\begin{aligned}\mathrm{Dom}(\Phi(t_i^{-1})\Phi(a)\Phi(t_i)) &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Saa^{-1}\}\Phi(t_i^{-1})^{-1} \\ &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in Saa^{-1}\}\Phi(t_i) \\ &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in S\phi_i(aa^{-1})\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathrm{Dom}(\Phi(\phi_i(a))) &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in S\phi_i(a)(\phi_i(a))^{-1}\} \\ &= \{X \in \mathsf{N} \mid \mathbf{z}(X) \in S\phi_i(aa^{-1})\}.\end{aligned}$$

Hence, $\mathrm{Dom}(\Phi(t_i^{-1})\Phi(a)\Phi(t_i)) = \mathrm{Dom}(\Phi(\phi_i(a)))$.

Take a normal sequence X from $\mathrm{Dom}(\Phi(t_i^{-1})\Phi(a)\Phi(t_i))$. Suppose that $X = (x_0, t_{i_1}^{\epsilon_1}, x_1, \ldots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, x_n)$. In computing $X\Phi(t_i^{-1})$ for X and t_i^{-1}, there are two possible cases; (2.11) and (2.12).

Suppose that (2.11) holds. Then we have $i = i_n$, $\epsilon_n = 1$ and $x_n \in B_{i_n} = B_i$. Remark that it cannot happen that $x_{n-1} \in A_i$, $i = i_n$ and $\epsilon_{n-1} = -1$ since $X \in \mathsf{N}$. Then $\phi_i(x_{n-1}^{-1}x_{n-1}) = \phi_{i_n}(x_{n-1}^{-1}x_{n-1}) = x_n x_x^{-1}$ by (C1). We have

$$X\Phi(t_i^{-1}) = (x_0, t_{i_1}^{\epsilon_1}, x_1, \ldots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}\phi_i^{-1}(x_n)).$$

Then we have

$$X\Phi(t_i^{-1})\Phi(a) = (x_0, t_{i_1}^{\epsilon_1}, x_1, \ldots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}\phi_i^{-1}(x_n)a).$$

By the remark above $t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}\phi_i^{-1}(x_n)at_i$ is not a pinch, and so, we have

$$X\Phi(t_i^{-1})\Phi(a)\Phi(t_i) = (x_0, t_{i_1}^{\epsilon_1}, x_1, \ldots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, y_{n-1}, t_i, \phi_i(c)),$$

where y_{n-1} is a $\sim_{A_i}$ representative and $y_{n-1}^{-1}y_{n-1} = cc^{-1}$, $c \in A_i$ and $x_{n-1}\phi_i^{-1}(x_n)a = y_{n-1}c$. Then both x_{n-1} and y_{n-1} are $\sim_{A_i}$ representative. Note that $x_{n-1}^{-1}x_{n-1}$ belongs to $E(A_i)$ by the condition (C1). Also note that $y_{n-1}^{-1}y_{n-1}$ belongs to $E(A_i)$ since $c \in A_i$ and $y_{n-1}^{-1}y_{n-1} = cc^{-1}$. Set $h = \phi_i^{-1}(x_n)ac^{-1}$. Then h belongs to A_i (as $x_n \in B_i$, $a \in A_i$

and $c \in A_i$). Since $x_n = \mathbf{z}(X) \in S\phi_i(aa^{-1})$, we have $x_n = s\phi_i(aa^{-1})$ for some $s \in S$. Then $x_n\phi_i(aa^{-1}) = x_n$, and so, $\phi_i^{-1}(x_n)aa^{-1} = \phi_i^{-1}(x_n)$. We have

$$x_{n-1}h = x_{n-1}\phi_i^{-1}(x_n)ac^{-1} = y_{n-1}cc^{-1} = y_{n-1}y_{n-1}^{-1}y_{n-1} = y_{n-1}$$

and

$$\begin{aligned} y_{n-1}h^{-1} &= y_{n-1}ca^{-1}\phi_i^{-1}(x_n^{-1}) = x_{n-1}\phi_i^{-1}(x_n)aa^{-1}\phi_i^{-1}(x_n^{-1}) \\ &= x_{n-1}\phi_i^{-1}(x_n)\phi_i^{-1}(x_n^{-1}) = x_{n-1}\phi_i^{-1}(x_nx_n^{-1}) = x_{n-1}x_{n-1}^{-1}x_{n-1} = x_{n-1}. \end{aligned}$$

Then $x_{n-1}x_{n-1}^{-1} = y_{n-1}y_{n-1}^{-1}$. Furthermore, we have

$$\begin{aligned} x_{n-1}^{-1}y_{n-1} &= x_{n-1}^{-1}x_{n-1}h = x_{n-1}^{-1}x_{n-1}\phi_i^{-1}(x_n)ac^{-1} \\ &= \phi_i^{-1}(x_nx_n^{-1})\phi_i^{-1}(x_n)ac^{-1} = \phi_i^{-1}(x_n)ac^{-1} = h \in A_i. \end{aligned}$$

It follows that $x_{n-1} \sim_{A_i} y_{n-1}$ by Lemma 2.3. Since both x_{n-1} and y_{n-1} are $\sim_{A_i}$ representative, we have $x_{n-1} = y_{n-1}$. Since $x_{n-1}\phi_i^{-1}(x_n)a = y_{n-1}c$ and $y_{n-1}^{-1}y_{n-1} = cc^{-1}$, we have $c = y_{n-1}^{-1}y_{n-1}c = x_{n-1}^{-1}x_{n-1}\phi_i^{-1}(x_n)a = \phi_i^{-1}(x_n)a$. Hence, $\phi_i(c) = \phi_i(\phi_i^{-1}(x_n)a) = x_n\phi_i(a)$. Consequently, we have

$$\begin{aligned} X\Phi(t_i^{-1})\Phi(a)\Phi(t_i) &= (x_0, t_{i_1}^{\epsilon_1}, x_1, \dots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, y_{n-1}, t_i, \phi_i(c)) \\ &= (x_0, t_{i_1}^{\epsilon_1}, x_1, \dots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_i, x_n\phi_i(a)) \\ &= X\Phi(\phi_i(a)). \end{aligned}$$

Next suppose that (2.12) holds. Then

$$X\Phi(t_i^{-1}) = (x_0, t_{i_1}^{\epsilon_1}, x_1, \dots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_n, t_i^{-1}, \phi_i^{-1}(b)),$$

where $x_n = y_nb$ and y_n is a $\sim_{B_i}$ representative such that $y_n^{-1}y_n = bb^{-1}$ and $b \in B_i$. Then

$$X\Phi(t_i^{-1})\Phi(a) = (x_0, t_{i_1}^{\epsilon_1}, x_1, \dots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_n, t_i^{-1}, \phi_i^{-1}(b)a).$$

Since $t_i^{-1}\phi_i^{-1}(b)at_i$ is a pinch,

$$\begin{aligned} X\Phi(t_i^{-1})\Phi(a)\Phi(t_i) &= (x_0, t_{i_1}^{\epsilon_1}, x_1, \dots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_n\phi_i(\phi_i^{-1}(b)a)) \\ &= (x_0, t_{i_1}^{\epsilon_1}, x_1, \dots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, y_nb\phi_i(a)) \\ &= (x_0, t_{i_1}^{\epsilon_1}, x_1, \dots, x_{n-2}, t_{i_{n-1}}^{\epsilon_{n-1}}, x_{n-1}, t_{i_n}^{\epsilon_n}, x_n\phi_i(a)) \\ &= X\Phi(\phi_i(a)). \end{aligned}$$

We have proved (2.15).

Consequently, we can extend the mapping Φ to a homomorphism of S^* into Sym(N) by Proposition 1.1. Now suppose that x ($\in S^*$) has a normal form $x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}\cdots t_{i_n}^{\epsilon_n}x_n$. Then the sequence $(x_0, t_{i_1}^{\epsilon_1}, x_1, t_{i_2}^{\epsilon_2}, \dots, t_{i_n}^{\epsilon_n}, x_n)$ is normal, and so, it belongs to N. Set

$e = xx^{-1}$. By Lemma 2.6, we have $e = x_0x_0^{-1}$. Then it is easy to see that the normal sequence (e) is in $\mathrm{Dom}(\Phi(x))$ and

$$\begin{aligned}(e)\Phi(x) &= (e)\Phi(x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}\cdots t_{i_n}^{\epsilon_n}x_n)\\ &= (x_0, t_{i_1}^{\epsilon_1}, x_1, t_{i_2}^{\epsilon_2}, \ldots, t_{i_n}^{\epsilon_n}, x_n).\end{aligned}$$

This implies the uniqueness of a normal form of x. □

We have completed the proof of Theorem 2.7. A similar normal form is obtained for a full HNN extension of an inverse monoid with one stable letter and a locally full HNN extension of a semilattice in [15] and [17], respectively.

3. Applications of the normal form theorem

An immediate consequence of Theorem 2.7 is that an inverse semigroup S is strongly embedded into any locally full HNN extension S^*. Note that this follows from Proposition 1.2. Another easy consequence is that a formation of a locally full HNN extension yields no new idempotent. Conversely, if an HNN extension of an inverse semigroup has the same semilattice of idempotents as the original semigroup, then it must be locally full as we will see next.

COROLLARY 3.1. *An inverse semigroup S is strongly embedded into a locally full* HNN *extension S^* and $E(S^*) = E(S)$. Conversely, if S^* is an* HNN *extension of S with the property that $E(S^*) = E(S)$, then S^* is a locally full* HNN *extension of S.*

PROOF. Every element x of S has the normal form x since the sequence (x) satisfies the conditions (C1), (C2) and (C3). It follows that S can be naturally embedded into S^*.

Take $x \in t_i^{-1}St_i \cap S$. Then $x = t_i^{-1}st_i$, where $s \in S$. Using Lemma 2.8 we may assume that $t_i^{-1}st_i$ is a form. If $t_i^{-1}st_i$ is not a pinch, then x has the normal form $s_0t_i^{-1}s_1t_i$, where $s_0, s_1 \in S$. On the other hand, since $x \in S$, x is in the normal form. This contradicts the uniqueness of the normal form. Thus, $t_i^{-1}st_i$ must be a pinch, and so, $s \in A_i$ and $t_i^{-1}st_i = \phi_i(a) \in B_i$. It follows that $t_i^{-1}St_i \cap S = B_i$ and that S is strongly embedded in S^*.

Take any idempotent e from $E(S^*)$. Suppose that e has the normal form $x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}\cdots t_{i_n}^{\epsilon_n}x_n$. By Lemma 2.6, we have

$$e = ee^{-1} = (x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}\cdots t_{i_n}^{\epsilon_n}x_n)(x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}\cdots t_{i_n}^{\epsilon_n}x_n)^{-1} = x_0x_0^{-1} \in E(S)$$

since $x_0 \in S$. This shows that $E(S^*) \subset E(S)$. Clearly $E(S) \subset E(S^*)$ since S is embedded into S^*.

Conversely, we assume that the semilattice $E(S^*)$ is equal to $E(S)$. We show that each $E(A_i)$ coincides with $E(e_iSe_i) = E(S)e_i$. Take any $e \in E(S)e_i$. The element $t_i^{-1}et_i$ is an idempotent of S^*. Since $E(S^*) = E(S)$, $t_i^{-1}et_i$ is in $E(S)$. Then $t_i^{-1}et_i \in E(S) \cap t_i^{-1}(E(S)e_i)t_i \subset S \cap t_i^{-1}St_i$. On the other hand, we have $S \cap t_i^{-1}St_i = t_i^{-1}A_it_i = B_i$ by (1.6). It follows that $t_i^{-1}et_i \in t_i^{-1}A_it_i$ and that $e = t_it_i^{-1}et_it_i^{-1} \subset t_it_i^{-1}A_it_it_i^{-1} \subset A_i$ (as $t_it_i^{-1} = e_i \in A_i$ and $e \leq e_i$). Therefore, $E(A_i) = E(S)e_i$, and hence, S^* is a locally full HNN extension of S. □

COROLLARY 3.2. *If an element x of S^* has a reduced form*

$$x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}\cdots t_{i_n}^{\epsilon_n}x_n, \tag{3.1}$$

where $n \geq 1$, then $x \notin E(S^*) = E(S)$.

