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UNIQUE FACTORISATION IN P. I. GROUP-RINGS

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Abstract

We shall give necessary and sufficient conditions on the ring R and the group G for the group-ring RG to be a prime P. I. ring with the unique factorisation property as defined in [5].

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1. Introduction

The theory of unique factorisation domains in commutative algebra has been extended in several ways to non-commutative rings, and many genuinely non-commutative examples have been found. One such generalisation is the idea of a U. F. R. (unique factorisation ring) given in [5] for rings which are not assumed to be Noetherian. In the non-Noetherian case the best results in [5] were obtained for rings which satisfy a P. I. (polynomial identity). For instance if R is a P. I. ring and a U. F. R. then R is the intersection of a simple Artinian ring and a Noetherian ring, the invertible ideals of R are principal, and R satisfies the a. c. c. (ascending chain condition) for right ideals cR where c is a regular element of R. Examples include the group-ring RG where R is any prime P. I. ring which is a U. F. R. and G is any torsion-free Abelian group with the a. c. c. for cyclic subgroups [4].

This raised the following question: Assuming that RG is a prime P. I. ring, is it possible to give necessary and sufficient conditions on R and G for RG to be a U. F. R.? The purpose of this paper is to give an affirmative answer to the question. Not surprisingly, the appropriate condition on R is that it should be a prime P. I. ring and a U. F. R. Because RG must satisfy the a. c. c. for certain right ideals, it is not hard to show that G must have the a. c. c. for cyclic subgroups. It turns out that the appropriate extra condition on G is that G should be dihedral-free (see Section 2 for the definition), and this is much harder to prove. The precise answer to the above question is given in Theorem 2.1, but the following is an easily-stated special case: Suppose that R is a prime P. I. ring and that G is a torsion-free Abelian-by-finite group; then RG is a U. F. R. if and only if R is a U. F. R. and G satisfies the a. c. c. for cyclic subgroups.

The main theorem will be stated in Section 2, together with the relevant definitions. The proof is rather complicated, so it is spread over Sections 3 and 4. One of the main techniques is to reduce to finitely-generated subgroups of G which have the same properties as G, and the necessary group-theoretic results for this are given separately in Section 5.

2. Preliminaries

In this section we shall give some definitions and set the scene for the main theorem. For basic material on rings we refer to [3] or [6], and for group-rings [7]. All rings considered here will be associative with identity element.

Let R be a prime ring. An element p of R is *prime* if $pR = Rp$ and pR is a non-zero prime ideal of R. As in [5] we say that R is a *unique factorisation ring* (U. F. R. for short) if every non-zero prime ideal of R contains a prime element. We shall use P. I. as an abbreviation for 'polynomial identity'. Some general results about U. F. R.'s were proved in [5], but the theory was developed much further in Section 4 of that paper in the cases where the U. F. R. is left and right Noetherian or is a P. I. ring.

Let G be a multiplicative group. The *F. C. subgroup* of G, which we shall denote by A, is the set of elements of G which have only a finite number of distinct conjugates. Clearly A contains all proper finite normal subgroups of G. A fact which we shall rely on very heavily is that if G has no proper finite normal subgroups then A is torsion-free Abelian [7, Theorem 4.2.10]. A subgroup H of G is *orbital* in G if H has only a finite number of distinct conjugates by elements of G, or equivalently $N_G(H)$ has finite index in G where $N_G(H)$ is the normaliser of H in G. The infinite dihedral group is the group generated by a and b subject to the relations $b^2 = 1$ and $b^{-1}ab = a^{-1}$. As in [2] we say that G is *dihedral-free* if G has no orbital infinite dihedral subgroups.

Let R be a ring and let G be a group. We shall use RG to denote the corresponding group-ring. In this paper we shall only work with RG when it is a prime P. I. ring, equivalently when R is a prime P. I. ring, G has no proper finite normal subgroups, and G/A is finite where A is the F. C. subgroup of G.

