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A SIMPLE RING OVER WHICH PROPER CYCLICS ARE CONTINUOUS IS A PCI-RING

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ABSTRACT. It is shown that simple rings over which proper cyclic right modules are continuous coincide with simple right PCI-rings, introduced by Faith.

1. **Introduction.** Rings over which proper cyclics are injective (called PCI-rings) have been characterized by Faith [4] and Damiano [3] as semisimple artinian rings or simple right noetherian, right hereditary domains over which each proper cyclic module is semisimple. For an example of a right PCI-ring, one may refer to Cozzens [2]. Recently, Huynh, Jain and López-Permouth [9] showed that simple rings over which proper cyclics are quasi-injective (called PCQI-rings) are same as simple PCI-rings. In this paper we extend this result by showing that a simple ring over which proper cyclics are continuous is indeed a PCI-ring thus obtaining the latter result as a corollary. Our proof has a strong computational flavor, as it is based on looking at the form of the cyclic modules involved and on the precise computation of the intersection of certain pair of modules.

We first show that a simple ring over which proper cyclics are continuous is either simple artinian or a right Ore domain (Proposition 2.4). A particular instance of a lemma, which is essentially due to Stafford, is needed here. We provide a simple, self-contained proof of it (Lemma 2.2). In Theorem 2.5, we prove our main result.

A right R-module M is called *continuous* if it satisfies the conditions (C1): every submodule N of M is essential in a summand of M, and (C2): every submodule N of M which is isomorphic to a direct summand of M is itself a direct summand of M. M is called *quasi-continuous* (π-injective) if for all submodules N_1, N_2 of M with $N_1 \cap N_2 = (0)$, the projection map $\pi\colon N_1 \oplus N_2 \longrightarrow N_1$ can be lifted to an endomorphism of M (*c.g.* [13], p. 367), equivalently, if M satisfies the above condition (C1) and the condition (C3): for every direct summands N_1, N_2 of M such that $N_1 \cap N_2 = (0)$, $N_1 \oplus N_2$ is also a direct summand ([13], 41.21). By a proper cyclic R-module we mean a cyclic module that is not isomorphic to the ring R. For a right module M, soc(M), and $\hat{M}$ will respectively denote the socle, and injective hull of M. A ring R satisfies the condition ($*$) if:

($*$) every proper cyclic right R-module is continuous.

Throughout, all rings have unity and all modules are right unital, unless otherwise stated.

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2. **Rings satisfying ($*$).** Recall a right R-module M is said to be *CS* if every closed submodule is a direct summand of M, equivalently, if every submodule of M is essential in a direct summand, *i.e.*, M satisfies condition (C1) stated in the introduction. We will need the following.

LEMMA 2.1 ([9], THEOREM A). *If R is a simple ring such that every cyclic singular right R-module is* CS*, then R is right noetherian.*

Our next lemma is essentially due to Stafford ([1], Theorem 14.1). For the sake of completeness, we give here a short direct proof.

LEMMA 2.2. *Let R be a simple Goldie ring which is not artinian. Let M be a singular R-module. If $M = aR \oplus bR$ for some $a, b \in M$ and if bR is simple, then $M = (a + bx)R$ for some $x \in R$.*

PROOF. Since M is singular and R is prime Goldie ring, there exists a regular element $d \in R$ such that $ad = 0$. Now R is simple and $RdR \neq 0$, therefore, $RdR = R$. Thus $bR = bRdR$. Since $bR \neq 0$, $bxd \neq 0$ for some $x \in R$. Since bR is simple, $bR = bxdR$. Now $(a + bx)d = bxd$. Thus $bR = bxdR \subset (a + bx)R$. Also, as $a = (a + bx) - bx$ and $bR \subset (a + bx)R$, it follows that $aR \subset (a + bx)R$. Thus $M = (a + bx)R$.