PROOF. By Lemma 2.10, if the element x has a reduced form (3.1), then x has the normal form $u_0t_{i_1}^{\epsilon_1}u_1t_{i_2}^{\epsilon_2}\cdots t_{i_n}^{\epsilon_n}u_n$ for some $u_0, u_1, \ldots, u_n$ in S. Since $E(S^*) = E(S)$ by Corollary 3.1, every idempotent e has the normal form e. Since the expression is unique, we have $x \notin E(S)$. □

Note that Corollary 3.2 is considered as a generalization of Britton's lemma in group theory ([3, 10]).

Some algebraic properties are not preserved under a formation of a locally full HNN extension. For examples, the property of being a finite semigroup, being commutative, being free, being E-unitary, being factorizable, being combinatorial, being fundamental, being congruence-free, being completely semisimple or having a zero element is not necessarily preserved under a formation of a locally full HNN extension.

On the other hand, we show that several algebraic properties are preserved under a formation of a locally full HNN extension. It is easy to see that an HNN extension of a monoid (group) is always a monoid (group).

COROLLARY 3.3. *Let S be an inverse semigroup. Let $\mathscr{P}$ be one of the properties listed below. Then a locally full* HNN *extension S^* satisfies $\mathscr{P}$ if S does.*

(1) *Having finitely many idempotents.*
(2) *Being torsion-free.*
(3) *Being bisimple.*
(4) *Being simple.*

PROOF. (1) By Corollary 3.1, $E(S) = E(S^*)$ and so the HNN extension S^* has finitely many idempotents if S does so. (2) Using Theorem 2.7, we can easily show that S^* is torsion-free if S is. (3) We note that an inverse semigroup is bisimple if

and only if all idempotents are $\mathscr{D}$-related. By Corollary 3.1, all idempotents of S^* are $\mathscr{D}$-related in S^* if all idempotents of S are $\mathscr{D}$-related in S. (4) We note that an inverse semigroup is simple if and only if all idempotents are $\mathscr{J}$-related. By Corollary 3.1, all idempotents of S^* are $\mathscr{J}$-related in S^* if all idempotents of S are $\mathscr{J}$-related in S. □

Under a certain condition on associated inverse submonoids A_i and B_i $(i \in I)$, algebraic properties are preserved.

COROLLARY 3.4. *Let S be an E-unitary inverse semigroup. Suppose that S^* is a locally full* HNN *extension of S associated with $\phi_i : A_i \to B_i$ $(i \in I)$. If A_i and B_i are closed in S for every $i \in I$, then S^* is E-unitary.*

PROOF. Suppose that S^* is not E-unitary. There exist $e \in E(S^*) = E(S)$ and $x \in S^* \setminus E(S)$ such that $e \leq x$. By Theorem 2.7, x has the normal form $x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots x_{n-1} t_{i_n}^{\epsilon_n} x_n$, where either $n \geq 1$ or $x_0 \notin E(S)$. Since $e \leq x$, there exists $f \in E(S)$ such that $fx = e$. Then it is easy to see that

$$fx = f x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots x_{n-1} t_{i_n}^{\epsilon_n} x_n = x_0' t_{i_1}^{\epsilon_1} x_1' t_{i_2}^{\epsilon_2} \cdots x_{n-1}' t_{i_n}^{\epsilon_n} x_n',$$

for some $x_0', x_1', \ldots, x_n'$ in S with $x_i' \leq x_i$ for every i. By Lemma 2.8, we may assume that $e = fx$ has the form $x_0' t_{i_1}^{\epsilon_1} x_1' t_{i_2}^{\epsilon_2} \cdots x_{n-1}' t_{i_n}^{\epsilon_n} x_n'$. Then either $x_i' \in A_i$ or $x_i' \in B_i$ holds for some i by Corollary 3.2. On the other hand, we have $x_i \notin A_i$ and $x_i \notin B_i$. This contradicts the fact that A_i (or B_i) is closed in S. Consequently S^* is E-unitary. □

Similarly any locally full HNN extension of an F-inverse monoid associated with closed inverse submonoids is F-inverse. We next discuss the word problem for a locally full HNN extension.

LEMMA 3.5. *Let S be an inverse semigroup, and let S^* be a locally full* HNN *extension of S associated with the isomorphisms $\phi_i : A_i \to B_i$ $(i \in I)$. If the membership problems for the semilattice $E(S)$, A_i and B_i $(i \in I)$ in S are solvable and the isomorphisms ϕ_i and ϕ_i^{-1} are effectively computable $(i \in I)$, then the membership problem for $E(S)$ in S^* is solvable.*

PROOF. For a given word w on generators of S^*, we can effectively obtain a reduced form for w by Lemma 2.8 and Lemma 2.9. Suppose that the reduced form for w is $x_0 t_{i_1}^{\epsilon_1} x_1 t_{i_2}^{\epsilon_2} \cdots t_{i_n}^{\epsilon_n} x_n$. If $n > 0$, then we can conclude that $w \notin E(S^*)$. If $n = 0$, w is written as a word on generators of S. We use the algorithm for the membership problem for $E(S)$ in S to determine whether or not w is in $E(S)$. If so, we can conclude that $w \in E(S^*)$, otherwise $w \notin E(S^*)$. □

THEOREM 3.6. *If the word problem for S is solvable and ϕ_i and ϕ_i^{-1} are effectively computable for every $i \in I$ and the membership problems for A_i and B_i in S are solvable for every $i \in I$, then the word problem for S^* is solvable.*

PROOF. Let w_1 and w_2 be given words on generators of S^*. We note that $w_1 = w_2$ if and only if $w_1 w_1^{-1} = w_2 w_2^{-1}$ and $w_2^{-1} w_1 \in E(S^*) = E(S)$. For, if $w_1 w_1^{-1} = w_2 w_2^{-1}$ and $w_2^{-1} w_1 \in E(S)$, then we have

$$\begin{aligned} w_1 &= w_1 w_1^{-1} w_1 = w_2 w_2^{-1} w_1 = w_2 w_2^{-1} w_1 w_2^{-1} w_1 \\ &= w_2 w_2^{-1} w_1 w_1^{-1} w_2 \qquad (\text{as } w_2^{-1} w_1 = (w_2^{-1} w_1)^{-1} = w_1^{-1} w_2) \\ &= w_2 w_2^{-1} w_2 w_2^{-1} w_2 = w_2. \end{aligned}$$

Hence, to solve the word problem for S^* it is enough to decide for given words w_1 and w_2 whether or not $w_1 w_1^{-1} = w_2 w_2^{-1}$ and $w_2^{-1} w_1 \in E(S)$.

Since the word problem for S is solvable, we can effectively determine whether or not $w = w^2$ for a given word w on the generators of S. Therefore, the membership problem for $E(S)$ in S is solvable. By Lemma 3.5, we can effectively determine whether or not $w_2^{-1} w_1 \in E(S)$. By Lemma 2.8 and Lemma 2.9, we can effectively find reduced form e_1 and e_2 for $w_1 w_1^{-1}$ and $w_2 w_2^{-1}$, respectively. Then $w_1 w_1^{-1}$ and $w_2 w_2^{-1}$ are written as words on the generators of S. By Corollary 3.1, e_1 and e_2 are in $E(S)$. Since the word problem for S is solvable, we can effectively decide whether or not $e_1 = e_2$ in S. Consequently, the word problem for S^* is solvable. □

COROLLARY 3.7. *A locally full* HNN *extension of a finite inverse semigroup has solvable word problem.*

PROOF. Obviously the word problem for a finite inverse semigroup is solvable, the membership problems for inverse subsemigroups are solvable and the partial isomorphisms are effectively computable. □

Margolis and Meakin [11] solved the word problem for an idempotent pure image of a free inverse monoid using Rabin's tree theorem. Using their result, the author showed that any HNN extension of a finite semilattice has solvable word problem ([17]). Note that any idempotent pure image of a free inverse semigroup is an HNN extension of a semilattice. However, a finitely generated idempotent pure image of a free inverse semigroup does not necessarily have solvable word problem. Jajcayova [8] solved the word problem for an HNN extension of a free inverse semigroup associated with finitely generated inverse subsemigroups using graphical methods. It is not known whether or not any HNN extension of a finite inverse semigroup has solvable word problem. We remark that Cherubini, Meakin and Piochi [2] proved the solvability of the word problem for an amalgamated free product of free inverse semigroups associated with finitely generated inverse subsemigroups.

4. Action of a maximal subgroup on a tree

Haataja, Margolis and Meakin [4] employed the Bass-Serre theory to obtain a group presentation of a maximal subgroup of a full amalgamated free product of regular semigroups. See [1, 8, 12, 15] for similar applications of the Bass-Serre theory to find presentations of maximal subgroups of free constructions of inverse semigroups. We obtain a group presentation of a maximal subgroup of a locally full HNN extension using the Bass-Serre theory. The reader is referred to [3, 14] for the detail of the Bass-Serre theory.

Let S^* be the locally full HNN extension given by (2.2). Let e be an idempotent in $E(S^*) = E(S)$. We denote the maximal subgroup (group $\mathscr{H}$-class) of S^* containing e by S_e^*. We construct a graph X_e on which S_e^* acts without inversion.

4.1. Forest X Let S be an inverse semigroup and H an inverse subsemigroup of S. For any x in S with $x^{-1}x \in E(H)$, we define a *coset* xH to be

$$xH = \{xh \mid x^{-1}x = hh^{-1}\ (h \in H)\}.$$

LEMMA 4.1. *The coset xH is identical to the $\sim_H$ class containing x if $x^{-1}x \in E(H)$.*

PROOF. Recall that $\sim_H$ is defined in (2.1). Suppose that $x^{-1}x \in E(H)$. Then, if $y \in xH$, then $y = xh$ for some $h \in H$ with $x^{-1}x = hh^{-1}$. By the definition of $\sim_H$, we have $x \sim_H y$ and so $xH \subset \overline{x}$. Conversely, we take $y \in \overline{x}$. Then, $y = xh$ with $h \in H$ and $x^{-1}x = hh^{-1}$. Hence, $y = xh \in xH$. Thus, $\overline{x} \subset xH$. □

LEMMA 4.2. *For $x, y \in S$ with $x^{-1}x, y^{-1}y \in E(H)$, $xH = yH$ if and only if $x\mathscr{R}y$ in S and $x^{-1}y \in H$.*

PROOF. This is an immediate consequence of Lemma 2.3 and Lemma 4.1. □

LEMMA 4.3. *Let $x, y \in S$ with $x^{-1}x, y^{-1}y \in E(H)$. If $xH = yH$, then $x\mathscr{R}y$ and $x^{-1}x\mathscr{D}y^{-1}y$ in H.*

PROOF. By Lemma 4.2, we have $x\mathscr{R}y$ in S and $x^{-1}y \in H$. Let $h = x^{-1}y$. Then we have

$$hh^{-1} = (x^{-1}y)(x^{-1}y)^{-1} = x^{-1}(yy^{-1})x = x^{-1}(xx^{-1})x = x^{-1}x.$$