We can now state the main theorem. The proof will be given in Sections 3 and

4. The cases in which G is Abelian or polycyclic-by-finite are already known [4, Theorem 4.2] and [1, Proposition 4.4]. The inspiration for much of this work is [2].

THEOREM 2.1. *Suppose that RG is a prime P. I. ring. Then RG is a U. F. R. if and only if R is a U. F. R. and G is dihedral-free with the ascending chain condition for cyclic subgroups.*

Unfortunately there is more than one definition of a U. F. R. in the literature. We therefore need the following technical result so that we can use material from [1].

LEMMA 2.2. *Let R be a prime P. I. ring, and suppose that R is a U. F. R. as defined above. Then R is also a U. F. R. as defined in* [1] *(this will be explained in the proof).*

PROOF. Let Q be the classical quotient ring of R and let S be a subset of Q. Set $S_\ell = \{q \in Q : qS \subseteq R\}$ and $S_r = \{q \in Q : Sq \subseteq R\}$. In order to show that R is a U. F. R. as defined in [1], we must show that certain ideals of R are principal and that R satisfies a weak form of the a. c. c. for one-sided ideals; these two conditions will be stated precisely and proved in the next two paragraphs.

Let P be a prime ideal of R such that $(P_l)_r = P$ or $(P_r)_l = P$; we must show that $P = pR = Rp$ for some p. This is immediate by Theorem 4.16 of [5].

Let X be a right ideal of R which is closed in the sense that $X = \{x \in R : xH \subseteq X$ for some right ideal H of R such that $(y^{-1}H)_l = R$ for all $y \in R\}$, where $y^{-1}H = \{r \in R : yr \in H\}$. We must show that R has the a. c. c. for such right ideals X. We shall follow the proof of Theorem 4.19 of [5] where a different definition of 'closed' is being used. There are right Noetherian over-rings S and T of R such that $R = S \cap T$. Because S and T are right Noetherian, it is enough to show that $XS \cap XT = X$. Let $b \in XS \cap XT$ and set $H = \{r \in R : br \in X\}$. Let $y \in R$. We shall show that $(y^{-1}H)_l = R$, and it will follow that $b \in X$ and that $XS \cap XT = X$. Set $K = \{r \in R : byr \in X\}$. It was shown in the proof of Theorem 4.19 of [5] that $K_l = R$. We have $byK \subseteq X$, so that $yK \subseteq H$ and $K \subseteq y^{-1}H$. Hence $(y^{-1}H)_l \subseteq K_l = R$, and clearly $R \subseteq (y^{-1}H)_l$. Therefore $(y^{-1}H)_l = R$, as required.

3. Necessary conditions for RG to be a U. F. R.

THEOREM 3.1. *Let R be a ring and let G be a group. Suppose that the group-ring RG is a prime P. I. ring and a U. F. R. Then*

(1) *R is a prime P. I. ring and a U. F. R.*
(2) *G is Abelian-by-finite with no proper finite normal subgroups.*
(3) *G satisfies the a. c. c. for cyclic subgroups.*
(4) *G is dihedral-free.*

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PROOF. We know by [4, proposition 4], that R is a U. F. R. Thus (1) and (2) follow from the standard material of Section 2.

Let A be the F. C. subgroup of G. Then A is torsion-free Abelian and G/A is finite. Hence if A satisfies the a. c. c. for cyclic subgroups then so also does G. Let $x \in A$. Then x is not a torsion element of G. Hence $1 - x$ is a regular element (that is, not a zero-divisor) of RG. But it was shown in [5, Corollary 4.14] that RG satisfies the a. c. c. for right ideals of the form cRG where c is a regular element of RG. Hence RG satisfies the a. c. c. for right ideals of the form $(1-x)RG$ with $x \in A$, from which it follows that A has the a. c. c. for cyclic subgroups, This proves (3).

The remainder of this section is devoted to proving (4). This we do by a series of reductions to get to the case in which R is a field and G is finitely-generated, at which point we shall be able to use the material of [2].

