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On the definition of saturated formations of groups

John Cossey and Sheila Oates Macdonald

We exhibit a closure operation which serves to define saturated formations of finite soluble groups.

1. Introduction

Classes of groups defined in terms of closure operations have proved both interesting and useful. Some classes originally defined in terms of more than one operation can be defined in terms of a single closure operation which is a product of the original operations, for example, varieties and formations. In this paper we attempt to do the same for saturated formations, with partial success. In Theorem 3.1 we obtain a closure operation for saturated formations of soluble groups. This particular operation is unusual in that it uses twice each of the original closure operations. We do not know if this is best possible, although it is not difficult to check that it is not possible to define such an operation using each of the original operations once only. Our techniques rely heavily on the solubility of the groups involved; we are unable to say anything in the insoluble case.

2. Notation and preliminary results

For notation and basic facts about formations we refer the reader to [2] or [3]. For the remainder of this paper all groups considered will be finite and soluble.

If X is a class of groups, fX will denote the formation and sfX

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the saturated formation generated by X . If $X = \{G\}$ we will write fG and sfG .

If F is a formation, G^F will denote the F-residual of G , that is,

$$G^F = \cap\{N : N \triangleleft G \text{ and } G/N \in F\} .$$

If F and G are formations, then

$$FG = \{G : G^G \in F\}$$

is also a formation.

S_p denotes the formation of p-groups, p a prime.

A closure operation is a map P defined on classes of groups which satisfies:-

(i) $X \leq Y \Rightarrow PX \leq PY$;

(ii) $X \leq PX = P(PX)$.

We have:-

LEMMA 2.1. *Q, R_0 , and Φ are closure operations where:-*

$QX = \{H : G/N \simeq H$ *and* $G \in X\}$ *;*

$$R_0X = \left\{H : N_i \triangleleft H,\ i = 1, \ldots, n;\ \bigcap_{i=1}^{n} N_i = 1 \text{ and } H/N_i \in X\right\} ;$$

$\Phi X = \{H : H/\Phi(H) \simeq G/\Phi(G)$ *and* $G \in X\}$ *where* $\Phi(H)$ *is the Frattini subgroup of* H *.*

We shall need the following well known fact (see, for example, the introduction to [*1*]).

LEMMA 2.2. $R_0QX \leq QR_0X = fX$.

The socle of a group G is denoted by σG . If G has a unique minimal normal subgroup we call G monolithic and σG its monolith.

We say that a group with trivial Frattini subgroup is Φ-free. We need:-

LEMMA 2.3. *If* F *is a saturated formation and* X *the class of* Φ*-free monolithic groups in* F *, then*

$$F = \Phi R_0 X .$$

Proof. Let $G \in F$, then $H = G/\Phi(G)$ is Φ-free. Let $\sigma H = N_1 \times \ldots \times N_r$ be the decomposition of the socle of H into minimal normal subgroups of H ; then $H \in R_0\{H/K_i, i = 1, \ldots, r\}$ where K_i is a maximal normal subgroup of H containing $N_1 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_r$ and avoiding N_i . Clearly $H/K_i \in F$ and is monolithic (with monolith isomorphic to N_i). But it is also Φ-free since it splits over its monolith.

LEMMA 2.4. *Let* $\mathcal{D}$ *be the set of* Φ*-free monolithic quotients of the group* G *and let* F_p *be the formation generated by* $\{D/\sigma D : D \in \mathcal{D}$ *and* $p \mid |\sigma D|\}$ *for each prime* p *dividing* $|G|$ *. Then if* H *is a monolithic group in* sfG *whose monolith is a* p*-group,* $H \in S_pF_p$ *.*

Proof. Let $F = \bigcup_{p \mid |G|} S_pF_p$. Clearly $sfG = \bigcup_n (\Phi QR_0)^n F$. Let H be a monolithic group in $(\Phi QR_0)^n F$ whose monolith is a p-group. We prove by induction on n that $H \in S_pF_p$. For $n = 0$, we have $H \in F$ and so either $H \in S_pF_p$ or $H \in F_q$ for some $q \neq p$. In the latter case

$$H \in QR_0\{D/\sigma D : q \mid |\sigma D|\} .$$

As in [4] we can obtain an expression for H as a quotient of a subdirect product of monolithic groups whose monoliths are p-groups. It follows that $H/\sigma H \in F_p$ and so $H \in S_pF_p$.

Now take $n \geq 1$ and assume the lemma true for $n - 1$. Since σH is a p-group so also is $\sigma(H/\Phi(H))$. But $H/\Phi(H) \in QR_0(\Phi QR_0)^{n-1}F$. Again, using the methods of [4] we obtain that $H/\Phi(H) \in QR_0\{A_i\}$ where the A_i are monolithic with σA_i a p-group and belong to $(\Phi QR_0)^{n-1}F$. By the induction hypothesis A_i and hence $H/\Phi(H)$ belongs to S_pF_p . Thus

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$H \in S_p F_p$ as required.

The next lemma is rather technical in nature, and may be regarded as a generalisation of [3] VI 7.22. For notation and results about varieties of groups we refer the reader to [6].