Hence, $h\mathscr{R}x^{-1}x$ in H. Similarly we can show $h\mathscr{L}y^{-1}y$ in H. Consequently, $x^{-1}x\mathscr{D}y^{-1}y$ in H. □

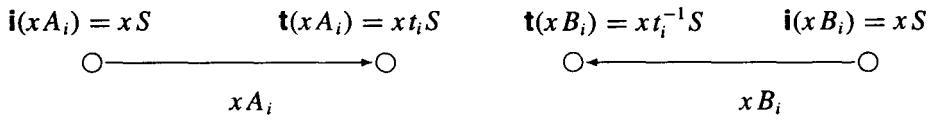

FIGURE 1. Edge and initial and terminal vertex

FIGURE 2. Inverse edge

The graph X consists of the set $\mathrm{Vert}(X)$ of vertices and the set $\mathrm{Edge}(X)$ of edges. The set $\mathrm{Vert}(X)$ consists of cosets xS:

$$\mathrm{Vert}(X) = \{xS \mid x \in S^*\}. \tag{4.1}$$

Note that $x^{-1}x \in E(S^*) = E(S)$ for every $x \in S^*$. The set $\mathrm{Edge}_+(X)$ of the positively oriented edges of X consists of cosets xA_i $(i \in I)$:

$$\mathrm{Edge}_+(X) = \{xA_i \mid x \in S^*,\ x^{-1}x \le e_i = t_it_i^{-1} \text{ for } i \in I\}. \tag{4.2}$$

The set $\mathrm{Edge}_-(X)$ of the negatively oriented edges of X consists of cosets xB_i $(i \in I)$:

$$\mathrm{Edge}_-(X) = \{xB_i \mid x \in S^*,\ x^{-1}x \le f_i = t_i^{-1}t_i \text{ for } i \in I\}. \tag{4.3}$$

We make a convention that xA_i and xA_j are distinct for distinct i and j even if A_i coincides with A_j, and similarly, xB_i and xB_j are distinct for distinct i and j even if B_i coincides with B_j. The set $\mathrm{Edge}(X)$ is the union of $\mathrm{Edge}_+(X)$ and $\mathrm{Edge}_-(X)$:

$$\mathrm{Edge}(X) = \mathrm{Edge}_+(X) \cup \mathrm{Edge}_-(X) \tag{4.4}$$

Here we make a convention that $\mathrm{Edge}_+(X)$ and $\mathrm{Edge}_-(X)$ are disjoint (even if $A_i = B_j$ for some i and j).

We define the initial vertex, the terminal vertex and the inverse edge. The initial vertex of xA_i and xB_i are defined by

$$\mathbf{i}(xA_i) = xS, \qquad \mathbf{i}(xB_i) = xS. \tag{4.5}$$

https://doi.org/10.1017/S1446788700002639 Published online by Cambridge University Press

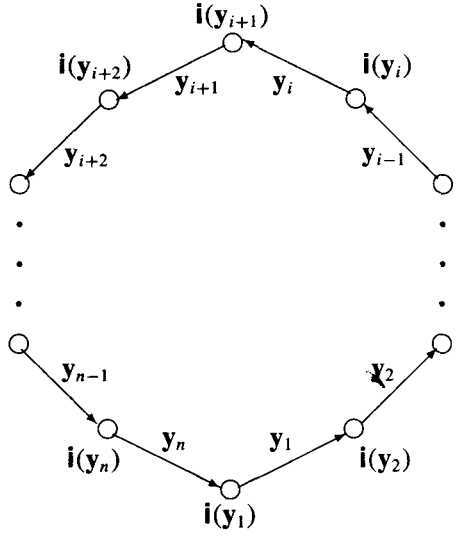

FIGURE 3. Circuit

The terminal vertex of xA_i and xB_i are defined by

$$\mathbf{t}(xA_i) = xt_iS, \qquad \mathbf{t}(xB_i) = xt_i^{-1}S. \tag{4.6}$$

The inverse edge of xA_i and xB_i are defined by

$$\overline{xA_i} = xt_iB_i, \qquad \overline{xB_i} = xt_i^{-1}A_i \tag{4.7}$$

See Figure 1 and Figure 2. It is routine to verify that $\mathrm{Vert}(X)$ and $\mathrm{Edge}(X)$ forms a graph in the sense of Serre [14].

THEOREM 4.4. *The graph X defined by* (4.1)–(4.7) *is a forest.*

PROOF. Suppose that X is not a forest. Then there exists a circuit without backtracking. Suppose that the edges $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ $(n \geq 1)$ form a circuit without backtracking in this order. See Figure 3. We have $\mathbf{t}(\mathbf{y}_k) = \mathbf{i}(\mathbf{y}_{k+1})$ for each $k = 1, 2, \ldots, n-1$ and $\mathbf{t}(\mathbf{y}_n) = \mathbf{i}(\mathbf{y}_1)$. We also have $\mathbf{y}_k \neq \overline{\mathbf{y}_{k+1}}$ for $k = 1, 2, \ldots, n-1$ and $\mathbf{y}_1 \neq \overline{\mathbf{y}_n}$.

We assume that $n \geq 2$ and derive a contradiction. In the case of $n = 1$, we can similarly derive a contradiction and so we do not discuss the case. If $\mathbf{y}_k \in \mathrm{Edge}_+(X)$, then we have $\mathbf{y}_k = z_{k-1}A_{i_k}$ for some $z_{k-1} \in S^*$ with $z_{k-1}^{-1}z_{k-1} \in A_{i_k}$. Then we define ϵ_k to be 1. We note that $\mathbf{i}(\mathbf{y}_k) = z_{k-1}S$ and $\mathbf{t}(\mathbf{y}_k) = z_{k-1}t_{i_k}S = z_{k-1}t_{i_k}^{\epsilon_k}S$. If $\mathbf{y}_k \in \mathrm{Edge}_-(X)$, then we have $\mathbf{y}_k = z_{k-1}B_{i_k}$ for some $z_{k-1} \in S^*$ with $z_{k-1}^{-1}z_{k-1} \in B_{i_k}$. Then we define ϵ_k to be -1. We note that $\mathbf{i}(\mathbf{y}_k) = z_{k-1}S$ and $\mathbf{t}(\mathbf{y}_k) = z_{k-1}t_{i_k}^{-1}S = z_{k-1}t_{i_k}^{\epsilon_k}S$. In the both cases, we have $\mathbf{i}(\mathbf{y}_k) = z_{k-1}S$ and $\mathbf{t}(\mathbf{y}_k) = z_{k-1}t_{i_k}^{\epsilon_k}S$ for every $k = 1, 2, \ldots, n$.

Since $\mathbf{t}(\mathbf{y}_k) = \mathbf{i}(\mathbf{y}_{k+1})$ for every $k = 1, 2, \dots, n-1$, we have $z_{k-1}t_{i_k}^{\epsilon_k}S = z_kS$. There exists $x_k \in S$ such that $z_k = z_{k-1}t_{i_k}^{\epsilon_k}x_k$ with $(z_{k-1}t_{i_k}^{\epsilon_k})^{-1}(z_{k-1}t_{i_k}^{\epsilon_k}) = x_kx_k^{-1}$ for every $k = 1, 2, \dots, n-1$. Similarly there exists $x_n \in S$ such that $z_0 = z_{n-1}t_{i_n}^{\epsilon_n}x_n$ with $(z_{n-1}t_{i_n}^{\epsilon_n})^{-1}(z_{n-1}t_{i_n}^{\epsilon_n}) = x_nx_n^{-1}$, since we have $\mathbf{t}(\mathbf{y}_n) = \mathbf{i}(\mathbf{y}_1)$. Then we have

$$z_k^{-1}z_k = x_k^{-1}t_{i_k}^{-\epsilon_k}z_{k-1}^{-1}z_{k-1}t_{i_k}^{\epsilon_k}x_k = x_k^{-1}x_kx_k^{-1}x_k = x_k^{-1}x_k$$

for every $k = 1, 2, \dots, n-1$. Therefore,

$$x_kx_k^{-1} = (z_{k-1}t_{i_k}^{\epsilon_k})^{-1}(z_{k-1}t_{i_k}^{\epsilon_k}) = t_{i_k}^{-\epsilon_k}z_{k-1}^{-1}z_{k-1}t_{i_k}^{\epsilon_k} = t_{i_k}^{-\epsilon_k}x_{k-1}^{-1}x_{k-1}t_{i_k}^{\epsilon_k} = \phi_{i_k}^{\epsilon_k}(x_{k-1}^{-1}x_{k-1}).$$

for every $k = 2, 3, \dots, n$ and also $x_1x_1^{-1} = \phi_{i_1}^{\epsilon_1}(z_0^{-1}z_0)$. We note that if $\epsilon_k = 1$, then we have $x_{k-1}^{-1}x_{k-1} \in E(A_{i_k})$ and that if $\epsilon_k = -1$, then we have $x_{k-1}^{-1}x_{k-1} \in E(B_{i_k})$. Using the equations $z_k = z_{k-1}t_{i_k}^{\epsilon_k}x_k$ $(k = 1, 2, \dots, n-1)$ and $z_0 = z_{n-1}t_{i_n}^{\epsilon_n}x_n$, we can inductively derive $z_{k-1} = z_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}x_2\cdots t_{i_{k-1}}^{\epsilon_{k-1}}x_{k-1}$, for every $k = 2, 3, \dots, n$, and hence, we have $z_0 = z_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}x_2\cdots t_{i_{n-1}}^{\epsilon_{n-1}}x_{n-1}t_{i_n}^{\epsilon_n}x_n$. Set $f = z_0^{-1}z_0$. Then we have $f = z_0^{-1}z_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}x_2\cdots t_{i_{n-1}}^{\epsilon_{n-1}}x_{n-1}t_{i_n}^{\epsilon_n}x_n$. Since $x_kx_k^{-1} = \phi_{i_k}^{\epsilon_k}(x_{k-1}^{-1}x_{k-1})$ holds for every $k = 2, 3, \dots, n$ and $x_1x_1^{-1} = \phi_{i_1}^{\epsilon_1}(z_0^{-1}z_0)$, f has a form

$$z_0^{-1}z_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}x_2\cdots t_{i_{n-1}}^{\epsilon_{n-1}}x_{n-1}t_{i_n}^{\epsilon_n}x_n.$$

Since $z_0^{-1}z_0 \in E(S^*) = E(S)$, the form $z_0^{-1}z_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}x_2\cdots t_{i_{n-1}}^{\epsilon_{n-1}}x_{n-1}t_{i_n}^{\epsilon_n}x_n$ is not reduced by Corollary 3.2.