Step (a). To reduce to the case where R is a finite-dimensional central simple algebra. Let Q be the classical quotient ring of R and let F be the centre of Q. Because R is a prime P. I. ring we know that Q is a finite-dimensional central simple F-algebra. Also Q can be formed from R by inverting the non-zero central elements of R. Thus we can regard QG as being the partial quotient ring of RG formed by inverting certain central elements of RG. It is now routine to show that because RG is a U. F. R. then so also is QG.

Step (b). We can now suppose that R is a finite-dimensional central simple F-algebra and we will reduce to the case where R is the full $n \times n$ matrix ring $M_n(F)$ for some positive integer n. Let S be the opposite ring of R and set $T = R \otimes_F S$. Then $T \cong M_n(F)$ for some positive integer n. Because S is a matrix ring over a division ring, the ideals of $RG \otimes_F S$ are of the form $I \otimes_F S$ where I is an ideal of RG (see for instance Case 1 of the proof of [7, Theorem 7.3.9]). It follows easily from this that $RG \otimes_F S$ is a U. F. R. But $RG \otimes_F S \cong (R \otimes_F S)G = TG$. Therefore TG is a U. F. R.

Step (c). We can now suppose that $R = M_n(F)$ for some field F and some positive integer n, and we will reduce to the case where $R = F$. Thus $M_n(F)G$ is a U. F. R., so that $M_n(FG)$ is a U. F. R. At this point we have to be careful because it does not follow in general that FG is a U. F. R.

We shall therefore abandon the U. F. R.-property and work instead with the following weaker condition.

DEFINITION 3.2. Let R be a prime Goldie ring with quotient ring Q. We shall say that R is an *M-ring* if it satisfies the following condition:
If I is a non-zero ideal of R and $q \in Q$ with $qI \subseteq I$ or $Iq \subseteq I$ then $q \in R$.

REMARK 3.3. The terminology of 3.2 is not standard but it is convenient for our purposes. If R is a commutative integral domain then R is an M-ring if and only if R is completely integrally-closed.

Returning to the proof of Step (c), we are supposing that $M_n(FG)$ is a U. F. R. Therefore $M_n(FG)$ is an M-ring, by [5,Corollary 4.8]. It follows by routine arguments that FG is also an M-ring.

Step (d). We can suppose that FG is an M-ring (as defined in 3.2) for some field F, and we will show that FK is also an M-ring where K is a suitable subgroup of G. To be precise, let K be a normal subgroup of G such that K has no proper finite normal subgroups. Then FK is a prime P. I. ring. We shall use $Q(W)$ to denote the classical quotient ring of a ring W. Let T be a transversal for K in G. Then FG is a free FK-module (on both the right and the left) with T as a basis. It follows easily that the regular elements of FK are also regular as elements of FG, so that we can take $Q(FK)$ to be a subring of $Q(FG)$. Let $x \in FG \cap Q(FK)$. Then $xc \in FK$ for some non-zero central element c of FK. Because FG is a free FK-module we have $FGc \cap FK = FKc$. But $xc \in FGc \cap FK$. Hence $xc \in FKc$ so that $x \in FK$. Therefore $FG \cap Q(FK) = FK$. Now let I be a non-zero ideal of FK and let $q \in Q(FK)$ with $qI \subseteq I$. Because K is a normal subgroup of G we know that IG is a two-sided ideal of FG. But $qIG \subseteq IG$ with $q \in Q(FG)$. Because FG is an M-ring it follows that $q \in FG$. Thus $q \in FG \cap Q(FK)$ so that $q \in FK$. Therefore FK is an M-ring.

Step (e). We now make the further assumption that G is not dihedral-free and we will obtain a contradiction. Thus G has an orbital infinite dihedral subgroup D. Let A be the F. C. subgroup of G. Then A has a finite subset X such that if K is any subgroup of G which contains X then $K \cap A$ is the F. C. subgroup of K (see 5.4). Let $D_1, \ldots, D_k$ be the finitely-many distinct conjugates of D in G. Each D_i can be generated by two elements a_i and b_i. Set $Y = \{g^{-1}xg : x \in X, g \in G\}$. Then Y is a finite set because X is a finite subset of the F. C. subgroup of G. Let K be the subgroup of G generated by the set Y and all the elements a_i and b_i. Then K is a finitely-generated normal subgroup of G. Let B be the F. C. subgroup of K. Then $B = K \cap A$ because $X \subseteq K$. Hence B is torsion-free because A is, so that K has no proper finite normal subgroups.

