2 The Standard Model

The interactions of the Standard Model give rise to the phenomena of our day to day experience. They explain virtually all the particles and interactions which have been observed in accelerators. Yet the underlying laws can be summarized in a few lines. In this chapter we describe the ingredients of this theory and some of its important features. Many dynamical questions will be studied in subsequent chapters. For detailed comparisons of theory and experiment there are a number of excellent texts, described in the suggested reading at the end of the chapter.

2.1 Yang–Mills theory

By the early 1950s physicists were familiar with approximate global symmetries such as isospin. Yang and Mills argued that the lesson of Einstein's general theory was that symmetries, if exact, should be local. In ordinary electrodynamics the gauge symmetry is a local Abelian symmetry. Yang and Mills explained how to generalize this to a non-Abelian symmetry group. Let's first review the case of electrodynamics. The electron field $\psi(x)$ transforms under a gauge transformation as follows:

$$\psi(x) \to e^{i\alpha(x)}\psi(x) = g_\alpha(x)\psi(x). \tag{2.1}$$

We can think of $g_\alpha(x) = e^{i\alpha(x)}$ as a group element in the group $U(1)$. The group is Abelian: $g_\alpha g_\beta = g_\beta g_\alpha = g_{\alpha+\beta}$. Quantities such as $\bar\psi\psi$ are gauge invariant, but derivative terms such as $i\bar\psi\ \partial\!\!\!/\psi$, are not. In order to write down the derivative terms in an action or equation of motion, one needs to introduce a gauge field A_μ transforming under the symmetry transformation as

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu\alpha \\ &= A_\mu + ig(x)\partial_\mu g^{-1}(x). \end{aligned} \tag{2.2}$$

This second form allows more immediate generalization to the non-Abelian case. Given A_μ and its transformation properties, we can define a covariant derivative,

$$D_\mu\psi = (\partial_\mu - iA_\mu)\psi. \tag{2.3}$$

This derivative has the property that it transforms like ψ itself under the gauge symmetry:

$$D_\mu\psi \to g(x)D_\mu\psi. \tag{2.4}$$

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We can also form a gauge-invariant object from the gauge fields A_μ themselves. A simple way to do this is to construct the commutator of two covariant derivatives,

$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.5}$$

This form of the gauge transformations may be somewhat unfamiliar. Note in particular that the charge of the electron, e (the gauge coupling) does not appear in the transformation laws. Instead, the gauge coupling appears when we write down a gauge-invariant Lagrangian:

$$\mathcal{L} = i\bar{\psi}\, \not{D}\psi - m\bar{\psi}\psi - \frac{1}{4e^2}F_{\mu\nu}^2, \tag{2.6}$$

where the "slash" notation is defined by $\not{a} = a^\mu \gamma_\mu$. The more familiar formulation is obtained if we make the replacement

$$A_\mu \to eA_\mu. \tag{2.7}$$

In terms of this new field the gauge transformation law is

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha \tag{2.8}$$

and the covariant derivative is

$$D_\mu \psi = (\partial_\mu - ieA_\mu)\psi. \tag{2.9}$$

We can generalize this to a non-Abelian group, $\mathcal{G}$, by taking ψ to be a field (fermion or boson) in some representation of the group; $g(x)$ is then a matrix which describes a group transformation acting in this representation. Formally, the transformation law is the same as before,

$$\psi \to g(x)\psi(x), \tag{2.10}$$

but the group composition law is more complicated:

$$g_\alpha g_\beta \neq g_\beta g_\alpha. \tag{2.11}$$

The gauge field A_μ is now a matrix-valued field, transforming in the adjoint representation of the gauge group:

$$A_\mu \to gA_\mu g^{-1} + ig(x)\partial_\mu g^{-1}(x). \tag{2.12}$$

Formally, the covariant derivative also looks exactly as before:

$$D_\mu \psi = (\partial_\mu - iA_\mu)\psi, \quad D_\mu\psi \to g(x)D_\mu\psi. \tag{2.13}$$

Like A_μ, the field strength is a matrix-valued field:

$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \tag{2.14}$$

Note that $F_{\mu\nu}$ is not gauge *invariant* but, rather, covariant:

$$F_{\mu\nu} \to gF_{\mu\nu}g^{-1}, \tag{2.15}$$

i.e. it transforms like a field in the adjoint representation, with no inhomogeneous term.

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The gauge-invariant action $\mathcal{L}$ is formally almost identical to that of the $U(1)$ theory:

$$\mathcal{L} = i\bar{\psi}\, \not{D}\psi - m\bar{\psi}\psi - \frac{1}{2g^2}\,\mathrm{Tr}\, F_{\mu\nu}^2. \tag{2.16}$$

Here we have changed the letter we use to denote the coupling constant: we will usually reserve e for the electron charge and use g for a generic gauge coupling. Note also that it is necessary to take the trace of F^2 to obtain a gauge-invariant expression.

The matrix form for the fields may be unfamiliar, but it is very powerful. One can recover expressions in terms of more conventional fields by defining

$$A_\mu = A_\mu^a T_a, \tag{2.17}$$

where T_a are the group generators in the representation appropriate to ψ. Then, for $SU(N)$, for example, if the T_as are in the fundamental representation, we have

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}, \quad [T^a, T^b] = if^{abc}T^c, \tag{2.18}$$

where f^{abc} are the structure constants of the group and

$$A_\mu^a = 2\,\mathrm{Tr}(T_a A^\mu), \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc}A_\mu^a A_\nu^b. \tag{2.19}$$

While they are formally almost identical, there are great differences between the Abelian and non-Abelian theories. Perhaps the most striking is that the equations of motion for the A_μs are non-linear in non-Abelian theories. This behavior means that, unlike the case of Abelian gauge fields, a theory of non-Abelian fields without matter is a non-trivial, interacting, theory with interesting properties. With and without matter fields, this will lead to much richer behavior even classically. For example, we will see that non-Abelian theories sometimes contain solitons, localized finite-energy solutions of the classical equations. The most interesting of these are the magnetic monopoles. At the quantum level these non-linearities lead to properties such as asymptotic freedom and confinement.