We assume that $t_{i_k}^{\epsilon_k}x_kt_{i_{k+1}}^{\epsilon_{k+1}}$ is a pinch for some $k = 1, 2, \dots n-1$. There are two possible cases; (1) $i_k = i_{k+1}$, $\epsilon_k = -1$, $\epsilon_{k+1} = 1$ and $x_k \in A_{i_{k+1}}$ or (2) $i_k = i_{k+1}$, $\epsilon_k = 1$, $\epsilon_{k+1} = -1$ and $x_k \in B_{i_{k+1}}$. We now suppose that case (1) holds. Then we have $\mathbf{y}_k = z_{k-1}B_{i_k}$ and $\mathbf{y}_{k+1} = z_kA_{i_{k+1}}$. We show that $z_{k-1}t_{i_k}^{\epsilon_k}x_kA_{i_k} = z_{k-1}t_{i_k}^{\epsilon_k}A_{i_k}$. Note that $(z_{k-1}t_{i_k}^{\epsilon_k})^{-1}(z_{k-1}t_{i_k}^{\epsilon_k}) = x_kx_k^{-1}$ and so $z_{k-1}t_{i_k}^{\epsilon_k}x_k\mathscr{R}z_{k-1}t_{i_k}^{\epsilon_k}$. We have

$$\begin{aligned}(z_{k-1}t_{i_k}^{\epsilon_k})^{-1}(z_{k-1}t_{i_k}^{\epsilon_k}x_k) &= x_kx_k^{-1}x_k = x_k \in A_{i_{k+1}} = A_{i_k},\\ (z_{k-1}t_{i_k}^{\epsilon_k}x_k)^{-1}(z_{k-1}t_{i_k}^{\epsilon_k}x_k) &= x_k^{-1}(z_{k-1}t_{i_k}^{\epsilon_k})^{-1}(z_{k-1}t_{i_k}^{\epsilon_k})x_k\\ &= x_k^{-1}x_kx_k^{-1}x_k = x_k^{-1}x_k \in E(A_{i_k})\end{aligned}$$

and

$$(z_{k-1}t_{i_k}^{\epsilon_k})^{-1}(z_{k-1}t_{i_k}^{\epsilon_k}) = x_kx_k^{-1} \in E(A_{i_k}).$$

Hence, we have $z_{k-1}t_{i_k}^{\epsilon_k}x_kA_{i_k} = z_{k-1}t_{i_k}^{\epsilon_k}A_{i_k}$ by Lemma 4.2. Then we have

$$\begin{aligned}\mathbf{y}_{k+1} &= z_kA_{i_{k+1}} = z_{k-1}t_{i_k}^{\epsilon_k}x_kA_{i_{k+1}} = z_{k-1}t_{i_k}^{\epsilon_k}x_kA_{i_k}\\ &= z_{k-1}t_{i_k}^{\epsilon_k}A_{i_k} = z_{k-1}t_{i_k}^{-1}A_{i_k} = \overline{z_{k-1}B_{i_k}} = \overline{\mathbf{y}_k},\end{aligned}$$

which contradicts the assumption that there exists no backtracking in the circuit. In the case (2), we can similarly derive a contradiction. Consequently, there exists no circuit in the graph X. □

$$\begin{tikzcd} eS = x_0S \arrow[r, "\mathbf{y}_1"'] & x_0t_{i_1}^{\epsilon_1}S \arrow[r, "\mathbf{y}_2"'] & x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}S \arrow[r, "\mathbf{y}_3"'] & \cdots\cdots \arrow[r, "\mathbf{y}_n"'] & xS = x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}\cdots t_{i_n}^{\epsilon_n}S \end{tikzcd}$$

FIGURE 4. Path form eS to xS

4.2. Connected component For an idempotent $e \in E(S^*)$, we denote the connected component of X containing the vertex eS by X_e. We now describe the tree X_e.

LEMMA 4.5. *Let e be an idempotent in $E(S^*) = E(S)$. For $x \in S^*$, the vertex xS (the edge xA_i provided that $x^{-1}x \in E(A_i)$ or the edge xB_i provided that $x^{-1}x \in E(B_i)$) is in X_e if and only if $xx^{-1} = e$.*

PROOF. Suppose that $xx^{-1} = e$. Suppose that x has the normal form $x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}x_2\cdots t_{i_{n-1}}^{\epsilon_{n-1}}x_{n-1}t_{i_n}^{\epsilon_n}x_n$. By Lemma 2.6, we have $e = xx^{-1} = x_0x_0^{-1}$. By Lemma 4.2, $eS = x_0S$ since we have $e, x_0 \in S$. Similarly, we have $x_0t_{i_1}^{\epsilon_1}x_1\cdots t_{i_k}^{\epsilon_k}S = x_0t_{i_1}^{\epsilon_1}x_1\cdots t_{i_k}^{\epsilon_k}x_kS$ for every $k = 1, 2, \ldots, n$. For each $k = 2, 3, \ldots, n$, we define $\mathbf{y}_k$ to be the edge $x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}\cdots t_{i_{k-1}}^{\epsilon_{k-1}}x_{k-1}A_{i_k}$ if $\epsilon_k = 1$ and to be the edge $x_0t_{i_1}^{\epsilon_1}x_1t_{i_2}^{\epsilon_2}\cdots t_{i_{k-1}}^{\epsilon_{k-1}}x_{k-1}B_{i_k}$ if $\epsilon_k = -1$. We define $\mathbf{y}_1$ to be $x_0A_{i_1}$ or $x_0B_{i_1}$ according to $\epsilon_1 = \pm 1$. Then the path connecting eS and xS is given in Figure 4.

Note that $x_0t_{i_1}^{\epsilon_1}x_1\cdots t_{i_k}^{\epsilon_k}S = x_0t_{i_1}^{\epsilon_1}x_1\cdots t_{i_k}^{\epsilon_k}x_kS$ for every $k = 1, 2, \ldots, n$. Hence, xS is in the connected component X_e.

Let xA_i ($x^{-1}x \in E(A_i)$) be a positively oriented edge such that $xx^{-1} = e$. Suppose that $\mathbf{t}(xA_i) = zS$. Note that $x^{-1}x \leq e_i = t_it_i^{-1}$, $\mathbf{i}(xA_i) = xS$ and $\mathbf{t}(xA_i) = xt_iS$. Moreover, we have $(xt_i)(xt_i)^{-1} = xt_it_i^{-1}x^{-1} = xx^{-1} = e$ since $x^{-1}x \leq t_it_i^{-1}$. On the other hand, by Lemma 4.2, $zz^{-1} = (xt_i)(xt_i)^{-1}$ since we have $xt_iS = zS$. Hence, $zz^{-1} = e$. Similarly, we can show that for a negatively oriented edge xB_i such that $\mathbf{t}(xB_i) = zS$ we have $zz^{-1} = e$. It follows that if a vertex zS is connected to a vertex xS such that $xx^{-1} = e$ then we have $zz^{-1} = e$. Since X_e is connected, every vertex zS in X_e has the property that $zz^{-1} = e$.

We have shown the result for vertices. It is similarly shown that xA_i (or xB_i) is in X_e if and only if $xx^{-1} = e$. □

4.3. Action of a maximal subgroup We now consider a maximal subgroup S_e^* of S^*, where $e \in E(S^*)$. Take a vertex xS from X_e and w from S_e^*. We have $xx^{-1} = e$ by Lemma 4.5. Then we have

$$(wx)(wx)^{-1} = wxx^{-1}w^{-1} = wew^{-1} = ww^{-1} = e$$

(as $ww^{-1} = w^{-1}w = e$). This implies that wxS is in X_e. Similarly, we can show that wxA_i and wxB_i are in X_e if xA_i and xB_i are in X_e. Furthermore, we have

$\mathbf{i}(s\mathbf{y}) = s\mathbf{i}(\mathbf{y})$, $\mathbf{t}(s\mathbf{y}) = s\mathbf{t}(\mathbf{y})$ and $\overline{(s\mathbf{y})} = s\bar{\mathbf{y}}$ for $s \in S_e^*$ and $\mathbf{y} \in \mathrm{Edge}(X_e)$, that is to say, the group S_e^* acts on the tree X_e from the left. The action preserves the orientation of X_e: $S_e^* \mathrm{Edge}_+(X_e) \subset \mathrm{Edge}_+(X_e)$ and $S_e^* \mathrm{Edge}_-(X_e) \subset \mathrm{Edge}_-(X_e)$. Hence, S_e^* acts on X_e without inversion. Denote the *orbit* of xS (xA_i and xB_i) by $\mathrm{Orb}(xS)$ ($\mathrm{Orb}(xA_i)$ and $\mathrm{Orb}(xB_i)$). Denote the *stabilizer* of xS (xA_i and xB_i) by $\mathrm{Stab}(xS)$ ($\mathrm{Stab}(xA_i)$ and $\mathrm{Stab}(xB_i)$). We describe the orbits and the stabilizers of the vertices and the edges of X_e.

LEMMA 4.6. *Let x be an element in S^* such that $xx^{-1} = e$.*

(1) *We have*

$$\mathrm{Orb}(xS) = \{wS \mid ww^{-1} = e,\ x^{-1}x\mathscr{D}_S w^{-1}w\}.$$

(2) *If $x^{-1}x \in E(A_i)$, then we have*

$$\mathrm{Orb}(xA_i) = \{wA_i \mid ww^{-1} = e,\ w^{-1}w \in E(A_i),\ x^{-1}x\mathscr{D}_{A_i} w^{-1}w\}.$$

(3) *If $x^{-1}x \in E(B_i)$, then we have*

$$\mathrm{Orb}(xB_i) = \{wB_i \mid ww^{-1} = e,\ w^{-1}w \in E(B_i),\ x^{-1}x\mathscr{D}_{B_i} w^{-1}w\}.$$

Here $\mathscr{D}_S$, $\mathscr{D}_{A_i}$ and $\mathscr{D}_{B_i}$ denote Green's $\mathscr{D}$-relation of S, A_i and B_i, respectively.

PROOF. Take wS from $\mathrm{Orb}(xS)$. There exists z in S_e^* such that $z(xS) = wS$. Note that

$$(zx)^{-1}zx = x^{-1}z^{-1}zx = x^{-1}ex = x^{-1}xx^{-1}x = x^{-1}x.$$

By Lemma 4.3, we have $zx\mathscr{R}w$ and $(zx)^{-1}(zx)\mathscr{D}_S w^{-1}w$. Since $(zx)^{-1}zx = x^{-1}x$, we have $x^{-1}x\mathscr{D}_S w^{-1}w$. Moreover, we have

$$ww^{-1} = (zx)(zx)^{-1} = zxx^{-1}z^{-1} = zez^{-1} = e.$$

Hence, we have shown that $ww^{-1} = e$ and $x^{-1}x\mathscr{D}_S w^{-1}w$.

Conversely, we assume that $ww^{-1} = e$ and $x^{-1}x\mathscr{D}_S w^{-1}w$. There exists $a \in S$ such that $w^{-1}w\mathscr{R}_S a\mathscr{L}_S x^{-1}x$. Let $d = wax^{-1}$. Then we have

$$\begin{aligned} dd^{-1} &= wax^{-1}(wax^{-1})^{-1} = wax^{-1}xa^{-1}w^{-1} \\ &= waa^{-1}aa^{-1}w^{-1} = waa^{-1}w^{-1} = ww^{-1}ww^{-1} = ww^{-1} = e \end{aligned}$$

and

$$\begin{aligned} d^{-1}d &= (wax^{-1})^{-1}(wax^{-1}) = xa^{-1}w^{-1}wax^{-1} \\ &= xa^{-1}aa^{-1}ax^{-1} = xa^{-1}ax^{-1} = xx^{-1}xx^{-1} = e. \end{aligned}$$

Hence, $d \in S_e^*$. We now note that

$$(wa)(wa)^{-1} = waa^{-1}w^{-1} = ww^{-1}ww^{-1} = ww^{-1}.$$

Hence, $wa\mathscr{R}w$. Since $a \in S$ and $w^{-1}w \in E(S)$, we have $w^{-1}wa \in S$. By Lemma 4.2, we have $wS = waS$. On the other hand, we have

$$dxS = wax^{-1}xS = waa^{-1}aS = waS.$$

Hence, $wS = dxS$ and so $wS \in \mathrm{Orb}(xS)$. It follows that

$$\mathrm{Orb}(xS) = \{wS \mid ww^{-1} = e,\ x^{-1}x\mathscr{D}_S w^{-1}w\}.$$

We can similarly show the other two statements. □

LEMMA 4.7. (1) *For $x \in S^*$ with $xx^{-1} = e$ and $x^{-1}x = f$ we have*

$$\mathrm{Stab}(xS) = xH_S(f)x^{-1},$$

where $H_S(f)$ is the group $\mathscr{H}$-class of S containing f.

