Can. J. Math., Vol. XXVIII, No. 4, 1976, pp. 875–878

A MINIMAL REGULAR SPACE THAT IS NOT STRONGLY MINIMAL REGULAR

DIX H. PETTEY

1. Introduction. A regular T_1 space is said to be *R-closed* if there is no regular T_1 space in which it can be embedded as a nonclosed subspace. A regular T_1 space is said to be *minimal regular* if no regular T_1 topology on the underlying set is strictly weaker than the given topology. It is known (see [**1**, Theorem 4, p. 455]) that every minimal regular space is R-closed. An R-closed space, however, need not be minimal regular [**3**, Example 2, p. 288].

In [**5**] R. M. Stephenson introduced the notion of strong minimal regularity. A regular T_1 space X is said to be *strongly minimal regular* if there is a base $\mathscr{V}$ for the topology of X such that for each V in $\mathscr{V}$, $X - V$ is an R-closed subspace of X. Stephenson noted [**5**, p. 287] that every strongly minimal regular space is minimal regular but left unanswered the question as to whether the converse is true. The main purpose of this paper is to give an example of a minimal regular space which is not strongly minimal regular. This will be done in Section 3.

2. Preliminaries. A filterbase on a topological space X is said to be *open* if each of its members is an open set in X.

An open filterbase $\mathscr{B}$ on a topological space is called a *regular* filterbase if for every member B of $\mathscr{B}$ there is a member B' of $\mathscr{B}$ such that $\bar{B}' \subseteq B$.

It is well-known (see [**1**, Theorem 2, p. 454]) that a regular T_1 space X is minimal regular if and only if every regular filterbase on X having a unique cluster point is convergent. With this characterization, we obtain the following result, of which we shall make later use.

(2.1) *If an R-closed space X is the union of countably many minimal regular subspaces, then X is minimal regular.*

Proof. Assume the contrary. Then there is a regular filterbase $\mathscr{B}$ on X such that $\mathscr{B}$ has a unique cluster point x but does not converge to x. Since X is regular, there is an open neighborhood V of x such that $\bar{V}$ contains no member of $\mathscr{B}$. Now by our hypothesis, X is the union of minimal regular subspaces $X_1, X_2, X_3, \ldots$. For each i, either $x \in X_i$—in which case $\{B \cap X_i | B \in \mathscr{B}\}$ is a regular filterbase on X_i having x as its only cluster point and therefore converging to x—or some member of $\mathscr{B}$ fails to intersect X_i. Thus, we can choose a nested sequence $B_1, B_2, B_3, \ldots$ of members of $\mathscr{B}$ such that for each i, $\bar{B}_i \cap X_i \subseteq V$. Since $\bar{V}$ contains no member of $\mathscr{B}$, the collection $\{B_i - \bar{V}\}_{i=1}^{\infty}$

Received December 15, 1975.

is an open filterbase on X. From [**5**, Lemma 2.4(i), p. 287] it follows that for some m, $\{B_i - \bar{V}\}_{i=1}^{\infty}$ has a cluster point in X_m. Clearly this cluster point must lie in $X_m - V$. But for each i greater than or equal to m, $\bar{B}_i \cap X_m \subseteq \bar{B}_m \cap X_m \subseteq V$. Hence, we have a contradiction and the proof is complete.

We shall also make use of the known results which are listed below.

(2.2) *If X is a minimal regular space and Y a compact T_2 space then $X \times Y$ is minimal regular* [**4**, Corollary 3.17, p. 139].

(2.3) *A topological space is R-closed if it is the image, under a perfect continuous function, of an R-closed space.* (This follows immediately from [**2**, Theorem 5.2, p. 235] and [**4**, Theorem 3.6, p. 137].)

(2.4) *Every completely regular R-closed space is compact* [**1**, Theorem 3, p. 455].

(2.5) *Every open and closed subspace of an R-closed space is R-closed* [**5**, Lemma 2.2, p. 287].

3. Example. Throughout this section we shall let J denote the set of all integers. The symbols ω and Ω will be used to denote, respectively, the first infinite ordinal and the first uncountable ordinal. For each ordinal α, $[0, \alpha]$ will denote the set of all ordinals less than or equal to α. All ordered sets will be regarded as ordered topological spaces.

As building blocks in the construction of our example, we shall use copies of the noncompact minimal regular space Z described by Berri and Sorgenfrey in [**1**]. This space can be obtained as follows.