Using the form in which we have written the action, the matter fields ψ can appear in any representation of the group; one just needs to choose appropriate matrices T^a. We can also consider scalars, as well as fermions. For a scalar field ϕ, we define the covariant derivative $D_\mu\phi$ as before and add to the action a term $|D_\mu\phi|^2$ for a complex field or $(D_\mu\phi)^2/2$ for a real field.

2.2 Realizations of symmetry in quantum field theory

The most primitive exercise we can do with the Yang–Mills Lagrangian is to set $g = 0$ and examine the equations of motion for the fields A^μ. If we choose the gauge $\partial_\mu A^{\mu a} = 0$, all the gauge fields obey

$$\partial^2 A_\mu^a = 0. \tag{2.20}$$

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So, like the photon, all the gauge fields A^a_μ of the Yang–Mills theory are massless. At first sight there is no obvious place for these fields in either the strong or the weak interactions. But it turns out that in non-Abelian theories the possible ways in which the symmetry may be realized are quite rich. First, the symmetry can be realized in terms of massless gauge bosons; this is known as the *Coulomb phase*. This possibility is not relevant to the Standard Model but will appear in some of our more theoretical considerations later. A second way is known as the *Higgs phase*. In this phase, the gauge bosons are massive. In the third, the *confinement phase*, there are no physical states with the quantum numbers of isolated quarks (particles in the fundamental representation), and the gauge bosons are also massive. The second phase is relevant to the weak interactions; the third, confinement, phase to the strong interactions.[1]

2.2.1 The Goldstone phenomenon

Before introducing the Higgs phase it is useful to discuss global symmetries. While we will frequently argue, like Yang and Mills, that global symmetries are less fundamental than local ones, they are important in nature. Examples are isospin, the chiral symmetries of the strong interactions and baryon number. We can represent the action of such a symmetry much as we represented the symmetry action in Yang–Mills theory:

$$\Phi \to g_\alpha \Phi, \tag{2.21}$$

where Φ is some set of fields and g is now a constant matrix, independent of spatial position. Such symmetries are typically accidents of the low-energy theory. Isospin, for example, as we will see arises because the masses of the u and d quarks are small compared with other scales of quantum chromodynamics. Then g is the matrix

$$g_{\vec{\alpha}} = e^{i\vec{\alpha}\cdot\vec{\sigma}/2} \tag{2.22}$$

acting on the u and d quark doublet. Note that $\vec{\alpha}$ is not a function of space but a continuous parameter, so we will refer to such symmetries as continuous global symmetries. In the case of isospin it is also important that the electromagnetic and weak interactions, which violate this symmetry, are small perturbations on the strong interactions.

The simplest model of a continuous global symmetry is provided by a complex field ϕ transforming under a $U(1)$ symmetry,

$$\phi \to e^{i\alpha}\phi. \tag{2.23}$$

We can take for the Lagrangian for this system

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2 - \frac{1}{2}\lambda|\phi|^4. \tag{2.24}$$

If $m^2 > 0$ and λ is small, this is simply a theory of a weakly interacting, complex scalar. The states of the theory can be organized as states of definite $U(1)$ charge. This is the unbroken

[1] The differences between the confinement and Higgs phases are subtle, as was first stressed by Fradkin, Shenker and 't Hooft. But we now know that the Standard Model is well described by a weakly coupled field theory in the Higgs phase.

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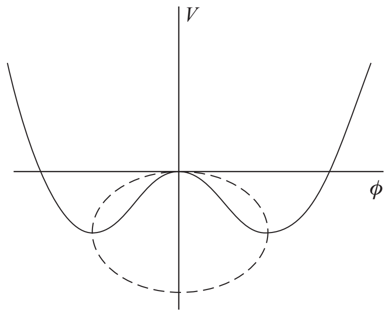

Fig. 2.1 Scalar potential with negative mass-squared. The stable minimum leads to broken symmetry.

phase. However, m^2 is just a parameter and we can ask what happens if $m^2 = -\mu^2 < 0$. In this case the potential,

$$V(\phi) = -\mu^2|\phi|^2 + \lambda|\phi|^4, \tag{2.25}$$

looks as in Fig. 2.1. There is a set of degenerate minima,

$$\langle\phi\rangle_\alpha = \frac{\mu}{\sqrt{2\lambda}}e^{i\alpha}. \tag{2.26}$$

These ground states are obtained from one another by symmetry transformations; in somewhat more mathematical language, we say that there is a manifold of vacuum states. Quantum mechanically it is necessary to choose a particular value of α. As will be explained in the next section, if one chooses α then no local operator, e.g. no small perturbation, will take the system into a state of different α. To simplify the writing, take $\alpha = 0$. Then we can parameterize the complex field ϕ in terms of real fields σ and π:

$$\phi = \frac{1}{\sqrt{2}}[v + \sigma(x)]e^{i\pi(x)/v} \approx \frac{1}{\sqrt{2}}[v + \sigma(x) + i\pi(x)]. \tag{2.27}$$

Here $v = \mu/\sqrt{\lambda}$ is known as the *vacuum expectation value* (vev) of the field ϕ. In terms of σ and π, the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2 - 2\mu^2\sigma^2 + \mathcal{O}(\sigma,\pi)^3]. \tag{2.28}$$

So we see that σ is an ordinary real, scalar field of mass-squared $2\mu^2$, while the π field is massless. The fact that it is massless is not a surprise: the mass represents the energy cost of turning on a zero-momentum excitation of π, but such an excitation is just a symmetry transformation $v \to ve^{i\pi(0)}$ of ϕ. So there *is* no energy cost.

The appearance of massless particles when a symmetry is broken is quite general and is known as the Nambu–Goldstone phenomenon; π is called a Nambu–Goldstone boson. In any theory with scalars, the choice of a minimum may break some symmetry. This means that there is a manifold of vacuum states. The broken-symmetry generators are those which transform the system from one point on this manifold to another. Because there is no energy cost associated with such a transformation, there is a massless particle associated with each broken-symmetry generator. This result is very general. Symmetries can be broken not only

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by the expectation values of scalar fields but also by the expectation values of composite operators, and the theorem holds. A proof of this result is provided in Appendix B. In nature there are a number of excitations which can be identified as Goldstone or almost-Goldstone ("pseudo-Goldstone") bosons. These include spin waves in solids and the pi mesons. We will have much more to say about pions later.