(2) *Suppose that xA_i (xB_i) is in the tree X_e. Then we have*

$$\mathrm{Stab}(xA_i) = xH_{A_i}(f)x^{-1} \qquad (\mathrm{Stab}(xB_i) = xH_{B_i}(f)x^{-1}),$$

where $H_{A_i}(f)$ ($H_{B_i}(f)$) is the group $\mathscr{H}$-class of A_i (B_i) containing f.

PROOF. We show that $\mathrm{Stab}(xS) = xH_S(f)x^{-1}$. Take an element z from $\mathrm{Stab}(xS)$. Then $zxS = xS$. Note that $zz^{-1} = z^{-1}z = e = xx^{-1}$. By Lemma 4.2, we have $zx\mathscr{R}x$ and $x^{-1}(zx) \in S$. Set $g = x^{-1}zx$. Then we have

$$gg^{-1} = x^{-1}zx(x^{-1}zx)^{-1} = x^{-1}zxx^{-1}z^{-1}x = x^{-1}zz^{-1}zz^{-1}x = x^{-1}xx^{-1}x = x^{-1}x = f$$

and

$$g^{-1}g = (x^{-1}zx)^{-1}x^{-1}zx = x^{-1}z^{-1}xx^{-1}zx = x^{-1}z^{-1}zz^{-1}zx = x^{-1}xx^{-1}x = x^{-1}x = f.$$

Hence, $g \in H_S(f)$. Then $z = xx^{-1}zxx^{-1} = xgx^{-1} \in xH_S(f)x^{-1}$. Thus, $\mathrm{Stab}(xS) \subset xH_S(f)x^{-1}$.

Conversely, we take an element v from $H_S(f)$. Then we have

$$(xvx^{-1})xS = xvx^{-1}xS = xvfS = xvS$$

(as $v \in H_S(f)$). We next show that $xv\mathscr{R}x$ and $x^{-1}(xv) \in S$. Note that

$$(xv)(xv)^{-1} = xvv^{-1}x^{-1} = xfx^{-1} = xx^{-1}xx^{-1} = xx^{-1}.$$

Hence, $xv\mathscr{R}x$. Since $x^{-1}xv = fv = v$ (as $v \in H_S(f)$), we have $x^{-1}xv \in S$ (as $v \in S$). By Lemma 4.2, we have $xvS = xS$. It follows that $(xvx^{-1})xS = xS$. Hence, $xvx^{-1} \in \mathrm{Stab}(xS)$. This shows that $xH_S(f)x^{-1} \subset \mathrm{Stab}(xS)$. Consequently, $\mathrm{Stab}(xS) = xH_S(f)x^{-1}$. Similarly it can be shown that $\mathrm{Stab}(xA_i) = xH_{A_i}(f)x^{-1}$ and $\mathrm{Stab}(xB_i) = xH_{B_i}(f)x^{-1}$. □

5. Presentation of a maximal subgroup

Using the action of S_e^* on X_e, we obtain a group presentation of S_e^*.

5.1. Graph of groups $S_e^* \backslash\backslash X_e = (K, Z_e)$ We describe the graph of groups induced from the action of S_e^* on the tree X_e. Let $S_e^* \backslash\backslash X_e = (K, Z_e)$ be the graph of groups obtained from the action of S_e^* on X_e in the Bass-Serre theory. We recall the construction of (K, Z_e).

(1) Graph Z_e The graph Z_e is formed by the set of vertices

$$\mathrm{Vert}(Z_e) = \{\mathrm{Orb}(xS) \mid xx^{-1} = e\}$$

and the set of edges

$$\mathrm{Edge}(Z_e) = \mathrm{Edge}_+(Z_e) \cup \mathrm{Edge}_-(Z_e),$$

where

$$\mathrm{Edge}_+(Z_e) = \{\mathrm{Orb}(xA_i) \mid xx^{-1} = e,\ x^{-1}x \in E(A_i),\ i \in I\}$$

is the set of positively oriented edges and

$$\mathrm{Edge}_-(Z_e) = \{\mathrm{Orb}(xB_i) \mid xx^{-1} = e,\ x^{-1}x \in E(B_i),\ i \in I\}$$

is the set of negatively oriented edges. The initial vertex and the terminal vertex of $\mathrm{Orb}(xA_i)$ are $\mathrm{Orb}(xS)$ and $\mathrm{Orb}(xt_iS)$, respectively. The initial vertex and the terminal vertex of $\mathrm{Orb}(xB_i)$ are $\mathrm{Orb}(xS)$ and $\mathrm{Orb}(xt_i^{-1}S)$, respectively. The inverse edge of $\mathrm{Orb}(xA_i)$ is $\mathrm{Orb}(xt_iB_i)$ and the inverse edge of $\mathrm{Orb}(xB_i)$ is $\mathrm{Orb}(xt_i^{-1}A_i)$. It is easy to verify that Z_e forms a graph.

(2) Groups K Let T be a maximal subtree of Z_e, that is, a spanning tree of Z_e. There exists a lifting $j : T \to X_e$ because T is a tree (see [14]). We extend j to all edges of Z_e so that we can have $j(\mathbf{i}(\mathbf{y})) = \mathbf{i}(j(\mathbf{y}))$ for every $\mathbf{y} \in \mathrm{Edge}_+(Z_e)$. Then j is no longer a graph morphism. The groups associated with $\mathrm{Orb}(xS)$, $\mathrm{Orb}(xA_i)$ and $\mathrm{Orb}(xB_i)$ $(i \in I)$ are defined by

$$K_{\mathrm{Orb}(xS)} = \mathrm{Stab}(j(\mathrm{Orb}(xS))), \qquad K_{\mathrm{Orb}(xA_i)} = \mathrm{Stab}(j(\mathrm{Orb}(xA_i)))$$

and

$$K_{\mathrm{Orb}(xB_i)} = \mathrm{Stab}(j(\mathrm{Orb}(xB_i))).$$

If $s\mathbf{y} = \mathbf{y}$, then $s\overline{\mathbf{y}} = \overline{\mathbf{y}}$ for every $\mathbf{y} \in \mathrm{Edge}(X_e)$. Therefore, $K_{\overline{\mathrm{Orb}(xA_i)}} = K_{\mathrm{Orb}(xA_i)}$ and $K_{\overline{\mathrm{Orb}(xB_i)}} = K_{\mathrm{Orb}(xB_i)}$.

(3) Monomorphisms Suppose that $\mathbf{y}$ is a positively oriented edge of Z_e. Then $\mathbf{y} = \mathrm{Orb}(xA_i)$ for some $x \in S^*$ and $i \in I$ such that $xx^{-1} = e$ and $x^{-1}x \in E(A_i)$. We define group isomorphisms

$$\sigma_{\mathbf{y}} : K_{\mathbf{y}} \to K_{\mathbf{i}(\mathbf{y})} \quad \text{and} \quad \tau_{\mathbf{y}} : K_{\mathbf{y}} \to K_{\mathbf{t}(\mathbf{y})}.$$

https://doi.org/10.1017/S1446788700002639 Published online by Cambridge University Press

Suppose that $j(\mathbf{y}) = xA_i$. Then $j(\mathbf{i}(\mathbf{y})) = \mathbf{i}(j(\mathbf{y})) = xS$. We have

$$\begin{aligned} K_{\mathbf{i}(\mathbf{y})} &= \mathrm{Stab}(j(\mathbf{i}(\mathbf{y}))) = \mathrm{Stab}(\mathbf{i}(j(\mathbf{y}))) \\ &= \mathrm{Stab}(\mathbf{i}(xA_i)) = \mathrm{Stab}(xS) = xH_S(x^{-1}x)x^{-1}. \end{aligned}$$

We also have

$$K_{\mathbf{y}} = \mathrm{Stab}(j(\mathbf{y})) = \mathrm{Stab}(xA_i) = xH_{A_i}(x^{-1}x)x^{-1}.$$

Note that $xH_{A_i}(x^{-1}x)x^{-1} \subset xH_S(x^{-1}x)x^{-1}$. Then $\sigma_{\mathbf{y}} : K_{\mathbf{y}} \to K_{\mathbf{i}(\mathbf{y})}$ is defined to be the inclusion mapping.

Since we have

$$\mathrm{Orb}(j(\mathbf{t}(\mathbf{y}))) = \mathrm{Orb}(\mathbf{t}(j(\mathbf{y}))) = \mathrm{Orb}(\mathbf{t}(xA_i)) = \mathrm{Orb}(xt_iS),$$

we can choose an element $\gamma_{\mathbf{y}}$ from S_e^* such that

$$j(\mathbf{t}(\mathbf{y})) = \gamma_{\mathbf{y}} x t_i S. \tag{5.1}$$

Then we have

$$\begin{aligned} K_{\mathbf{t}(\mathbf{y})} &= \mathrm{Stab}(j(\mathbf{t}(\mathbf{y}))) = \mathrm{Stab}(\gamma_{\mathbf{y}} x t_i S) = (\gamma_{\mathbf{y}} x t_i) H_S((\gamma_{\mathbf{y}} x t_i)^{-1}(\gamma_{\mathbf{y}} x t_i))(\gamma_{\mathbf{y}} x t_i)^{-1} \\ &= \gamma_{\mathbf{y}} x t_i H_S(t_i^{-1}x^{-1}\gamma_{\mathbf{y}}^{-1}\gamma_{\mathbf{y}} x t_i) t_i^{-1}x^{-1}\gamma_{\mathbf{y}}^{-1} = \gamma_{\mathbf{y}} x t_i H_S(\phi_i(x^{-1}x)) t_i^{-1}x^{-1}\gamma_{\mathbf{y}}^{-1} \end{aligned}$$

because

$$t_i^{-1}x^{-1}\gamma_{\mathbf{y}}^{-1}\gamma_{\mathbf{y}} x t_i = t_i^{-1}x^{-1}ext_i = t_i^{-1}x^{-1}xx^{-1}xt_i = t_i^{-1}x^{-1}xt_i = \phi_i(x^{-1}x)$$

(as $x^{-1}x \in E(A_i) \subset A_i$). Then $\tau_{\mathbf{y}} : K_{\mathbf{y}} \to K_{\mathbf{t}(\mathbf{y})}$ is defined by

$$\tau_{\mathbf{y}}(s) = \gamma_{\mathbf{y}} s \gamma_{\mathbf{y}}^{-1} \tag{5.2}$$

for $s \in K_{\mathbf{y}}$. For $\mathbf{y} \in \mathrm{Edge}_-(Z_e)$, we define $\sigma_{\mathbf{y}}$ and $\tau_{\mathbf{y}}$ by $\sigma_{\mathbf{y}} = \tau_{\overline{\mathbf{y}}}$ and $\tau_{\mathbf{y}} = \sigma_{\overline{\mathbf{y}}}$.

5.2. Graph of groups (G, Y_e) We construct a graph of groups (G, Y_e) associated with S, A_i and B_i $(i \in I)$ for each idempotent $e \in E(S)$ using Green's $\mathscr{D}$-relation. We also show that (G, Y_e) is conjugate isomorphic to the graph of groups $S_e^* \backslash\backslash X_e = (K, Z_e)$. Then we can conclude that S_e^* is isomorphic to the fundamental group of (G, Y_e). Let us start to define the graph of groups (G, Y_e).