In ([9], Lemma 3.1), it is shown that if R is a simple right PCQI-domain, then every finitely generated artinian R-module is semisimple. With the aid of Lemma 2.2, and using an argument similar to the one used in ([9], Lemma 3.1), we obtain the following Lemma.

LEMMA 2.3. *If R is a simple domain satisfying ($*$), then every finitely generated artinian right R-module is semisimple.*

PROPOSITION 2.4. *A simple ring satisfying ($*$) is either simple artinian or a right Ore domain.*

PROOF. By Lemma 2.1, R is right noetherian and hence has finite right uniform dimension. Thus, R is a right Goldie ring. If R_R is uniform then either R is a division ring or a right Ore domain. Consider next the case when R_R is not uniform. Let the uniform dimension of R be $n \geq 2$ and $U_1, U_2, \ldots, U_n$ be uniform cyclic right ideals of R such that $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ is essential in R. We will show that R is simple artinian. It is sufficient to show that $\mathrm{soc}(R_R)$ is non-zero ([1], Theorem 1.24). Suppose that there exists $0 \neq U \subsetneqq U_1$. Since R is prime Goldie, U and U_1 are subisomorphic to each other ([11], p. 73, Lemma (ii)). Consequently there exists a monomorphism $\phi\colon U_1 \longrightarrow U$. Since R is not uniform, $U_1 \ncong R$. Therefore U_1, being cyclic, is continuous. But then, $\phi(U_1) \cong U_1$ implies $\phi(U_1) \subset^{\oplus} U_1$. Since U_1 is uniform and $\phi(U_1) \neq 0$, $U_1 = \phi(U_1) \subset U$, a contradiction. Thus U_1 is minimal. It follows that $\mathrm{soc}(R_R) \neq 0$, as desired.

Remember that a ring R is said to satisfy the right *restricted minimum condition* (RMC) if for every essential right ideal I, R/I is artinian. It is known that right PCI-ring has right RMC (c.f. [12], Corollary 5 and [8], Proposition 3.1).

THEOREM 2.5. *A simple ring satisfying ($*$) is a* PCI-*ring.*

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PROOF. Suppose R is a simple ring satisfying $(*)$. By Proposition 2.4, R is a either simple artinian or a right Ore domain. Therefore we only need to consider the latter case. We first show that it is sufficient to show that R satisfies right RMC. Suppose we have shown that R satisfies right RMC.Then for every non-zero right ideal I in R, R/I is artinian. By Lemma 2.3, R/I is semisimple. By ([8], Theorem 3.1), R/I is injective, as desired.

Assume on the contrary that R does not satisfy right RMC. Choose a non-zero right ideal A of R maximal with respect the property that R/A is not artinian. Since, by Lemma 2.1, R is right noetherian, the existence of A is guaranteed. By choice of A each proper factor module of R/A is artinian.

We show that R/A is uniform. Suppose, on the contrary that R/A is not uniform. Then there exists a non-essential proper uniform submodule U/A in R/A. Because R/A is continuous, $R/A = \mathrm{cl}(U/A) \oplus V/A$, where $\mathrm{cl}(U/A)$ denote a closure of U/A in R/A. Since every proper homomorphic image of R/A is artinian, it follows from the above decomposition of R/A, that R/A is itself artinian, a contradiction. Thus R/A is uniform. Note $\mathrm{soc}(R/A) = 0$, else $\mathrm{soc}(R/A)$ is simple and so, by Lemma 2.2, $R/A \times \mathrm{soc}(R/A)$ is cyclic and thus continuous. But then, by ([7], Proposition 1.11), $R/A \cong \mathrm{soc}(R/A)$, a contradiction.

Suppose B/A is a maximal submodule of R/A and let $K = R/A \times B/A$ be the external direct sum of R/A and B/A. We will show that K is quasi-continuous (π-injective).