2.2.2 Aside: choosing a vacuum

In quantum mechanics there is no notion of a spontaneously broken symmetry. If one has a set of degenerate classical configurations, the ground state will invariably involve a superposition of these configurations. If we took σ and π in Eq. (2.27) to be functions only of the time t then the σ–π system would just be an ordinary quantum mechanical system with two degrees of freedom. Here σ would correspond to an anharmonic oscillator of frequency $\omega = \sqrt{2}\mu$. Placing this particle in its ground state, one would be left with the coordinate π. Note that π, in Eq. (2.27), is an angle, like the azimuthal angle, in ordinary quantum mechanics. We could call its conjugate variable L_z. The lowest lying state would be the zero-angular-momentum state, a uniform superposition of all values of π. In field theory at finite volume, the situation is similar. The zero-momentum mode of π is again an angular variable, and the ground state is invariant under the symmetry. At infinite volume, however, the situation is different. One is forced to choose a value of π.

This issue is most easily understood by considering a different problem: rotational invariance in a magnet. Consider Fig. 2.2, which shows a ferromagnet with spins aligned at an angle θ. We can ask: what is the overlap of two states, one with $\theta = 0$, one at θ, i.e. what is $\langle\theta|0\rangle$? For a single site the overlap between the state $|+\rangle$ with $\theta = 0$ and the rotated state is

$$\left\langle + \left| e^{i\tau_1\theta/2} \right| + \right\rangle = \cos(\theta/2). \tag{2.29}$$

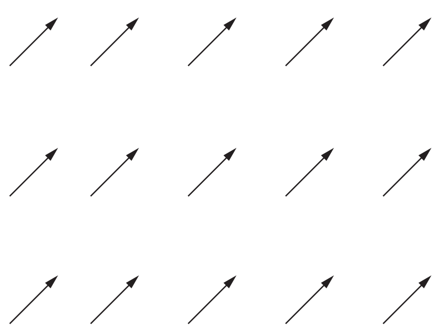

Fig. 2.2 In a ferromagnet the spins are aligned but their direction is arbitrary.

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If there are N such sites, the overlap behaves as follows:

$$\langle\theta|0\rangle \sim [\cos(\theta/2)]^N, \tag{2.30}$$

i.e. it vanishes exponentially rapidly with the "volume", N.

For a continuum field theory, states with differing values of the order parameter v also have no overlap in the infinite-volume limit. This is illustrated by the theory of a scalar field ϕ with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2. \tag{2.31}$$

For this system there is no potential, so the expectation value $\phi = v$ is not fixed. The Lagrangian has a symmetry, $\phi \to \phi + \delta$, for which the charge is just

$$Q = \int d^3x\, \Pi(\vec{x}) \tag{2.32}$$

where Π is the canonical momentum. So we want to study

$$\langle v|0\rangle = \langle 0|e^{iQ}|0\rangle. \tag{2.33}$$

We must be careful how we take the infinite-volume limit. We will insist that this be done in a smooth fashion, so we will define

$$\begin{aligned} Q &= \int d^3x\, \partial_0 \left(\phi e^{-\vec{x}^2/V^{2/3}}\right) \\ &= -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\left(\frac{V^{1/3}}{\sqrt{\pi}}\right)^3 e^{-\vec{k}^2V^{2/3}/4}[a(\vec{k}) - a^\dagger(\vec{k})]. \end{aligned} \tag{2.34}$$

Now one can evaluate the matrix element, using

$$e^{A+B} = e^A e^B e^{-[A,B]/2}$$

(provided that the commutator is a c-number), obtaining

$$\langle 0|e^{iQ}|0\rangle = e^{-cv^2V^{2/3}}, \tag{2.35}$$

where c is a numerical constant. So the overlap vanishes with the volume. You can convince yourself that the same holds for matrix elements of local operators. This result does not hold in 0+1 and 1+1 dimensions, because of the severe infrared behavior of theories in low dimensions. This is known to particle physicists as Coleman's theorem, and to condensed matter theorists as the Mermin–Wagner theorem. This theorem will make an intriguing appearance in string theory, where it is the origin of energy–momentum conservation.

2.2.3 The Higgs mechanism

Suppose that the $U(1)$ symmetry of the previous section is local. In that case, even a spatially varying $\pi(x)$ represents a symmetry transformation and, by a suitable gauge

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choice, it can be eliminated. In other words, by a gauge transformation we can bring the field ϕ to the form

$$\phi = \frac{1}{\sqrt{2}}[v + \sigma(x)]. \tag{2.36}$$

In this gauge, the gauge-invariant kinetic term for ϕ takes the form

$$|D_\mu \phi|^2 = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}A_\mu^2 v^2 + \cdots . \tag{2.37}$$

The second term is a mass term for the gauge field A_μ. To determine the actual value of the mass, we need to examine the kinetic term for the gauge fields,

$$-\frac{1}{2g^2}(\partial_\mu A^\nu)^2 + \cdots . \tag{2.38}$$

So the gauge field must have mass $m_A^2 = g^2 v^2$.

This phenomenon, that the gauge boson becomes massive when the gauge symmetry is spontaneously broken, is known as the Higgs mechanism. While formally quite similar to the Goldstone phenomenon, it is also quite different. The fact that there is no massless particle associated with motion along the manifold of ground states is not surprising – these states are all physically equivalent. Symmetry breaking, in fact, is a paradoxical notion in gauge theories, since gauge transformations describe entirely equivalent physics (gauge symmetry is often referred to as a redundancy in the description of a system). Perhaps the most important lesson here is that gauge invariance does not necessarily mean, as it does in electrodynamics, that the gauge bosons are massless.

2.2.4 Goldstone and Higgs phenomena for non-Abelian symmetries

Both the Goldstone and Higgs phenomena generalize to non-Abelian symmetries. In the case of global symmetries, for every generator of a broken global symmetry there is a massless particle. For local symmetries, each broken generator gives rise to a massive gauge boson.