(1) Graph Y Let $\mathrm{Vert}(Y)$ be the set of $\mathscr{D}$-classes of S, $\mathrm{Edge}_+(Y)$ be the set of $\mathscr{D}$-classes of A_i for all $i \in I$ and $\mathrm{Edge}_-(Y)$ be the set of $\mathscr{D}$-classes of B_i for all $i \in I$. We make a convention that $\mathrm{Edge}_+(Y)$ and $\mathrm{Edge}_-(Y)$ are disjoint although A_i may be equal to B_j for some $i, j \in I$. We also make a convention that for distinct $i, j \in I$ the set of $\mathscr{D}$-classes of A_i is disjoint from the set of $\mathscr{D}$-classes of A_j even though A_i may coincide with A_j. We make a similar convention for the set of $\mathscr{D}$-classes of B_i's. The set of edges $\mathrm{Edge}(Y)$ is the union of the set $\mathrm{Edge}_+(Y)$ of positively oriented edges and the set $\mathrm{Edge}_-(Y)$ of negatively oriented edges. Suppose that $D_{A_i} \in \mathrm{Edge}_+(Y)$, that is,

D_{A_i} is a $\mathscr{D}$-class of A_i for some $i \in I$. Then the initial vertex $\mathbf{i}(D_{A_i})$ is the $\mathscr{D}$-class of S including D_{A_i}, the terminal vertex $\mathbf{t}(D_{A_i})$ is the $\mathscr{D}$-class of S including $\phi_i(D_{A_i})$ and the inverse edge $\overline{D_{A_i}}$ is $\phi_i(D_{A_i})$. Note that $\phi_i(D_{A_i})$ is a $\mathscr{D}$-class of B_i. Suppose that $D_{B_i} \in \mathrm{Edge}_-(Y)$, that is, D_{B_i} is a $\mathscr{D}$-class of B_i for some $i \in I$. Then the initial vertex $\mathbf{i}(D_{B_i})$ is the $\mathscr{D}$-class of S including D_{B_i}, the terminal vertex $\mathbf{t}(D_{B_i})$ is the $\mathscr{D}$-class of S including $\phi_i^{-1}(D_{B_i})$ and the inverse edge $\overline{D_B}$ is $\phi_i^{-1}(D_{B_i})$. Note that $\phi_i^{-1}(D_{B_i})$ is a $\mathscr{D}$-class of A_i. It is routine to verify that Y forms a graph in the sense of Serre. We remark that Y is not necessarily connected.

(2) Groups G For each $D_S \in \mathrm{Vert}(Y)$, that is, D_S is a $\mathscr{D}$-class of S, we choose a group $\mathscr{H}$-class of S included in D_S and denote it by G_{D_S}. Similarly, for each $D_{A_i} \in \mathrm{Edge}_+(Y)$, we choose a group $\mathscr{H}$ class of A_i included in D_{A_i} and denote it by $G_{D_{A_i}}$. For each $D_{B_i} \in \mathrm{Edge}_-(Y)$, we set $G_{D_{B_i}} = G_{\phi_i^{-1}(D_{B_i})} = G_{\overline{D_{B_i}}}$.

(3) Monomorphisms Suppose $D_{A_i} \in \mathrm{Edge}_+(Y)$. There exists a unique $D_S \in \mathrm{Vert}(Y)$ such that $D_{A_i} \subset D_S$. Let K be a group $\mathscr{H}$-class of S including $G_{D_{A_i}}$. Then K and G_{D_S} are group $\mathscr{H}$-classes of S included in D_S. By Green's lemma there exists $n \in D_S$ such that $nKn^{-1} = G_{D_S}$. We choose such an element n from D_S and fix it. Then $\sigma_{D_{A_i}} : G_{D_{A_i}} \to G_{D_S}$ is defined by

$$\sigma_{D_{A_i}}(h) = nhn^{-1} \tag{5.3}$$

for $h \in G_{D_{A_i}}$. We note that $G_{D_S} = G_{\mathbf{i}(D_{A_i})}$ as $D_S = \mathbf{i}(D_{A_i})$. We now define $\tau_{D_{A_i}} : G_{D_{A_i}} \to G_{\mathbf{t}(D_{A_i})}$. There exists a unique $D_S \in \mathrm{Vert}(Y)$ such that $\phi_i(D_{A_i}) \subset D_S$. Let K be the group $\mathscr{H}$-class of S including $\phi_i(G_{D_{A_i}})$. Then K and G_{D_S} are group $\mathscr{H}$-classes included in D_S. By Green's lemma there exists $n \in D_S$ such that

$$nKn^{-1} = G_{D_S}. \tag{5.4}$$

We choose such an element n from D_S and fix it. Then $\tau_{D_{A_i}} : G_{D_{A_i}} \to G_{D_S}$ is defined by

$$\tau_{D_{A_i}}(h) = n\phi_i(h)n^{-1} \tag{5.5}$$

for $h \in G_{D_{A_i}}$. We note that $G_{D_S} = G_{\mathbf{t}(D_{A_i})}$ as $D_S = \mathbf{t}(D_{A_i})$. For $D_{B_i} \in \mathrm{Edge}_-(Y)$, $\sigma_{D_{B_i}}$ and $\tau_{D_{B_i}}$ are defined by $\sigma_{D_{B_i}} = \tau_{\overline{D_{B_i}}}$ and $\tau_{D_{B_i}} = \sigma_{\overline{D_{B_i}}}$.

(4) Connected component Y_e For $e \in E(S^*) = E(S)$, let Y_e be the connected component of Y containing the $\mathscr{D}$-class of S containing e. We denote the restriction of (G, Y) to Y_e by (G, Y_e). Then (G, Y_e) is a connected graph of groups.

5.3. Conjugate isomorphism We prove that (G, Y_e) is conjugate isomorphic to (K, Z_e). The reader is referred to [3] for the definition of a conjugate isomorphism of graphs of groups.

THEOREM 5.1. *The graph of groups (K, Z_e) is conjugate isomorphic to the graph of groups (G, Y_e).*

PROOF. We define a graph morphism $\Phi : Z_e \to Y_e$ by

$$\Phi(\mathrm{Orb}(xS)) = D_S(x^{-1}x), \qquad \Phi(\mathrm{Orb}(xA_i)) = D_{A_i}(x^{-1}x)$$

and

$$\Phi(\mathrm{Orb}(xB_i)) = D_{B_i}(x^{-1}x),$$

where $D_S(x^{-1}x)$ ($D_{A_i}(x^{-1}x)$ and $D_{B_i}(x^{-1}x)$) is the $\mathscr{D}$-class of S (A_i and B_i) containing $x^{-1}x$. By Lemma 4.6, Φ is a well-defined bijection. Note that

$$\Phi(\mathbf{i}(\mathrm{Orb}(xA_i))) = \Phi(\mathrm{Orb}(xS)) = D_S(x^{-1}x) = \mathbf{i}(D_{A_i}(x^{-1}x)) = \mathbf{i}(\Phi(\mathrm{Orb}(xA_i))).$$

Therefore, $\Phi(\mathbf{i}(\mathbf{y})) = \mathbf{i}(\Phi(\mathbf{y}))$ for every $\mathbf{y} \in \mathrm{Edge}_+(Z_e)$. Similarly we can show that $\Phi(\mathbf{t}(\mathbf{y})) = \mathbf{t}(\Phi(\mathbf{y}))$ for every $\mathbf{y} \in \mathrm{Edge}_+(Z_e)$ and that $\Phi(\mathbf{i}(\mathbf{y})) = \mathbf{i}(\Phi(\mathbf{y}))$ and $\Phi(\mathbf{t}(\mathbf{y})) = \mathbf{t}(\Phi(\mathbf{y}))$ for every $\mathbf{y} \in \mathrm{Edge}_-(Z_e)$. It is also easy to see that $\Phi(\overline{\mathbf{y}}) = \overline{(\Phi(\mathbf{y}))}$ for every $\mathbf{y} \in \mathrm{Edge}(Z_e)$. Thus Φ is a graph isomorphism of Z_e onto Y_e.

We define isomorphisms between the vertex groups. Let $\mathbf{v}$ be a vertex in Z_e. Suppose $j(\mathbf{v}) = xS$. Then $\mathbf{v} = \mathrm{Orb}(j(\mathbf{v})) = \mathrm{Orb}(xS)$ and so $\Phi(\mathbf{v}) = \Phi(\mathrm{Orb}(xS)) = D_S(x^{-1}x)$. Then we have

$$K_{\mathbf{v}} = \mathrm{Stab}(j(\mathbf{v})) = \mathrm{Stab}(xS) = xH_S(x^{-1}x)x^{-1}.$$

Since $H_S(x^{-1}x)$ and $G_{D_S(x^{-1}x)}$ are group $\mathscr{H}$-classes of S included in $D_S(x^{-1}x)$, there exists $m_x \in D_S(x^{-1}x)$ such that

$$m_x H_S(x^{-1}x)m_x^{-1} = G_{D_S(x^{-1}x)} \tag{5.6}$$

by Green's lemma. We choose such m_x and fix it. Then a mapping $\Phi_{\mathbf{v}} : K_{\mathbf{v}} \to G_{\Phi(\mathbf{v})} = G_{D_S(x^{-1}x)}$ is defined by

$$\Phi_{\mathbf{v}}(s) = m_x x^{-1} s x m_x^{-1} \tag{5.7}$$

for $s \in K_{\mathbf{v}} = xH_S(x^{-1}x)x^{-1}$. We show that $\Phi_{\mathbf{v}}$ is well defined. Since

$$x^{-1}sx \in x^{-1}xH_S(x^{-1}x)x^{-1}x = H_S(x^{-1}x),$$

we have

$$\Phi_{\mathbf{v}}(s) \in m_x H_S(xx^{-1})m_x^{-1} = G_{D_S(x^{-1}x)}.$$

Note that the choice of m_x depends on x. We show that $\Phi_{\mathbf{v}}$ can be defined so that any other choice of the coset representative z for xS can have an element m_z in $D_S(z^{-1}z)$ such that $m_z z^{-1} s z m_z^{-1} = m_x x^{-1} s x m_x^{-1}$ for $s \in K_{\mathbf{v}}$. Suppose that $xS = zS$. We define m_z to be $m_x x^{-1} z$. Then m_z belongs to S since m_x and $x^{-1}z$ belong to S.

Then $m_z \in D_S(z^{-1}z) = D_S(x^{-1}x)$ and $m_z z^{-1}szm_z^{-1} = m_x x^{-1}zz^{-1}sz(m_x x^{-1}z)^{-1} = m_x x^{-1}sxm_x^{-1}$. Therefore, $\Phi_{\mathbf{v}}$ is well defined. Clearly $\Phi_{\mathbf{v}}$ is an isomorphism.