LEMMA 2.5. *Let* M *be a faithful* $\mathrm{GF}(p)G$ *module of rank* r . *Then there exists a group* L *which is the split extension by* G *of a metabelian class* c *p-group* N *such that* $\gamma_c(N) \simeq M^{\otimes c}$ *, where* $\gamma_c(N)$ *is regarded as a* $\mathrm{GF}(p)G$*-module, and* $M^{\otimes c}$ *denotes the* c*-fold tensor power of* M .

Proof. Let F be the free group of rank rc in the variety consisting of the groups of class c in the product variety $\underline{\underline{A}}_p\underline{\underline{A}}_p$, that is,

$$F = F_{rc}\left(\underline{\underline{A}}_p\underline{\underline{A}}_p \cap \underline{\underline{N}}_c\right) .$$

Suppose $F = \langle x_{ij} : 1 \le i \le c,\ 1 \le j \le r\rangle$; then for each $g \in G$, we can define an action on the generators of F by letting g act on the sets

$$T_i = \{x_{ij} : 1 \le j \le r\}$$

as it does on a set of generators for M , and extending this action to an automorphism of F . We can thus regard G as a set of automorphisms of F . Now let K be the subgroup of F generated by all those left-normed commutators which have an entry from some T_i in more than one place or which are of weight c and have an entry in the first two places other than from T_1 and T_2 . Let J be the subgroup of F generated by those left-normed commutators of weight c of the form:-

$$\left[x_{1j_1}, \ldots, x_{cj_c}\right] .$$

Then it can be extracted from 4.05 of [5] that $K \cap J = 1$. Also it is readily checked that K and J are normal in F and are mapped to themselves by the automorphisms induced by G . Consider F/K . G can be considered as a subset of the automorphism group of F/K , and it is easy to check that it is actually a subgroup. We see that $N = F/K$ satisfies the required conditions; for $\gamma_c(N)$ is an elementary abelian p-group of rank rc and the mapping

$$\psi : M^{\otimes c} \to \gamma_c(N)$$

defined by

$$x_{1j_1} \otimes \ldots \otimes x_{cj_c}\psi = \left[x_{1j_c}, \ldots, x_{cj_c}\right]$$

is clearly a module isomorphism. Hence we may take L to be the split extension of N by G.

3. Proof of the theorem

THEOREM 3.1. $sfX = \left(\Phi QR_0\right)^2 X$.

Proof. We consider first the case in which $X = \{G\}$. By Lemma 2.3 $sfG = \Phi R_0 Y$ where Y is the set of Φ-free monolithic groups in sfG. Since $R_0 QR_0 = QR_0$, the proof will be complete if we show that

$$Y \leq QR_0 \Phi QR_0 G ,$$

and, by Lemma 2.4, it will suffice to prove that if H is a Φ-free monolithic group in $S_p F_p$ then

$$H \in QR_0 \Phi QR_0 G .$$

Since

$$F_p = QR_0\{D/\sigma D\} \leq QR_0 G$$

we can assume $H \notin F_p$; and so σH is a p-group. Since H is Φ-free this is also its Fitting subgroup, and so $H/\sigma H \in F_p$; say $H/\sigma H \simeq S/T$ where S is subdirect in $\prod_{i=1}^{s} A_i$, $A_i \simeq D_i/\sigma D_i$. Then $B = \sigma H$ is an irreducible $\mathrm{GF}(p)S/T$ module, and a fortiori, an irreducible S module. There is a natural epimorphism from $\prod D_i$ to $\prod A_i$. Let $\tilde{S}$ be the inverse image in $\prod D_i$ of S. Then, by the definition of the D_i, $\tilde{S}$

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is the split extension by S of a faithful $\mathrm{GF}(p)S$ module M, and $\tilde{S} \leq QR_0G$. By [3] VI 7.19, B is a composition factor of $M^{\otimes c}$ for some c. Consider the group L defined as in Lemma 2.5. Then

$$L/\Phi(L) \in R_0\tilde{S} \leq QR_0G$$

and so $L \in \Phi QR_0G$. By [3] VI 7.21 the split extension of B by S lies in fL and hence so does $B.S/T$. But this is H. Hence

$$H \in QR_0L \leq QR_0\Phi QR_0G$$

as required.

For the general case consider $K \in sf\mathfrak{X}$. Then K is obtained from $\mathfrak{X}$ by a finite number of applications of Φ, Q and R_0, say $K \in (\Phi QR_0)^n\mathfrak{X}$. We prove by induction on n that K is obtained from a finite number of elements of $\mathfrak{X}$; the result being true for $n = 0$. Now

$$K \in \Phi QR_0(\Phi QR_0)^{n-1}\mathfrak{X} ,$$

that is, $K/\Phi(K) \simeq S/T$ where $S \leq X_1 \times \ldots \times X_r$ and $X_i \in (\Phi QR_0)^{n-1}\mathfrak{X}$. By the induction hypothesis, each X_i arises from only finitely many members of $\mathfrak{X}$ and hence so also does K. Thus $K \in sfG$ where G is the direct product of these members of $\mathfrak{X}$ and hence

$$K \in (\Phi QR_0)^2G \leq (\Phi QR_0)^2\mathfrak{X} .$$

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School of General Studies,
The Australian National University,
Canberra, ACT,
and
The University of Queensland,
St Lucia, Queensland.