As an example, relevant both to the strong and the weak interactions, consider a theory with a symmetry $SU(2)_L \times SU(2)_R$. Take M to be a Hermitian matrix field,

$$M = \sigma I + i\vec{\pi} \cdot \vec{\sigma}. \tag{2.39}$$

Under the above symmetry, which we first take to be global, M transforms as follows:

$$M \to g_L M g_R \tag{2.40}$$

with g_L and g_R $SU(2)$ matrices. We can take the Lagrangian to be

$$\mathcal{L} = \mathrm{Tr}\,(\partial_\mu M^\dagger \partial^\mu M) - V(\mathrm{Tr}(M^\dagger M)). \tag{2.41}$$

This Lagrangian respects the symmetry. If the curvature of the potential at the origin is negative, M will acquire an expectation value. If we take:

$$\langle M \rangle = \langle \sigma \rangle I \tag{2.42}$$

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then some of the symmetry is broken. However, the expectation value of M is invariant under the subgroup of the full symmetry group with $g_L = g_R^\dagger$. In other words, the unbroken symmetry is $SU(2)$. Under this symmetry, the fields $\vec{\pi}$ transform as a vector. In the case of the strong interactions, this unbroken symmetry can be identified with isospin. In the case of the weak interactions, there is an approximate global symmetry reflected in the masses of the W and Z particles, as we will discuss later.

2.2.5 Confinement

There is still another possible realization of gauge symmetry: confinement. This is crucial to our understanding of strong interactions. As we will see, Yang–Mills theories, in the case where there is not too much matter, become weak at short distances and strong at large distances. This is just what is required to understand the qualitative features of the strong interactions: free-quark and free-gluon *behavior* at very large momentum transfers, but strong forces at larger distances so that there are in fact no free quarks or gluons. As is the case for the Higgs mechanism, there are no massless particles in the spectrum of hadrons: QCD is said to have a "mass gap." These features of strong interactions are supported by extensive numerical calculations, but they are hard to understand through simple analytical or qualitative arguments (indeed, if you can offer such an argument, you could win a Clay prize of \$1 million). We will have more to say about the phenomenon of confinement when we discuss lattice gauge theories.

One might wonder: what is the difference between the Higgs mechanism and confinement? This question was first raised by Fradkin and Shenker and by 't Hooft, who also gave an answer: there is often no qualitative difference. The qualitative features of a theory without massless gauge fields as a result of the Higgs phenomenon can be reproduced by a confined strongly interacting theory. However, the detailed predictions of the weakly interacting Weinberg–Salaam theory are in close agreement with experiment but those of the strongly interacting theory are not.

2.3 The quantization of Yang–Mills theories

In this book we will encounter a number of interesting classical phenomena in Yang–Mills theory but, in most of the situations in nature on which we are focusing, we will be concerned with the quantum behavior of the weak and strong interactions. Abelian theories such as QED already present considerable challenges. One can perform canonical quantization in a gauge, such as the Coulomb gauge or a light cone gauge, in which unitarity is manifest – all the states have positive norm. But, in such a gauge the covariance of the theory is hard to see. Or one can choose a gauge where Lorentz invariance is manifest, but not unitarity. In QED it is not too difficult to show, at the level of Feynman diagrams, that these gauge choices are equivalent. In non-Abelian theories, canonical quantization is still more challenging. Path integral methods provide a much more powerful approach to the quantization of these theories than the canonical methods mentioned above.

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A brief review of path integration appears in Appendix C. Here we discuss gauge fixing and derive the Feynman rules. We start with the gauge fields alone; adding the matter fields – scalars or fermions – is not difficult. The basic path integral is

$$\int [dA_\mu] e^{iS}. \tag{2.43}$$

The problem is that this integral includes a huge redundancy: the gauge transformations. To deal with this, we need to make a gauge choice, for example

$$G^a\left(A^a_\mu\right) = \partial_\mu A^{\mu a} = 0. \tag{2.44}$$

We insert unity in the form

$$1 = \int [dg] \delta\left(G\left(A^g_\mu\right)\right) \Delta[A]. \tag{2.45}$$

Here we have reverted to our matrix notation: G is a general gauge-fixing condition; A^g_μ denotes the gauge transform of A_μ by g. The quantity Δ is a functional determinant known as the Faddeev–Popov determinant. Note that Δ is gauge invariant: $\Delta[A^h] = \Delta[A]$. This follows from the definition

$$\int [dg] \delta\left(G\left(A^{hg'}_\mu\right)\right) = \int [dg] \delta\left(G\left(A^{g'}_\mu\right)\right), \tag{2.46}$$

where, in the last step, we have made the change of variables $g \to h^{-1} g$. We can write a more explicit expression for Δ as a determinant. To do this, we first need an expression for the variation of the As under an infinitesimal gauge transformation. Writing $g = 1 + i\omega$, and using the matrix form for the gauge field, we have

$$\delta A_\mu = \partial_\mu \omega + i[\omega, A_\mu]. \tag{2.47}$$

This can be written elegantly as a covariant derivative of ω, where ω can be thought of as a field in the adjoint representation:

$$\delta A_\mu = D_\mu \omega. \tag{2.48}$$

If we make the specific choice $G = \partial_\mu A^\mu$ then to evaluate Δ we need to expand G about the field A_μ for which $G = 0$:

$$G(A + \delta A) = \partial_\mu D^\mu \omega = \partial^2 \omega + i[A_\mu, \partial_\mu \omega] \tag{2.49}$$

or, in index form,

$$G\left(A^a_\mu\right) = (\partial^2 \delta^{ac} + f^{abc} A^{\mu\, b} \partial_\mu)\omega^c. \tag{2.50}$$

So

$$\Delta[A] = \det(\partial^2 \delta^{ac} + f^{abc} A^{\mu\, b} \partial_\mu)^{-1/2}. \tag{2.51}$$

We will discuss strategies to evaluate this determinant shortly.