We next define isomorphisms between the edge groups. Suppose $\mathbf{y}$ is in $\mathrm{Edge}_+(Z_e)$ and $j(\mathbf{y}) = xA_i$. Then we have

$$\mathbf{y} = \mathrm{Orb}(j(\mathbf{y})) = \mathrm{Orb}(xA_i)$$

and so

$$\Phi(\mathbf{y}) = \Phi(\mathrm{Orb}(xA_i)) = D_{A_i}(x^{-1}x).$$

We also have

$$K_{\mathbf{y}} = \mathrm{Stab}(j(\mathbf{y})) = \mathrm{Stab}(xA_i) = xH_{A_i}(x^{-1}x)x^{-1}.$$

Since $H_{A_i}(x^{-1}x)$ and $G_{D_{A_i}(x^{-1}x)}$ are group $\mathscr{H}$-classes of A_i included in $D_{A_i}(x^{-1}x)$, there exists $a_x \in D_{A_i}(x^{-1}x)$ such that

$$a_x H_{A_i}(x^{-1}x)a_x^{-1} = G_{D_{A_i}(x^{-1}x)} \tag{5.8}$$

by Green's lemma. Then a mapping $\Phi_{\mathbf{y}} : K_{\mathbf{y}} \to G_{\Phi(\mathbf{y})}$ is defined by

$$\Phi_{\mathbf{y}}(s) = a_x x^{-1}sxa_x^{-1} \tag{5.9}$$

for $s \in K_{\mathbf{y}} = x^{-1}H_A(xx^{-1})x$. For any z such that $xA_i = zA_i$ we can choose a_z from $D_{A_i}(z^{-1}z) = D_{A_i}(x^{-1}x)$ such that $a_z z^{-1}sza_z^{-1} = a_x x^{-1}sxa_x^{-1}$. Therefore $\Phi_{\mathbf{y}}$ is well defined. Clearly $\Phi_{\mathbf{y}}$ is an isomorphism. For any edge $\mathbf{y}$ in $\mathrm{Edge}_-(Z_e)$, we define $\Phi_{\overline{\mathbf{y}}}$ to be $\Phi_{\mathbf{y}}$.

We now prove that Φ is a conjugate isomorphism of (K, Z_e) onto (G, Y_e). Recall that Φ is a conjugate isomorphism if for every edge $\mathbf{y} \in \mathrm{Edge}(Z_e)$ the diagram

$$\begin{tikzcd}
K_{\mathbf{y}} \arrow[r, "\tau_{\mathbf{y}}"] \arrow[d, "\Phi_{\mathbf{y}}"'] & K_{\mathfrak{t}(\mathbf{y})} \arrow[d, "\Phi_{\mathfrak{t}(\mathbf{y})}"] \\
G_{\Phi(\mathbf{y})} \arrow[r, "\tau_{\Phi(\mathbf{y})}"'] & G_{\Phi(\mathfrak{t}(\mathbf{y}))}
\end{tikzcd}$$

is commutative up to conjugation by an element of $G_{\mathfrak{t}(\Phi(\mathbf{y}))} = G_{\Phi(\mathfrak{t}(\mathbf{y}))}$.

There are two cases: (1) $\mathbf{y} \in \mathrm{Edge}_+(Z_e)$ and (2) $\mathbf{y} \in \mathrm{Edge}_-(Z_e)$. We prove the first case and omit the second case.

Suppose that $\mathbf{y}$ is in $\mathrm{Edge}_+(Z_e)$ and $j(\mathbf{y}) = xA_i$. Then $\mathbf{y} = \mathrm{Orb}(j(\mathbf{y})) = \mathrm{Orb}(xA_i)$. We have

$$K_{\mathbf{y}} = \mathrm{Stab}(j(\mathbf{y})) = \mathrm{Stab}(xA_i) = xH_{A_i}(x^{-1}x)x^{-1}.$$

https://doi.org/10.1017/S1446788700002639 Published online by Cambridge University Press

Recall that $\gamma_{\mathbf{y}}$ satisfies the equation $j(\mathfrak{t}(\mathbf{y})) = \gamma_{\mathbf{y}} x t_i S$ by (5.1). Then we have

$$\begin{aligned} K_{\mathfrak{t}(\mathbf{y})} &= \mathrm{Stab}(j(\mathfrak{t}(\mathbf{y}))) = \mathrm{Stab}(\gamma_{\mathbf{y}} x t_i S) \\ &= \gamma_{\mathbf{y}} x t_i H_S((\gamma_{\mathbf{y}} x t_i)^{-1}(\gamma_{\mathbf{y}} x t_i))(\gamma_{\mathbf{y}} x t_i)^{-1} = \gamma_{\mathbf{y}} x t_i H_S(\phi_i(x^{-1}x))(\gamma_{\mathbf{y}} x t_i)^{-1}, \end{aligned}$$

since

$$(\gamma_{\mathbf{y}} x t_i)^{-1}(\gamma_{\mathbf{y}} x t_i) = t_i^{-1} x^{-1} \gamma_{\mathbf{y}}^{-1} \gamma_{\mathbf{y}} x t = t_i^{-1} x^{-1} x t = \phi_i(x^{-1}x).$$

Then $\tau_{\mathbf{y}} : K_{\mathbf{y}} \to K_{\mathfrak{t}(\mathbf{y})}$ is defined to be the mapping $s \mapsto \gamma_{\mathbf{y}} s \gamma_{\mathbf{y}}^{-1}$ for $s \in K_{\mathbf{y}}$ by (5.2) and $\Phi_{\mathbf{y}} : K_{\mathbf{y}} \to G_{\Phi(\mathbf{y})}$ is defined to be the mapping $s \mapsto a x^{-1} s x a^{-1}$ for $s \in K_{\mathbf{y}}$ by (5.9). Note that $a = a_x$ is the element of $D_{A_i}(x^{-1}x)$ defined in (5.8). Hence, we have $a_x H_{A_i}(x^{-1}x) a_x^{-1} = G_{D_{A_i}(x^{-1}x)}$. We recall that $\Phi_{\mathfrak{t}(\mathbf{y})} : K_{\mathfrak{t}(\mathbf{y})} \to G_{\Phi(\mathfrak{t}(\mathbf{y}))}$ is defined to be the mapping $s \mapsto m(\gamma_{\mathbf{y}} x t_i)^{-1} s (\gamma_{\mathbf{y}} x t_i) m^{-1}$ for $s \in K_{\mathfrak{t}(\mathbf{y})}$ by (5.7). Note that $m = m_{\gamma_{\mathbf{y}} x t_i}$ is the element of $D_S(\phi_i(x^{-1}x))$ defined in (5.6). Hence, we have $m H_S(\phi_i(x^{-1}x)) m^{-1} = G_{D_S(\phi_i(x^{-1}x))}$. Note that

$$\Phi(\mathfrak{t}(\mathbf{y})) = \Phi(\mathrm{Orb}(x t_i S)) = D_S((x t_i)^{-1}(x t_i)) = D_S(t_i^{-1} x^{-1} x t_i) = D_S(\phi_i(x^{-1}x)).$$

We also recall that $\tau_{\Phi(\mathbf{y})} : G_{\Phi(\mathbf{y})} \to G_{\mathfrak{t}(\Phi(\mathbf{y}))}$ is defined to be the mapping $s \mapsto n\phi_i(s)n^{-1}$ for $s \in G_{\Phi(\mathbf{y})}$ by (5.5). Note that n is the element of $D_S(\phi_i(x^{-1}x))$ defined by (5.4). Hence, we have $nKn^{-1} = G_{D_S(\phi(x^{-1}x))}$, where K is the group $\mathscr{H}$-class of S including $\phi_i(G_{D_{A_i}(x^{-1}x)})$. See Figure 5 for the $\mathscr{D}$-structure and the location of the elements m, n, a.

Take an element s from $H_{A_i}(x^{-1}x)$. Then $xsx^{-1} \in x H_{A_i}(x^{-1}x) x^{-1} = K_{\mathbf{y}}$. We have

$$\begin{aligned} \Phi_{\mathfrak{t}(\mathbf{y})}(\tau_{\mathbf{y}}(xsx^{-1})) &= \Phi_{\mathfrak{t}(\mathbf{y})}(\gamma_{\mathbf{y}} x s x^{-1} \gamma_{\mathbf{y}}^{-1}) = m(\gamma_{\mathbf{y}} x t_i)^{-1}(\gamma_{\mathbf{y}} x s x^{-1} \gamma_{\mathbf{y}}^{-1})(\gamma_{\mathbf{y}} x t_i) m^{-1} \\ &= m t_i^{-1} x^{-1} x s x^{-1} x t_i m^{-1} = m t_i^{-1} s t_i m^{-1} = m\phi_i(s)m^{-1}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \tau_{\Phi(\mathbf{y})}(\Phi_{\mathbf{y}}(xsx^{-1})) &= \tau_{\Phi(\mathbf{y})}(a x^{-1} x s x^{-1} x a^{-1}) = \tau_{\Phi(\mathbf{y})}(asa^{-1}) \\ &= n\phi_i(asa^{-1})n^{-1} = n\phi_i(a)\phi_i(s)(n\phi_i(a))^{-1}. \end{aligned}$$

Then we have

$$\begin{aligned} & n\phi_i(a)m^{-1}\Phi_{\mathfrak{t}(\mathbf{y})}(\tau_{\mathbf{y}}(xsx^{-1}))(n\phi_i(a)m^{-1})^{-1} \\ &\quad = n\phi_i(a)m^{-1}(m\phi_i(s)m^{-1})(n\phi_i(a)m^{-1})^{-1} \\ &\quad = n\phi_i(a)\phi_i(s)(n\phi_i(a))^{-1} = \tau_{\Phi(\mathbf{y})}(\Phi_{\mathbf{y}}(xsx^{-1})). \end{aligned}$$

We prove that $n\phi_i(a)m^{-1} \in G_{D_S(\phi_i(x^{-1}x))}$. By our choice of n and a, the element $n^{-1}n$ is the identity of K and the element $\phi_i(a)\phi_i(a^{-1})$ is the identity of $\phi_i(G_{D_{A_i}(x^{-1}x)})$. Since $\phi_i(G_{D_{A_i}(x^{-1}x)}) \subset K$, we have $n^{-1}n = \phi_i(a)\phi_i(a^{-1})$. Since

$a^{-1}a = x^{-1}x$, $\phi_i(a^{-1})\phi_i(a) = \phi_i(x^{-1}x)$. On the other hand, by the choice of m, we have $\phi_i(x^{-1}x)\mathscr{R}m^{-1}$. Hence, we have $m^{-1}m = \phi_i(x^{-1}x) = \phi_i(a^{-1}a)$. Then we have

$$\begin{aligned}(n\phi_i(a)m^{-1})^{-1}(n\phi_i(a)m^{-1}) &= m\phi_i(a^{-1})n^{-1}n\phi_i(a)m^{-1}\\ &= m\phi_i(a^{-1})\phi_i(a)m^{-1} = mm^{-1}mm^{-1} = mm^{-1}\end{aligned}$$

and so $n\phi_i(a)m^{-1}\mathscr{L}mm^{-1}$. By our choice of n and m, both nn^{-1} and mm^{-1} are the identity of $G_{D_S(\phi_i(x^{-1}x))}$. Therefore, $nn^{-1} = mm^{-1}$, that is, $n\mathscr{R}m$. Then

$$\begin{aligned}(n\phi_i(a)m^{-1})(n\phi_i(a)m^{-1})^{-1} &= n\phi_i(a)m^{-1}m\phi_i(a^{-1})n^{-1}\\ &= n\phi_i(a)\phi_i(a^{-1})\phi_i(a)\phi_i(a^{-1})n^{-1}\\ &= n\phi_i(a)\phi_i(a^{-1})n^{-1} = nn^{-1}nn^{-1} = nn^{-1} = mm^{-1}.\end{aligned}$$

It follows that $n\phi_i(a)m^{-1}\mathscr{R}mm^{-1}$. Hence, $n\phi_i(a)m^{-1}\mathscr{H}mm^{-1}$. This implies that $n\phi_i(a)m^{-1}$ is in the $\mathscr{H}$-class of mm^{-1}, that is, $G_{D_S(\phi_i(x^{-1}x))}$. Recall that $G_{D_S(\phi_i(x^{-1}x))} = G_{\mathfrak{t}(\Phi(\mathfrak{y}))}$. Consequently, for every $s \in H_{A_i}(x^{-1}x)$, the element $\Phi_{\mathfrak{t}(\mathfrak{y})}(\tau_{\mathfrak{y}}(xsx^{-1}))$ is conjugate to $\tau_{\Phi(\mathfrak{y})}(\Phi(xsx^{-1}))$ by the element $n\phi_i(a)m^{-1}$ of the group $G_{\mathfrak{t}(\Phi(\mathfrak{y}))}$. □

THEOREM 5.2. *S_e^* is isomorphic to the fundamental group $\pi(G, Y_e)$.*

PROOF. By the Bass-Serre theory, the group S_e^* is isomorphic to $\pi(K, Z_e)$. On the other hand, $\pi(G, Y_e)$ is isomorphic to $\pi(K, Z_e)$ since (G, Y_e) is conjugate isomorphic to (K, Z_e) (see [3, Lemma 21, page 202]). Therefore, S_e^* is isomorphic to $\pi(G, Y_e)$. □

We remark that the group action of the maximal subgroup on the corresponding tree can be integrated into an inverse semigroup action of S^* on the forest X. The Bass-Serre theory can be generalized to the class of inverse monoids acting on ordered forests; an inverse monoid acting on an ordered forest is characterized as a fundamental inverse monoid of a certain graph of inverse monoids and vice versa. See [12] for the inverse monoid actions on ordered forests, fundamental inverse monoids and graphs of inverse monoids.