Therefore FK is an M-ring, by Step (d). But K is finitely-generated, and K/B is finite because it embeds in G/A. Thus B is a subgroup of finite index in a finitely-generated group, and it is well known that this implies that B is finitely generated. Because B is Abelian it follows that B is polycyclic and that K is polycyclic-by-finite. Therefore FK is left and right Noetherian. Because FK is an M-ring it follows that FK is a maximal order. Hence K is dihedral-free, by [2, Theorem F]. But $D \subseteq K$ where D is orbital in G. Hence D is an orbital infinite dihedral subgroup of K, which contradicts the fact that K is dihedral-free.

4. Sufficient conditions for RG to be a U. F. R.

THEOREM 4.1. *Let R be a ring and let G be a group. Suppose that the group-ring RG is a prime P. I. ring; that R is a U. F. R.; and that G is dihedral-free and satisfies the a. c. c. for cyclic subgroups. Then RG is a U. F. R.*

The strategy is to use the known cases in which G is Abelian or polycyclic-by-finite. We will break the proof into several stages, but we start with some terminology.

DEFINITIONS 4.2. Let S be a ring. An element x of S is *normal* if $xS = Sx$. A subset I of a group-ring RG is *G-invariant* if I is closed under conjugation by elements of G. Let I be a G-invariant ideal of a subring S of RG; then I is said to be a *G-prime ideal* of S if, whenever U and V are G-invariant ideals of S such that $UV \subseteq I$, then $U \subseteq I$ or $V \subseteq I$.

For the remainder of this section we shall suppose that RG satisfies the hypotheses of 4.1. Also we shall use A to denote the F. C. subgroup of G. In these circumstances A is torsion-free Abelian with the a. c. c. for cyclic subgroups.

LEMMA 4.3. *Let y be a non-zero normal element of RG. Suppose that $y \in RA$ and that yRA is a G-prime ideal of RA. Let H be a finitely-generated dihedral-free subgroup of G such that $HA = G$, the F. C. subgroup of H is $H \cap A$, and $y \in RH$. Then RH is a U. F. R. and yRH is a prime ideal of RH.*

PROOF. We shall repeatedly use the fact that if K and L are subgroups of G with $K \subseteq L$ then RL is free as a left and right RK-module.

Set $B = H \cap A$. Then B is the F. C. subgroup of H, and B is torsion-free Abelian. Hence H has no proper finite normal subgroups. Also H/B is finite because G/A is finite. But H is finitely-generated. Hence B is finitely-generated. Therefore B is polycyclic so that H is polycyclic-by-finite. We are assuming that H is dihedral-free and that R is a U. F. R. Therefore RH is a U. F. R., by [1, Proposition 4.4] (the condition that plinths are centric is a consequence of the fact that A is torsion-free Abelian of finite index in G).

We have $yRG = RGy$. Hence $yRH = RH \cap yRG = RH \cap RGy = RHy$. Thus y is a non-zero normal element of RH, and RH is a U. F. R. Also $y \in RH \cap RA = R(H \cap A) = RB$. We have $yRB = RB \cap yRH = RB \cap RHy = RBy$. Because yRH and RB are H-invariant, so also is yRB.

We shall show that yRB is an H-prime ideal of RB. Let U and V be H-invariant ideals of RB with $UV \subseteq yRB$. Because A centralises RB it follows that UA and VA are two-sided ideals of RA. But U and V are H-invariant with $HA = G$. Therefore UA and VA are G-invariant. Because $UAVA = UVA \subseteq yRA$ where yRA is a

G-prime ideal of RA, it follows that $UA \subseteq yRA$ or $VA \subseteq yRA$. Hence $U \subseteq yRB$ or $V \subseteq yRB$.