At this stage, we have reduced the path integral to

$$Z = \int [dA_\mu] \delta(G(A)) \Delta[A] e^{iS} \tag{2.52}$$

and we can write down the Feynman rules. The δ-function remains rather awkward to deal with, though, and this expression can be simplified through the following trick. Introduce a function ω (not to be confused with the ω of Eq. (2.48)) and average over ω with a Gaussian weight factor:

$$Z = \int [d\omega] e^{i\int d^4x(\omega^2/\xi)} \sum \int [dA_\mu]\delta(G(A) - \omega)\Delta[A]e^{iS}. \tag{2.53}$$

We can do the integral over the δ-function. The quadratic terms in the exponent are now given by

$$\int d^4x\, A^{\mu a} \left[-\partial^2 \eta_{\mu\nu} + \partial_\mu \partial_\nu \left(1 - \frac{1}{\xi}\right)\right] A^{\nu a}. \tag{2.54}$$

We can invert this to find the propagator. In momentum space,

$$D_{\mu\nu} = -\frac{\eta_{\mu\nu} + (\xi - 1)k_\mu k_\nu / k^2}{k^2 + i\epsilon}. \tag{2.55}$$

To write down explicit Feynman rules, we need also to deal with the Faddeev–Popov determinant. Feynman long ago guessed that the unitarity problems of Yang–Mills theories could be dealt with by introducing fictitious scalar fields with the wrong statistics. Our expression for Δ can be reproduced by a functional integral for such particles:

$$\Delta = \int [dc^a][dc^{a\dagger}] \exp\left(i \int d^4x [c^{a\dagger}(\partial^2\delta^{ab} + f^{abc}A^{\mu\, c}\partial_\mu)c^b]\right). \tag{2.56}$$

From this we can read off the Feynman rules for Yang–Mills theories, including matter fields. They are summarized in Fig. 2.3.

2.3.1 Gauge fixing in theories with broken gauge symmetry

Gauge fixing in theories with broken gauge symmetries raises some new issues. We consider first a $U(1)$ gauge theory with a single charged scalar field ϕ. We suppose that the potential is such that $\langle\phi\rangle = v/\sqrt{2}$. We call e the gauge coupling and take the conventional scaling for the gauge kinetic terms. We can, again, parameterize the field ϕ as

$$\phi = \frac{1}{\sqrt{2}}[v + \sigma(x)]e^{i\pi/v}. \tag{2.57}$$

Then we can again choose a gauge in which $\pi(x) = 0$. This gauge is known as the unitary gauge since, as we have seen, in this gauge we have exactly the degrees of freedom we expect physically: a massive gauge boson and a single real scalar. But this gauge is not convenient for calculations. The gauge boson propagator in this gauge is

$$\langle A_\mu A_\nu \rangle = -\frac{i}{k^2 - M_V^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{M_V^2}\right). \tag{2.58}$$

Because of the momentum factors in the second term, individual Feynman diagrams have a bad high-energy behavior. A more convenient set of gauges, known as R_ξ gauges, avoids this difficulty at the price of keeping the π field (sometimes misleadingly called the

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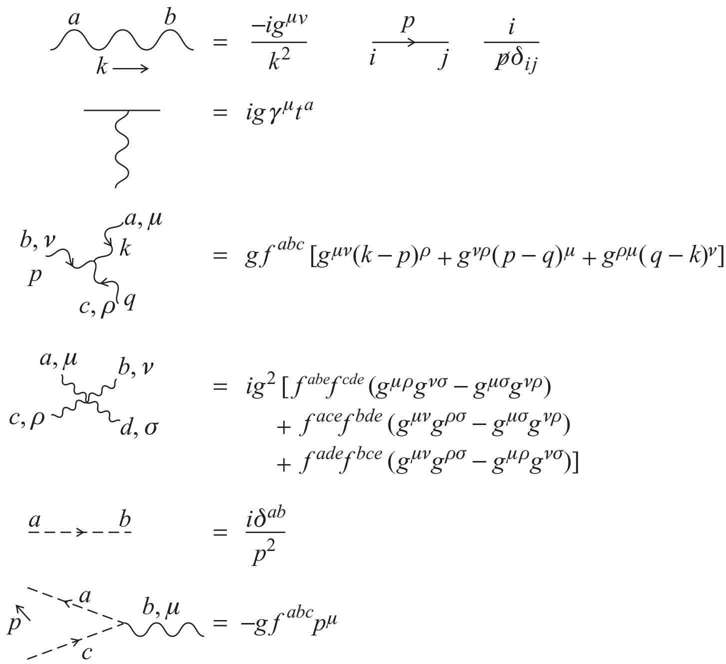

$$= \frac{-ig^{\mu\nu}}{k^2} \qquad\qquad \frac{i}{\not p\,\delta_{ij}}$$

$$= ig\gamma^\mu t^a$$

$$= gf^{abc}\,[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu]$$

$$\begin{aligned} = ig^2\,[\,& f^{abe}f^{cde}\,(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ace}f^{bde}\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+ f^{ade}f^{bce}\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})] \end{aligned}$$

$$= \frac{i\delta^{ab}}{p^2}$$

$$= -gf^{abc}p^\mu$$

Fig. 2.3 Feynman rules for Yang–Mills theory.

Goldstone particle) in the Feynman rules. We take, in the path integral, the gauge-fixing function

$$G = \frac{1}{\sqrt{\xi}}[\partial_\mu A^\mu \xi - ev\pi(x)]. \tag{2.59}$$

The extra term has been judiciously chosen so that when we exponentiate the gauge condition, as in Eq. (2.53), the $A^\mu \partial_\mu \pi$ terms in the action cancel. Explicitly, we have

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}A_\mu \left[\eta^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu - (e^2v^2)\eta^{\mu\nu}\right]A_\nu \\ &+ \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}(\partial_\mu\pi)^2 - \frac{\xi}{2}(ev)^2\pi^2 + \mathcal{O}(\phi^3). \end{aligned} \tag{2.60}$$

If we choose $\xi = 1$ (corresponding to the 't Hooft–Feynman gauge), the propagator for the gauge boson is then simply

$$\langle A_\mu A_\nu \rangle = \frac{-i}{k^2 - M_V^2}\eta_{\mu\nu} \tag{2.61}$$

with $M_V^2 = e^2v^2$, but we have also the field π explicitly in the Lagrangian, and it has the propagator

$$\langle \pi\pi \rangle = \frac{i}{k^2 - M_V^2}. \tag{2.62}$$

The mass here is just the mass of the vector boson (for other choices of ξ, this is not true).