6. Examples

EXAMPLE 6.1. Let S be an inverse semigroup. Let A_i be an inverse subsemigroup such that $e_i \in A_i \subset e_iSe_i$ and $E(A_i) = E(e_iSe_i)$ for every $i \in I$, where $e_i \in E(S)$. Suppose that B_i is an inverse subsemigroup of S such that $e_i \in B_i \subset e_iSe_i$ and $E(B_i) = E(e_iSe_i)$ for every $i \in I$. Let ϕ_i be an isomorphism of A_i onto B_i such that $\phi_i(f)\mathscr{D}f$ for all $f \in E(A_i)$ for every $i \in I$. It is easy to see that the graph of groups

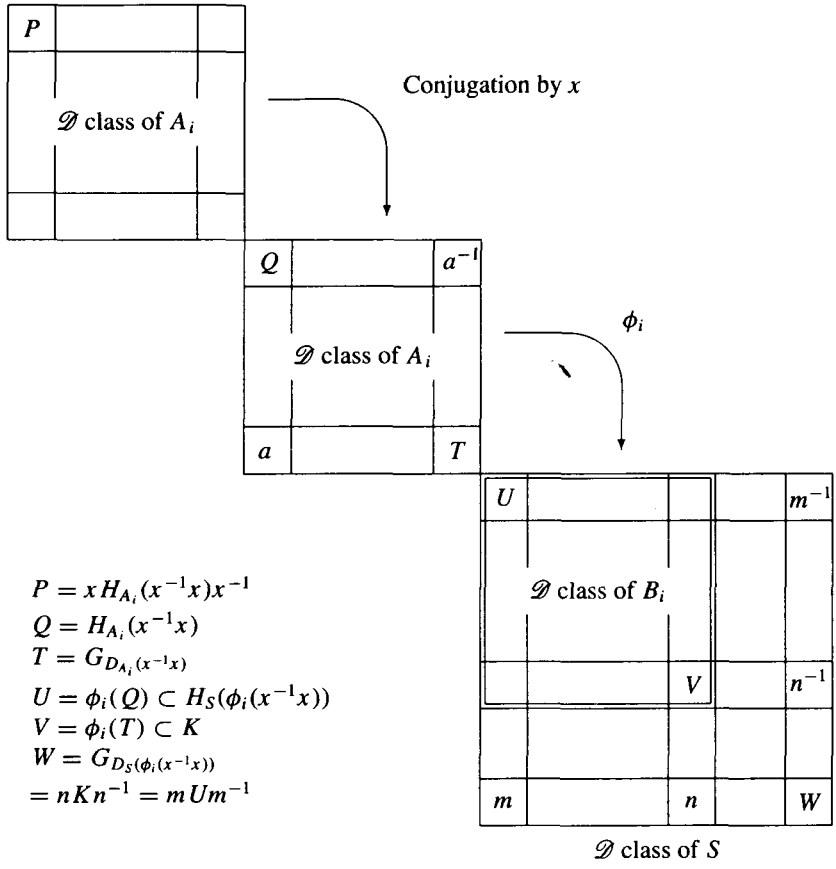

FIGURE 5. $\mathscr{D}$-structure of S, A_i and B_i

(G, Y_e) has the property that Y_e is a bouquet, that is, a graph consisting of only one vertex and several edges. It follows that each maximal subgroup of S^* is an HNN extension of the maximal subgroup of S.

EXAMPLE 6.2. Let B be the bicyclic monoid. Let ϕ be the identity mapping on $E(B)$. Let B^* be the full HNN extension of B associated with ϕ. We can conclude that the group $\mathscr{H}$-class B_1^* of B^* is a free group of infinite rank since the graph of groups defined from B is just a bouquet with infinitely many edges such that every vertex group is trivial and the fundamental group of such a graph of groups is a free group generated by infinite generators.

EXAMPLE 6.3. Let S be a strong semilattice $\mathscr{S}(E, S_e, \rho_{e_m}^{e_n})$, where E is the ω-chain,

https://doi.org/10.1017/S1446788700002639 Published online by Cambridge University Press

that is, $E = \{e_0, e_1, e_2, \dots\}$ (with the order $e_m \leq e_n$ if and only if $m \geq n$), S_{e_n} is a group for every $e_n \in E$ and $\rho_{e_m}^{e_n} : S_{e_n} \to S_{e_m}$ is a transition homomorphism for $e_n, e_m \in E$ with $m \leq n$. Let A be an inverse subsemigroup of S such that $E(S) = E(A)$. Let B be an inverse subsemigroup of S such that $E(B) = E(S) \setminus \{e_0\}$. Suppose that there exists an isomorphism ϕ of A onto B. Then it is easy to see that the locally full HNN extension S^* of S presented by

$$\mathrm{Inv}(S, t \mid t^{-1}at = \phi(a) \text{ for every } a \in A,\ t^{-1}t = e_1, tt^{-1} = e_0)$$

is a bisimple regular ω-semigroup, that is, a Reilly extension of a group. The maximal subgroup containing e_0 is isomorphic to the fundamental group of the graph of groups (G, Y_{e_0}), where Y_{e_0} is a half chain, that is, a graph consisting of countably many vertices $\mathbf{v}_n$ and countably many edges $\mathbf{y}_n$ with $\mathbf{i}(\mathbf{y}_n) = \mathbf{v}_{n-1}$ and $\mathbf{t}(\mathbf{y}_n) = \mathbf{v}_n$ for $n = 1, 2, 3, \dots$. Therefore, S^* is a Reilly extension of the fundamental group $\pi(G, Y_{e_0})$ of the graph of groups (G, Y_{e_0}).

In the case that every S_{e_n} is the trivial group, the fundamental group of the graph of groups (G, Y_{e_0}) is trivial. Then, S^* is a Reilly extension of the trivial group, that is, S^* is the bicyclic semigroup.

EXAMPLE 6.4. Let S be the free Clifford semigroup on a set X. Let E be a free semilattice on a set $\{e_x \mid x \in X\}$. For every e_x $(x \in X)$, we let ϕ_x be the identity mapping on the principal ideal Ee_x generated by e_x. Clearly, ϕ_x is an isomorphism of Ee_x onto itself. It is easy to see that the locally full HNN extension E^* presented by

$$\mathrm{Inv}(E, t_x (x \in X) \mid t_x^{-1}at_x = \phi_x(a) = a\ \forall a \in Ee_x,\ t_x^{-1}t_x = t_x t_x^{-1} = e_x\ \forall x \in X)$$

is isomorphic to S. Take an idempotent e from E. Then $e = e_{x_1}e_{x_2}\cdots e_{x_n}$ for some $x_1, x_2, \dots x_n \in X$ such that x_i's are distinct each other. We consider the graph of groups (G, Y_e). Clearly, the graph Y_e is a bouquet with n edges since all ϕ_x are identity mappings. Every vertex group is trivial since E is combinatorial. It follows that the fundamental group of (G, Y_e) is a free group of rank n. Therefore, the maximal subgroup of the free Clifford semigroup containing the idempotent

$$e = e_{x_1}e_{x_2}\cdots e_{x_n} = x_1x_1^{-1}x_2x_2^{-1}\cdots x_nx_n^{-1}$$

is a free group of rank n.

EXAMPLE 6.5. It is known that a universally E-unitary inverse semigroup is a locally full HNN extension of a semilattice of idempotent rank zero ([17]). Hence, a maximal subgroup is a fundamental group of the graph of groups (G, Y_e), where every vertex group $G(\mathbf{v})$ is trivial. This implies that the fundamental group of the graph of group is a free group. Hence, a maximal subgroup of a universally E-unitary inverse semigroup is a free group. The rank of the maximal subgroup can be determined by the shape of the graph Y_e, in fact, the rank is given by the number of circuits in the graph Y_e.

Acknowledgements

This paper is based on the author's Ph.D. Thesis. The author thanks J. C. Meakin, his thesis advisor, for valuable discussions, advice and encouragement. He is also grateful to the anonymous referee for constructive advice.

References

[1] P. Bennett, 'On the structure of inverse semigroup amalgams', *Int. J. Algebra Comput. (5)* **7** (1997), 577–604.

[2] A. Cherubini, J. C. Meakin and B. Piochi, 'Amalgams of free inverse semigroups', *Semigroup Forum* **54** (1997), 199–220.

[3] D. E. Cohen, *Combinatorial group theory: A topological approach* (Cambridge University Press, Cambridge, 1989).

[4] S. Haataja, S. W. Margolis and J. C. Meakin, 'On the structure of full regular semigroup amalgams', *J. Algebra* **183** (1996), 38–54.

[5] T. E. Hall, 'Free products with amalgamation of inverse semigroups', *J. Algebra* **34** (1975), 375–385.

[6] ———, 'Amalgamation of inverse and regular semigroups', *Trans. Amer. Math. Soc.* **246** (1978), 395–406.

[7] J. M. Howie, 'Embedding theorems for semigroups', *Quart. J. Math.* **14** (1963), 254–258.

[8] T. Jajcayova, HNN *extensions of inverse semigroups* (Ph.D. Thesis, University of Nebraska-Lincoln, 1997).

[9] M. V. Lawson, *Inverse semigroups* (World Scientific, Singapore, 1998).

[10] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory* (Springer, New York, 1976).

[11] S. W. Margolis and J. C. Meakin, 'Inverse monoids, trees and context-free languages', *Trans. Amer. Math. Soc.* **335** (1993), 259–276.

[12] J. C. Meakin and A. Yamamura, 'Bass-Serre theory and inverse monoids', in: *Semigroups and Applications* (eds. J. M. Howie and N. Ruškuc) (World Scientific, Singapore, 1998) pp. 125–140.

[13] M. Petrich, *Inverse semigroups* (Wiley, New York, 1984).

[14] J.-P. Serre, *Trees* (Springer, New York, 1980).

[15] A. Yamamura, HNN *extensions of inverse semigroups* (Ph.D. Thesis, University of Nebraska-Lincoln, 1996).

[16] ———, 'HNN extensions of inverse semigroups and applications', *Int. J. Algebra Comput. (5)* **7** (1997), 605–624.

[17] ———, 'HNN extensions of semilattices', *Int. J. Algebra Comput. (5)* **9** (1999), 555–596.

[18] ———, 'Presentations of Bruck-Reilly extensions and decision problems', *Semigroup Forum* (to appear).

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