Let S denote the space obtained from

$$[0, \omega] \times [0, \Omega] \times J - \{(\omega, \Omega, i) | i \in J\}$$

by making the following identifications: for each even i in J and each ordinal α less than Ω, let $(\omega, \alpha, i) = (\omega, \alpha, i + 1)$; for each odd i in J and each ordinal n less than ω, let $(n, \Omega, i) = (n, \Omega, i + 1)$. We note here, for later use, the fact that S is a locally compact T_2 space and is therefore completely regular. Now for each i in J, let T_i denote the set $[0, \omega] \times [0, \Omega] \times \{i\}$ in S. Let p and q be two points not in S and let Z denote the space $\{p\} \cup \{q\} \cup S$ with the following topology: a set V in Z is open if and only if (1) $V \cap S$ is open in S, (2) if $p \in V$ then V contains T_i for all but finitely many positive i, and (3) if $q \in V$ then V contains T_i for all but finitely many negative i. For a proof that Z is minimal regular, the reader is referrred to [**1**].

We shall now construct a space M which is minimal regular but not strongly minimal regular.

Let P and Q denote, respectively, the sets $\{p\} \cup \bigcup_{i=0}^{\infty} T_i$ and $\{q\} \cup \bigcup_{i=1}^{\infty} T_{-i}$ in Z. Let $F = P \cap Q$. (Then F is the noncompact closed set $\{(n, \Omega, 0) | n < \omega\}$.) We obtain the space M from $Z \times [0, \omega + 1]$ by first identifying (p, ω) with (q, ω) and then, for each z in F, identifying (z, ω) with $(z, \omega + 1)$.

To prove that M is minimal regular, we first observe that, by (2.2), $Z \times [0, \omega + 1]$ is minimal regular. The quotient projection from $Z \times [0, \omega + 1]$ onto M is closed and has compact point inverses, so it follows from (2.3) that M is R-closed. Now in the space M, each of the sets $P \times \{\omega\} \cup Q \times \{\omega + 1\}$ and $P \times \{\omega + 1\} \cup Q \times \{\omega\}$ is homeomorphic to Z and therefore minimal regular. Since M is the union of these two sets and all sets $Z \times \{n\}$ $(n < \omega)$, we have M the union of countably many minimal regular subspaces. Hence, by (2.1), M is minimal regular.

We now assert that M is not strongly minimal regular. For suppose that V is an open neighborhood of (p, ω) $(= (q, \omega))$ such that V does not intersect the closed set $F \times [0, \omega]$. For some m less than ω, each of (p, m) and (q, m) is in V. Therefore, $F \times \{m\} \subseteq Z \times \{m\} - V \subseteq S \times \{m\}$. Since $F \times \{m\}$ is noncompact and closed and since $S \times \{m\}$ is completely regular, $Z \times \{m\} - V$ must be noncompact and completely regular. Thus, by (2.4), $Z \times \{m\} - V$ is not R-closed. But $Z \times \{m\} - V$ is both open and closed in $M - V$, so it follows from (2.5) that $M - V$ is not R-closed. We conclude, then, that M is not strongly minimal regular.

4. Questions concerning product spaces. Since we have now shown that the class of strongly minimal regular spaces differs from the class of minimal regular spaces, it may be of interest to briefly consider some questions concerning strongly minimal regular spaces and products.

In [**5**] Stephenson observed that the product of a strongly minimal regular space and a compact T_2 space is always strongly minimal regular. In the same paper he gave an example of a noncompact strongly minimal regular space T such that for each strongly minimal regular space Y, $T \times Y$ is strongly minimal regular.

The following questions remain open.

Q1. Must the product of strongly minimal regular spaces be strongly minimal regular?

Q2. If each of X and Y is a topological space and if $X \times Y$ is strongly minimal regular then must one of the spaces X and Y be strongly minimal regular?

Question Q1 is analogous to open questions concerning R-closed spaces and minimal regular spaces (see [**4**, pp. 137-138]). It is quite easy to show that if every product of R-closed spaces is R-closed then Q1 has an affirmative answer.

Questions similar to Q2 have also been considered for R-closed spaces and minimal regular spaces. In fact, it is known (see [**4**, 3.7, p. 137 and 3.9, p. 138]) that if a product space is R-closed [minimal regular, respectively] then each factor is R-closed [minimal regular]. For strongly minimal regular spaces, however, the problem appears to be more difficult.

REFERENCES

1. Manuel P. Berri and R. H. Sorgenfrey, *Minimal regular spaces*, Proc. Amer. Math. Soc. *14* (1963), 454–458.

2. James Dugundji, *Topology* (Allyn and Bacon, Boston, Mass., 1966).
3. Horst Herrlich, *T_v-Abgeschlossenheit und T_v-minimalität*, Math. Z. *88* (1965), 285–294.
4. C. T. Scarborough and A. H. Stone, *Products of nearly compact spaces*, Trans. Amer. Math. Soc. *124* (1966), 131–147.
5. R. M. Stephenson, *Two R-closed spaces*, Canad. J. Math. *24* (1972), 286–292.

University of Missouri,
Columbia, Missouri