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This gauge choice is readily extended to non-Abelian theories with similar results: the gauge bosons have simple propagators, like those of massive scalars but multiplied by $\eta_{\mu\nu}$. The Goldstone bosons appear explicitly in perturbation theory, with propagators appropriate to massive fields. The Faddeev–Popov ghosts have couplings to the scalar fields.

2.4 The particles and fields of the Standard Model: gauge bosons and fermions

We are now in a position to write down the Standard Model. It is amazing that, at a microscopic level, almost everything we know about nature is described by such a simple structure. The gauge group is $SU(3)_{\rm c} \times SU(2)_{\rm L} \times U(1)_Y$. The subscript c denotes color, L means left-handed and Y is the hypercharge. Corresponding to these different gauge groups, there are gauge bosons: A^a_μ, $a = 1, \ldots, 8$; W^i_μ, $i = 1, 2, 3$; and B_μ.

One of the most striking features of the weak interactions is the violation of parity. In terms of four-component fields, this means that factors of $1 - \gamma_5$ appear in the couplings of fermions to the gauge bosons. In such a situation it is more natural to work with two-component spinors. For the reader unfamiliar with such spinors, a simple introduction appears in Appendix A. These spinors are the basic building blocks of the four-dimensional spinor representations of the Lorentz group. All spinors can be described as two-component quantities, with various quantum numbers. For example, quantum electrodynamics, which is parity invariant and has a massive fermion, can be described in terms of two left-handed fermions, e and $\bar{e}$, with electric charges $-e$ and $+e$ respectively. The Lagrangian takes the form

$$\mathcal{L} = ie\sigma^\mu D_\mu e^* + i\bar{e}\sigma^\mu D_\mu \bar{e}^* - m\bar{e}e - m\bar{e}^* e^*. \tag{2.63}$$

The covariant derivatives are those appropriate to fields of charge e and $-e$. Parity is symmetry under $\vec{x} \to -\vec{x}$, $e \leftrightarrow \bar{e}^*$ and $\vec{A} \to -\vec{A}$.

We can specify the fermion content of the Standard Model by giving the gauge quantum numbers of the left-handed spinors. So, for example, there are quark doublets which are in the 3 (fundamental) representation of color and doublets of $SU(2)$ and which have hypercharge $1/3$: $Q = (3, 2)_{1/3}$. The appropriate covariant derivative is:

$$D_\mu Q = \left(\partial_\mu - ig_s A^a_\mu T^a - igW^i_\mu T^i - i\frac{g'}{2}\frac{1}{3}B_\mu\right) Q, \tag{2.64}$$

where g_s is the strong coupling constant. Here the T^is are the generators of $SU(2)$; $T^i = \sigma^i/2$. These are normalized as follows:

$$\mathrm{Tr}(T^i T^j) = \frac{1}{2}\delta^{ij}. \tag{2.65}$$

The T^a are the generators of $SU(3)$; in terms of Gell-Mann's $SU(3)$ matrices, $T^a = \lambda^a/2$. They are normalized in the same way as the $SU(2)$ matrices: Tr $(T^a T^b) = (1/2)\delta^{ab}$.

Table 2.1 Fermions of the Standard Model and their quantum numbers

	$SU(3)$	$SU(2)$	$U(1)_Y$
Q_f	3	2	$1/3$
$\bar{u}_f$	$\bar{3}$	1	$-4/3$
$\bar{d}_f$	$\bar{3}$	1	$2/3$
L_f	1	2	-1
$\bar{e}_f$	1	1	2

We have followed the customary definition in coupling B_μ to half the hypercharge current. We have also scaled the fields so that the couplings appear in the covariant derivative and have labeled the $SU(3)_c$, $SU(2)_L$, and $U(1)_Y$ coupling constants as g_s, g, and g', respectively. Using matrix-valued fields, defined with the couplings in front of the gauge kinetic terms, this covariant derivative can be written in a very compact manner:

$$D_\mu Q = \left(\partial_\mu - iA_\mu - iW_\mu - \frac{i}{2}\frac{1}{3}B_\mu\right)Q. \tag{2.66}$$

As another example, the Standard Model contains lepton fields L with no $SU(3)$ quantum numbers but which are $SU(2)$ doublets with hypercharge -1. The covariant derivative is

$$D_\mu L = \left(\partial_\mu - igW^i_\mu T^i - \frac{ig'}{2}B_\mu\right)L. \tag{2.67}$$

We have summarized the fermion content in the Standard Model in Table 2.1. Here f labels the quark or lepton flavor, i.e. the generation number: $f = 1, 2, 3$. For example,

$$L_1 = \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \quad L_2 = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \quad L_3 = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}. \tag{2.68}$$

The reason why there is this repetitive structure, these three generations, is one of the great puzzles of the Standard Model, to which we will return. In terms of these two-component fields (indicated generically by ψ_i), the gauge-invariant kinetic terms have the form

$$\mathcal{L}_{f,k} = -i\sum_i \psi_i D_\mu \sigma^\mu \psi_i^*, \tag{2.69}$$

where the covariant derivatives are those appropriate to the representation of the gauge group.

Unlike QED (where, in two-component language, parity interchanges e and $\bar{e}^*$), the model does not have a parity symmetry. The fields Q and $\bar{u}$, $\bar{d}$ transform under different representations of the gauge group. There is simply no discrete symmetry that one can find which is the analog of the parity symmetry in QED.