Because RH is a U. F. R., we know by [5, Lemma 3.4] that $yRH = P_1P_2\dots P_k$ for some height-1 prime ideals P_i of RH. For each i set $Q_i = RB \cap P_i$. By [1, Lemma 4.2] we have $Q_iRH = P_i$. Clearly each Q_i is H-invariant. Also $Q_1Q_2\dots Q_k = Q_1Q_2\dots Q_kRH \cap RB = P_1P_2\dots P_k \cap RB = yRH \cap RB = yRB$, where yRB is an H-prime ideal of RB. Hence for some i we have $yRB = Q_i$ so that $yRH = Q_iRH = P_i$. Therefore yRH is a prime ideal of RH.

LEMMA 4.4. *Let y be a non-zero normal element of RG such that $y \in RA$ and yRA is a G-prime ideal of RA. Then yRG is a prime ideal of RG.*

PROOF. Suppose that yRG is not a prime ideal of RG. Then there are elements u and v of RG such that $uRGv \subseteq yRG$, $u \notin yRG$, $v \notin yRG$. Let H be any finitely-generated subgroup of G such that $u, v, y \in RH$. Without loss of generality we can suppose that H contains a transversal for A in G so that $HA = G$. By 5.10 we can also suppose without loss of generality that H is dihedral-free and that $H \cap A$ is the F. C. subgroup of H. We know by 4.3 that yRH is a prime ideal of RH. But $uRHv \subseteq uRGv \cap RH \subseteq yRG \cap RH = yRH$, where $u \notin yRH$ and $v \notin yRH$. This is a contradiction.

PROOF OF 4.1. By [4, Theorem 4.2] we know that RA is a U. F. R. Let x be a prime element of RA. Because G/A is finite, the ideal xRA has only finitely-many distinct conjugates by elements of G, and each of them is a principal ideal of RA. Let I be their intersection. Then I is G-invariant, and $I = yRA$ for some non-zero normal element y of RA by [5, Theorem 3.5]. Because I is G-invariant we have $IG = GI$. Hence $yRG = yRAG = IG = GI = RGy$. Thus y is a non-zero normal element of RG.

We shall show that yRA is a G-prime ideal of RA. Let U and V be G-invariant ideals of RA such that $UV \subseteq yRA$. Because $yRA \subseteq xRA$ and xRA is a prime ideal of RA, it follows that $U \subseteq xRA$ or $V \subseteq xRA$. Suppose that $U \subseteq xRA$. Because U is G-invariant it follows that $U \subseteq g^{-1}xRAg$ for all $g \in G$. Therefore $U \subseteq I = yRA$. Similarly if $V \subseteq xRA$ then $V \subseteq yRA$.

Therefore yRG is a prime ideal of RG, by 4.4. Thus corresponding to each prime element x of RA there is a prime element y of RG such that $y \in xRA$. Let S be the partial quotient ring of RA formed by inverting all such elements y. Then every prime element of RA is a unit of S. Let P be a non-zero prime ideal of RA. Because RA is a U. F. R. we have $x \in P$ for some prime element x of RA. Hence $PS = S$, and it follows that S is a simple ring. But S is a P. I. ring because it is contained in the classical quotient ring of the prime P. I. ring RA. Therefore S is simple Artinian, by Kaplansky's theorem.

Because S was formed from RA by inverting certain normal elements of RG, we can take S to be a subring of the classical quotient ring Q of RG. For the rest of this proof we shall use SG to denote the subring of Q generated by S and G, and similarly for SA. Because G/A is finite we know that SG is finitely-generated as an SA-module. But $SA = S$ and S is Artinian. Therefore SG is Artinian. But $RG \subseteq SG \subseteq Q$. Because SG is Artinian it follows that the regular elements of RG are units of SG and hence that $SG = Q$. Let W be a non-zero prime ideal of RG. Then $QW = Q$ because W contains a regular element. Hence $1 \in QW$ where $QW = SGW = SW$. Hence there are prime elements $y_1, \ldots, y_n$ of RG such that $y_1 y_2 \ldots y_n \in W$. Because W is a prime ideal of RG we have $y_i \in W$ for some i. Therefore RG is a U. F. R.