Let C/A be a maximal submodule of B/A. Since R/A is cyclic, $\frac{R/A}{C/A}$ is also cyclic. Indeed $\frac{R/A}{C/A} = (\bar{1} + C/A)R$. Also $\frac{B/A}{C/A}$, being simple, is cyclic. Let $\frac{B/A}{C/A} = (\bar{b} + C/A)R$. By Lemma 2.2,

$$\begin{aligned}\frac{R/A}{C/A} \times \frac{B/A}{C/A} &= [(\bar{1} + C/A, \bar{0}) + (\bar{0}, \bar{b} + C/A)\alpha]R \\ &= (\bar{1} + C/A, \bar{b}\alpha + C/A)R\end{aligned}$$

for some $\alpha \in R$. But then $R/A \times B/A = (\bar{1}, \bar{b}\alpha)R + (C/A \times C/A)$, *i.e.*, $(R/A \times 0) \oplus (0 \times B/A) = (\bar{1}, \bar{b}\alpha)R + (C/A \times 0) + (0 \times C/A)$.

Suppose $g = (\bar{1}, \bar{b}\alpha)$ and let $L = gR$. Note $\mathrm{u.dim}(L) \leq 2$. We claim that $\mathrm{u.dim}(L) = 2$. This is true in case $L \cap (C/A \times 0)$ and $L \cap (0 \times C/A)$ both are non-zero. We proceed to prove neither $L \cap (C/A \times 0)$ nor $L \cap (0 \times C/A)$ can be zero. This is accomplished by considering three possible cases given below. The precise computations of the intersections of L with $(C/A \times 0)$, and L with $(0 \times C/A)$ will play a key role in these cases. We note

(1) $L \cap (C/A \times 0) = g\big(C \cap (b\alpha)^{-1}A\big)$, and

(2) $L \cap (0 \times C/A) = g\big(A \cap (b\alpha)^{-1}C\big)$.

CASE 1. $L \cap (C/A \times 0) = 0$ and $L \cap (0 \times C/A) \neq 0$.

Since R/A is uniform, $L \cap (C/A \times 0) = 0$ implies $L \cap (R/A \times 0) = 0$. Thus L embeds in B/A. Since $\frac{L}{L \cap (0 \times C/A)} \cong \frac{L + C/A}{C/A} \subset \frac{R/A}{C/A} \cong R/C$ and $L \cap (0 \times C/A) \neq 0$, we have $\frac{L}{L \cap (0 \times C/A)}$ is artinian. By Lemma 2.3 $\frac{L}{L \cap (0 \times C/A)}$ is semisimple. It follows that $\frac{L}{L \cap (0 \times C/A)}$ embeds in

$\mathrm{soc}(\frac{R/A\times B/A}{0\times C/A})$. But $\frac{R/A\times B/A}{0\times C/A} \cong R/A \times B/C$. Since $\mathrm{soc}(R/A) = 0$ and B/C is simple, it follows that $\frac{L}{L\cap(0\times C/A)}$ is also simple. Since $\frac{L}{L\cap(0\times C/A)} \cong \frac{L\times(0\times C/A)}{0\times C/A}$, $\frac{L\times(0\times C/A)}{0\times C/A}$ is simple, a contradiction to the fact that $\frac{gC\times(0\times C/A)}{0\times C/A}$ is a proper submodule of $\frac{L\times(0\times C/A)}{0\times C/A}$.

CASE 2. $L\cap(C/A\times 0) \neq 0$ and $L\cap(0\times C/A) = 0$.