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2.5 The particles and fields of the Standard Model: Higgs scalars and the complete Standard Model

In order to account for the masses of the W and Z bosons and those of the quarks and leptons, the simplest approach is to include a scalar, ϕ, which transforms as a $(1,2)_1$ representation of the Standard Model gauge group. This Higgs field possesses both self-couplings and also Yukawa couplings to the fermions. Its kinetic term is simply

$$\mathcal{L}_{\phi,k} = |D_\mu \phi|^2. \tag{2.70}$$

The Higgs potential is similar to that of our toy model (2.24):

$$V(\phi) = \mu^2|\phi|^2 + \lambda|\phi|^4. \tag{2.71}$$

This is completely gauge invariant. But if μ^2 is negative, the gauge symmetry is broken as before. We will describe this breaking, and the mass matrix of the gauge bosons, shortly.

We could consider a more complicated Higgs sector. For example, we could include multiple Higgs doublets. Or, as we will see in Chapter 8, electroweak symmetry breaking might be the result of some new strong dynamics. But the single Higgs doublet is truly the *simplest* possibility, in the sense that it represents the smallest number of degrees of freedom we can include that will give rise to the observed pattern of gauge boson masses. As of this writing, at the level of precision of the two major LHC experiments, there is evidence for one such doublet and no evidence for additional doublets. Any additional scalars are likely to be heavy compared with the observed Higgs particle and so, if discovered or required by some other theoretical considerations, they can properly be referred to as Beyond the Standard Model physics.

At this point we have written down the most general renormalizable self-couplings of the scalar fields. Renormalizability and gauge invariance permit one other set of couplings in the Standard Model: Yukawa couplings of the scalars to the fermions. The most general such couplings are given by

$$\mathcal{L}_{\rm Yuk} = y^U_{f,f'} Q_f \bar{u}_{f'} \sigma_2 \phi^* + y^D_{f,f'} Q_f \bar{d}_{f'} \phi + y^L_{f,f'} L_f \bar{e}_{f'} \phi. \tag{2.72}$$

Here y^U, y^D and y^L are general matrices in the space of flavors.

We can simplify the Yukawa coupling matrices significantly by redefining fields. Any 3×3 matrix can be diagonalized by separate left and right $U(3)$ matrices. To see this, suppose that one has some matrix M, not necessarily Hermitian. The matrices

$$A = MM^\dagger, \quad B = M^\dagger M \tag{2.73}$$

will be Hermitian; A can be diagonalized by a unitary transformation $U_{\rm L}$, say, and B by a unitary transformation $U_{\rm R}$. In other words

$$U_{\rm L} M U_{\rm R}^\dagger, \quad U_{\rm R} M^\dagger U_{\rm L}^\dagger \tag{2.74}$$

are diagonal. By redefining fields, we can take y_U as diagonal and $M_d = V_{\rm CKM} M'_d$ as diagonal; $V_{\rm CKM}$ is the Cabibbo–Kobayashi–Maskawa (CKM) matrix. This matrix is not unique, and we will present various conventional forms in Section 3.3.

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To summarize, the entire Lagrangian of the Standard Model consists of the following:

1. gauge-invariant kinetic terms for the gauge fields,

$$\mathcal{L}_a = -\frac{1}{4g_s^2}G_{\mu\nu}^2 - \frac{1}{4g^2}W_{\mu\nu}^2 - \frac{1}{4g'^2}F_{\mu\nu}^2 \tag{2.75}$$

   (here we have returned to our scaling with the couplings in front and $G_{\mu\nu}$, $W_{\mu,\nu}$ and $F_{\mu\nu}$ are the $SU(3)$, $SU(2)$ and $U(1)$ field strengths);
2. gauge-invariant kinetic terms for the fermion and Higgs fields, $\mathcal{L}_{f,k}$, $\mathcal{L}_{\phi,k}$;
3. Yukawa couplings of the fermions to the Higgs field, $\mathcal{L}_{\text{Yuk}}$;
4. the potential for the Higgs field, $V(\phi)$.

If we require renormalizability, i.e. that all the terms in the Lagrangian be of dimension four or less, then this is all that we can write down. It is extraordinary that this simple structure incorporates over a century of investigation of elementary particles.

2.6 The gauge boson masses

The field ϕ has an expectation value, which we can take to be as follows:

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}, \tag{2.76}$$

where $v = \mu/\sqrt{\lambda}$. Expanding around this expectation value, the Higgs field can be written as

$$\phi = e^{i\vec{\pi}(x)\cdot\vec{\sigma}/2v}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+\sigma(x)\end{pmatrix}. \tag{2.77}$$

By a gauge transformation we can set $\vec{\pi} = 0$. Not all the gauge symmetry is broken by $\langle\phi\rangle$. It is invariant under the $U(1)$ symmetry generated by

$$Q = T_3 + \frac{Y}{2}. \tag{2.78}$$

This is the electric charge. If we write:

$$L = \begin{pmatrix}\nu\\ e\end{pmatrix}, \quad Q = \begin{pmatrix}u\\ d\end{pmatrix} \tag{2.79}$$

then ν has charge 0 and e has charge -1; u has charge $2/3$ and d has charge $-1/3$. The charges of the singlets also work out correctly.

With this gauge choice we will examine the scalar kinetic terms in order to determine the gauge boson masses. Keeping only terms quadratic in the fluctuating fields (σ and the gauge fields), these now have the form

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\sigma^2) + \frac{1}{2}(0\ v)\left(igW_\mu^i\frac{\sigma^i}{2} + \frac{ig'}{2}B_\mu\right)\left(-igW^{\mu j}\frac{\sigma^j}{2} - \frac{ig'}{2}B^\mu\right)\begin{pmatrix}0\\ v\end{pmatrix}. \tag{2.80}$$

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It is convenient to define the complex fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \pm i W_\mu^2\right) \tag{2.81}$$

These are states of definite charge, since they carry zero hypercharge and $T_3 = \pm 1$. In terms of these fields, the gauge boson mass and kinetic terms take the form

$$\begin{aligned} &\partial_\mu W_\nu^+ \partial^\mu W^{\nu -} + \frac{1}{2}\partial_\mu W_\nu^3 \partial^\mu W^{\nu 3} + \frac{1}{2}\partial_\mu B_\nu \partial^\mu B^\nu \\ &+ \frac{1}{4} g^2 v^2 W_\mu^+ W^{\mu -} + \frac{1}{8} v^2 \left(g W_\mu^3 - g' B_\mu\right)^2. \end{aligned} \tag{2.82}$$