5. Group-theoretic results

We prove here the purely group-theoretic results which were used in Sections 3 and 4. We shall establish in 5.1 some notation and assumptions which will be used subsequently without further explanation. Some of what follows can be proved more generally, but it is convenient to work with the unified assumptions of 5.1.

CONTEXT 5.1. G is a multiplicative group; (x) denotes the cyclic subgroup generated by an element x; D is an infinite dihedral subgroup of G and D is generated by a and b with $b^2 = 1$ and $b^{-1}ab = a^{-1}$; $N = N_G(D)$ is the normaliser of D in G; $C_G(x) = \{g \in G : gx = xg\}$; A is the F. C. subgroup of G; we assume that A is torsion-free Abelian and that G/A is finite.

LEMMA 5.2. *Let H be a group with a torsion-free Abelian normal subgroup B, and let $x \in H$. Then the group $B/C_B(x)$ is torsion-free.*

PROOF. Set $C = C_B(x)$, and $f(y) = y^{-1}x^{-1}yx$ for all $y \in B$. Then f is a homomorphism from B to B, and $\operatorname{Ker}(f) = C$. Therefore B/C is isomorphic to a subgroup of B and so is torsion-free.

LEMMA 5.3. *There are finitely-many subgroups $C_1, \ldots, C_n$ of A such that*

(1) *for each i we have $C_i \neq A$ and A/C_i is torsion-free, and*

(2) *for each $x \in G$ with $x \notin A$ we have $C_A(x) = C_i$ for some i.*

PROOF. Let $g_0, g_1, \ldots, g_n$ be a transversal for A in G with $g_0 = 1$. For each i from 1 to n set $C_i = C_A(g_i)$. Then A/C_i is torsion-free, by 5.2. Suppose that $C_i = A$ for

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some i. Then $A \subseteq C_G(g_i)$. But G/A is finite. Hence $C_G(g_i)$ has finite index in G, so that g_i has only finitely-many conjugates in G. Thus $g_i \in A$. This is a contradiction because $i \neq 0$ and $g_0, g_1, \ldots, g_n$ form a transversal for A in G with $g_0 = 1$.

Now let $x \in G$ with $x \notin A$. We have $x \in g_i A$ for some $i \neq 0$. Hence $C_A(x) = C_A(g_i) = C_i$.

PROPOSITION 5.4. *There is a finite subset S of A such that if K is any subgroup of G which contains S then $K \cap A$ is the F. C. subgroup of K.*

PROOF. Let $C_1, \ldots, C_n$ be as in 5.3. By 5.2 and 5.3 we can fix, for each i, an element s_i of A such that the image of s_i in A/C_i has infinite order. Set $S = \{s_1, \ldots, s_n\}$. Let K be any subgroup of G which contains S, and let B be the F. C. subgroup of K. Clearly $K \cap A \subseteq B$. Let $x \in K$ with $x \notin A$. By 5.3 we have $C_A(x) = C_i$ for some i. For this value of i set $C = C_i$ and $s = s_i$. We have $s \in S$ so that $s \in K$. Hence $s^{-j}xs^j \in K$ for every integer j. Suppose that $s^{-j}xs^j = s^{-k}xs^k$ for some integers j and k. Then $s^{j-k} \in C$. But the image of s in A/C has infinite order. Therefore $j = k$. It follows that x has infinitely-many distinct conjugates in K, that is, $x \notin B$. Hence $B \subseteq K \cap A$.

LEMMA 5.5. *Let D, a, b, N be as in* 5.1 *and let $x \in N$. Then $x^{-1}ax = a$ or a^{-1}.*

PROOF. Let (a) be the cyclic subgroup of D generated by a. Then the non-identity elements of (a) have infinite order, and the elements of D which are not in (a) have order 2. Hence any automorphism of D restricts to an automorphism of (a). Therefore conjugation by x induces an automorphism of (a). It follows that $x^{-1}ax$ is a generator of (a), that is, $x^{-1}ax = a$ or a^{-1}.

LEMMA 5.6. *Let $x \in N$. Then $x^4a^j \in C_G(b)$ for some integer j.*

PROOF. By 5.5 we have $x^{-1}ax = a$ or a^{-1}. In both cases it follows that $x^{-2}ax^2 = a$, that is, $ax^2 = x^2a$. Because b is an element of D of order 2, so also is $x^{-2}bx^2$. Hence $x^{-2}bx^2 = ba^i$ for some integer i. Thus $x^{-4}bx^4 = x^{-2}ba^ix^2 = x^{-2}bx^2a^i = ba^{2i}$. Therefore $bx^4a^{-i} = x^4ba^i = x^4a^{-i}b$, that is, $x^4a^{-i} \in C_G(b)$.

LEMMA 5.7. *Suppose that D is orbital in G. Then $a \in A$.*

PROOF. We are in effect assuming that N has finite index in G. Hence there are finitely-many elements $g_1, \ldots, g_k$ of G such that every element of G belongs to Ng_i for some i. Let $g \in G$. Then $g = xg_i$ for some $x \in N$ and some i. By 5.5 we have $x^{-1}ax = a$ or a^{-1}. Therefore $g^{-1}ag = g_i^{-1}ag_i$ or $g_i^{-1}a^{-1}g_i$ so that there are only finitely-many possibilities for $g^{-1}ag$.

PROPOSITION 5.8. *Let H be a torsion-free Abelian group which satisfies the a. c. c. for cyclic subgroups, let U be a cyclic subgroup of H, and let V be the inverse image in H of the torsion subgroup of H/U. Then V is cyclic.*

PROOF. For each positive integer n set $V_n = \{x \in V : x^n \in U\}$. Then $V = \bigcup_{n=1}^{\infty} V_n$. Clearly if r and s are positive integers such that r divides s then $V_r \subseteq V_s$. We shall show that each V_n is cyclic. It will then follow easily from the a. c. c. for cyclic subgroups that V is cyclic.

For the rest of this proof n denotes a fixed positive integer. Set $W = \{x^n : x \in V_n\}$. Then W is a subgroup of the cyclic group U so that W is cyclic. We fix $x \in V_n$ such that x^n generates W. Let $y \in V_n$. Then $y^n \in W$ so that $y^n = (x^n)^k$ for some integer k. Thus $(yx^{-k})^n = 1$. But H is torsion-free. Therefore $y = x^k$, and it follows that x generates V_n.

LEMMA 5.9. *Suppose that G is dihedral-free and satisfies the a. c. c. for cyclic subgroups. Then $A/C_A(b)$ is torsion-free of rank at least 2.*

PROOF. Set $C = C_A(b)$. We know by 5.2 that A/C is torsion-free. Because G/A is finite we have $a^k \in A$ for some positive integer k. The subgroup of G generated by a^k and b is infinite dihedral, so that without loss of generality we may suppose that $a \in A$.

Set $U = (a)$. For every non-zero integer n we have $b^{-1}a^n b = a^{-n} \neq a^n$, that is $a^n \notin C$. Thus $U \cap C = 1$. Hence $UC/C \cong U$ so that UC/C is infinite cyclic. Thus UC/C is an infinite cyclic subgroup of A/C. In order to show that A/C has rank at least 2, it is enough to show that $(A/C)/(UC/C)$ has an infinite cyclic subgroup, that is that A/UC is not torsion.