Examining the terms involving the neutral fields, B_μ and W_μ^3, it is natural to redefine

$$A_\mu = \cos\theta_{\rm W}\, B_\mu + \sin\theta_{\rm W}\, W_\mu^3, \quad Z_\mu = \sin\theta_{\rm W}\, B_\mu + \cos\theta_{\rm W}\, W_\mu^3 \tag{2.83}$$

where

$$\sin\theta_{\rm W} = \frac{g'}{\sqrt{g^2 + g'^2}} \tag{2.84}$$

is known as the Weinberg angle. The field A_μ is massless, while the Ws and Zs have the following masses:

$$M_W^2 = \frac{1}{4} g^2 v^2, \quad M_Z^2 = \frac{1}{4}(g^2 + g'^2) v^2 = \frac{M_W^2}{\cos^2\theta_{\rm W}}. \tag{2.85}$$

We can immediately see that A_μ couples to the current

$$\begin{aligned} j_{\rm em}^\mu &= g' \cos\theta_{\rm W} \frac{1}{2} j_\mu^Y + g \sin\theta_{\rm W} j_\mu^3 \\ &= e\left(\frac{1}{2} j_\mu^Y + j_\mu^3\right), \end{aligned} \tag{2.86}$$

where

$$e = \frac{g g'}{\sqrt{g^2 + g'^2}} \tag{2.87}$$

is the electric charge. So A_μ couples precisely as we expect the photon to couple and $W^{\mu\pm}$ couple to the charged currents of the four-fermion theory. The Z boson couples to:

$$j_\mu^Z = -g' \sin\theta_{\rm W} \frac{1}{2} j_\mu^Y + g \cos\theta_{\rm W} j_\mu^3. \tag{2.88}$$

2.7 Quark and lepton masses

On substituting the expectation value for the Higgs field into the expression for the quark and lepton Yukawa couplings, Eq. (2.72) leads directly to masses for the quarks and

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leptons. The lepton masses and the masses for the u quarks follow immediately:

$$m_{ef} = y_{ef}\frac{v}{\sqrt{2}}, \quad m_{uf} = y_{uf}\frac{v}{\sqrt{2}}. \tag{2.89}$$

So, for example, the Yukawa coupling of the electron is $m_e\sqrt{2}/v$.

The masses for the d quarks are somewhat more complicated. Because y_D is not diagonal, we have a matrix in flavor space for the d quark masses:

$$(m_d)_{ff'} = (y_d)_{ff'}\frac{v}{\sqrt{2}}. \tag{2.90}$$

As we have seen, any matrix can be diagonalized by separate unitary transformations acting on from left or the right. So we can diagonalize this matrix by separate rotations of the d quarks (within the quark doublets) and of the $\bar{d}$ quarks. The rotation of the $\bar{d}$ quarks corresponds to a simple redefinition of these fields. But the rotation of the d quarks is more significant, since it does not commute with $SU(2)_{\rm L}$. In other words the quark masses are not diagonal in a basis in which the W boson couplings are diagonal. The basis in which the mass matrix is diagonal is known as the *mass basis* (the corresponding fields are often called mass eigenstates).

The unitary matrix V acting on the d quarks is known as the Cabibbo–Kobayashi–Maskawa, or CKM, matrix. In terms of this matrix the coupling of the quarks to the $W^\pm$ fields can be written as

$$W^-_\mu u_f \sigma^\mu d^*_{f'} V_{ff'} + W^+_\mu d_f \sigma^\mu u^*_{f'} V^*_{f'f}. \tag{2.91}$$

There is a variety of parameterizations of V, which we will discuss shortly. One interesting feature of the model is the Z couplings. Because V is unitary, these are diagonal in flavor. This explains why Z bosons do not mediate processes which change flavor, such as $K_{\rm L} \to \mu^+\mu^-$. The suppression of these *flavor-changing neutral currents* was one of the early, and critical, successes of the Standard Model.

2.8 The Higgs field and its couplings

In the simplest Higgs theory, the couplings of the Higgs are fixed. This includes the couplings to gauge bosons, to fermions and to the Higgs field itself. At tree, or classical, level these can be read off the Lagrangian, as follows.

1. There is a Higgs–ZZ coupling and a Higgs–W^+W^- coupling arising from the replacement of ϕ by $\frac{1}{\sqrt{2}}(v+\sigma)$ in the Higgs kinetic term.
2. There is a Yukawa coupling to all fermions, which is proportional to their masses.
3. There are cubic and quartic self-couplings of the Higgs.

We will discuss these couplings in the context of the Higgs search in the next chapter.

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Suggested reading

There are a number of textbooks with good discussions of the Standard Model, including those of Peskin and Schroeder (1995), Weinberg (1995), Cottingham and Greenwood (1998), Donoghue *et al.* (1992) and Seiden (2005). We cannot give a full bibliography of the Standard Model here, but the reader may want to examine some original papers, including the discovery of non-Abelian gauge theory by Yang and Mills (1954); the Higgs mechanism by Englert and Brout (1964), Guralnik *et al*. (1964) and Higgs (1964); Salam and Ward (1964), Weinberg (1967) and Glashow *et al*. (1970) on weak interaction theory; 't Hooft (1971), Gross and Wilczek (1973) and Politzer (1973) on asymptotic freedom of the strong interactions. For discussion of the various phases found in gauge theories, see 't Hooft (1980) and Fradkin and Shenker (1979).

Exercises

(1) *The Georgi–Glashow model* Consider a gauge theory based on $SU(2)$, with the Higgs field $\vec{\phi}$ in the adjoint representation. Assuming that ϕ attains an expectation value, determine the gauge boson masses. Identify the photon and the $W^{\pm}$ bosons. Is there a candidate for the Z boson?

(2) Consider the Standard Model with two generations. Show that there is no CP violation and that the CKM matrices can be described in terms of a single angle, known as the Cabibbo angle.

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