With the aim of obtaining a contradiction, we suppose that A/UC is torsion. Let V be the inverse image in A of the torsion subgroup of A/U. By 5.8 we have $V = (v)$ for some v. Conjugation by b gives automorphisms of both A and U and hence also of V. Therefore $b^{-1}vb = v$ or v^{-1}. But $a \in (v)$ and $ba \neq ab$. Hence $bv \neq vb$, so that $b^{-1}vb = v^{-1}$. In particular $bV = Vb$. Let H be the subgroup of G generated by b and A. Then V is a normal subgroup of H. Set $C' = \{x \in A : xV \text{ commutes with } bV \text{ in } H/V\}$. Then C' is a subgroup of A and $V \subseteq C'$. Also $C'/V = C_{A/V}(bV)$. Because A/V is torsion-free, we know by 5.2 that $(A/V)/(C'/V)$ is torsion-free. Hence A/C' is torsion-free. Clearly $C \subseteq C'$. Also $U \subseteq V \subseteq C'$. Thus $UC \subseteq C'$. We are assuming that A/UC is torsion, and we have just shown that A/C' is torsion-free. Therefore $A/C' = 1$, that is $A = C'$. Therefore $A/V = C_{A/V}(bV)$, from which it follows that H/V is Abelian.

At this point we have shown that H/V is Abelian, where H is the subgroup of G generated by b and A. We showed above that $b^{-1}vb = v^{-1}$. Let E be the subgroup of G generated by v and b. Then E is infinite dihedral. We have $V \subseteq E \subseteq H$ with

H/V Abelian. Therefore E is a normal subgroup of H, so that $H \subseteq N_G(E)$. But $A \subseteq H$ and G/A is finite. Therefore $N_G(E)$ has finite index in G, which contradicts the fact that G is dihedral-free.

THEOREM 5.10. *Let G be a dihedral-free group with the a. c. c. for cyclic subgroups, and suppose that the F. C. subgroup A of G is torsion-free Abelian with G/A finite. Then there is a finite subset X of A with the following property: If K is any subgroup of G which contains X then K is dihedral-free and $K \cap A$ is the F. C. subgroup of K.*

PROOF. Let $C_1, \ldots, C_n$ be as in 5.3. For each i we know that A/C_i is torsion-free of positive rank. We define elements s_i and t_i of A as follows: if A/C_i has rank at least 2 we choose s_i and t_i so that their images in A/C_i are linearly independent; if A/C_i has rank 1 set $t_i = 1$ and choose s_i so that its image in A/C_i has infinite order. Set $X = \{s_i, t_i : 1 \leq i \leq n\}$.

Let K be a subgroup of G which contains X and set $B = K \cap A$. We have already shown in the proof of 5.4 that B is the F. C. subgroup of K. We must show that K is dihedral-free. Suppose that K has an infinite dihedral subgroup D generated by a and b with $b^2 = 1$ and $b^{-1}ab = a^{-1}$. Because b has infinitely-many conjugates in D, we have $b \notin A$. Hence $C_A(b) = C_i$ for some i, by 5.3. This value of i is fixed for the rest of the proof. By 5.9 we know that the rank of A/C_i is at least 2. Hence the images of s_i and t_i in A/C_i are linearly independent. Set $s = s_i$, $t = t_i$, $C = C_i = C_A(b)$, and $C' = B \cap C = C_B(b)$. We have $s, t \in X$ with $X \subseteq K \cap A$. Therefore $s, t \in B$.

We wish to show that D is not orbital in K, and this is immediate by 5.7 if $a \notin B$. From now on we suppose that $a \in B$. Because sC and tC are linearly independent in A/C, it follows easily that sC' and tC' are linearly independent in B/C'. Thus the rank of B/C' is at least 2. But aC' has infinite order in B/C'; this is because if k is a non-zero integer then a^k does not commute with b. Therefore we can fix $z \in B$ such that aC' and zC' are linearly independent in B/C'. Let j and k be integers such that $z^{-j}Dz^j = z^{-k}Dz^k$. Then $z^{j-k} \in N_K(D)$. By 5.6 we have $(Z^{j-k})^4 a^r \in C_K(b)$ for some integer r. Hence $(z^{j-k})^4 a^r \in C_B(b) = C'$. But zC' and aC' are linearly independent in B/C'. Therefore $j = k$. Thus for distinct values of j the conjugates $z^{-j}Dz^j$ are distinct, so that D is not orbital in K.

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