Since B/A is uniform, C/A is essential in B/A. Thus $L\cap(0\times B/A) = 0$. Consequently, L embeds in R/A. As in Case 1, $\frac{L}{L\cap(C/A\times 0)}$ is artinian and hence semisimple. Then $\frac{L}{L\cap(C/A\times 0)}$ embeds in $\mathrm{soc}(\frac{R/A\times B/A}{C/A\times 0})$. But $\frac{L}{L\cap(C/A\times 0)} \cong \frac{L+(C/A\times 0)}{C/A\times 0} \subset \frac{R/A\times B/A}{C/A\times 0} \cong R/C\times B/A$. Note that $\mathrm{soc}(B/A) = 0$ and $\mathrm{soc}(R/C) = B/C$. Thus $\frac{L}{L\cap(C/A\times 0)}$ must be simple. Since $\frac{L}{L\cap(C/A\times 0)} \cong \frac{L+(C/A\times 0)}{C/A\times 0}$, it follows that $\frac{L+(C/A\times 0)}{C/A\times 0}$ is simple, a contradiction to the fact that $\frac{gB+(C/A\times 0)}{C/A\times 0}$ is a proper submodule of $\frac{L+(C/A\times 0)}{C/A\times 0}$.

CASE 3. $L\cap(C/A\times 0) = 0$ and $L\cap(0\times C/A) = 0$.

As in Case 1, L embeds in B/A. Also, as $L\cap(0\times C/A) = 0$ we have, from (2), $g(A\cap(b\alpha)^{-1}C) = 0$. Thus $A\cap(b\alpha)^{-1}C \subset \mathrm{r.ann}_R(g)$. But $\mathrm{r.ann}_R(g) = A\cap(b\alpha)^{-1}A \subset A\cap(b\alpha)^{-1}C$. It follows, then, $\mathrm{r.ann}_R(g) = A\cap(b\alpha)^{-1}A = A\cap(b\alpha)^{-1}C$. Notice that $A\cap(b\alpha)^{-1}C \neq 0$, because R is uniform. Since $L \cong \frac{R}{\mathrm{r.ann}_R(g)}$ and by assumption L is uniform, we have $\frac{R}{A\cap(b\alpha)^{-1}C}$ is uniform. But then $A\cap(b\alpha)^{-1}C = A$ or $(b\alpha)^{-1}C$, for otherwise $\frac{A}{A\cap(b\alpha)^{-1}C} \oplus \frac{(b\alpha)^{-1}C}{A\cap(b\alpha)^{-1}C} \subset \frac{R}{A\cap(b\alpha)^{-1}C}$, contradicting the uniformity of $\frac{R}{A\cap(b\alpha)^{-1}C}$.

If $A\cap(b\alpha)^{-1}C = A$, then $L \cong R/A$. But L embeds in B/A. Therefore R/A embeds in B/A, a contradiction since R/A is continuous.

If $A\cap(b\alpha)^{-1}C = (b\alpha)^{-1}C$, then, because R is a domain, we have $L \cong \frac{R}{(b\alpha)^{-1}C} \cong \frac{(b\alpha)R+C}{C} \subset R/C$. Since R/C is artinian, L is artinian, a contradiction again.

Thus, neither $L\cap(C/A\times 0)$ nor $L\cap(0\times C/A)$ can be zero. Consequently, $\mathrm{u.dim}\,(L)$ is 2. Hence L is essential in K. By hypothesis, L is continuous and hence quasi-continuous (π-injective). Since L is essential in K, for every idempotent $\phi \in \mathrm{End}(\hat{K})$, $\phi(L) \subset L$ ([7], Theorem 1.1). Since $\mathrm{End}(\hat{K}) = \mathrm{End}(\widehat{R/A} \times \widehat{B/A}) = \mathrm{End}(\widehat{R/A} \times \widehat{R/A})$, for every $f \in \mathrm{End}(\widehat{R/A})$, $\begin{pmatrix} 0 & 0 \\ f & 1 \end{pmatrix}$ is an idempotent in $\mathrm{End}(\hat{K})$. Thus, $\begin{pmatrix} 0 & 0 \\ f & 1 \end{pmatrix}\begin{pmatrix} 1r \\ \bar{b}\alpha r \end{pmatrix} = \begin{pmatrix} \bar{0} \\ f(\bar{r}) + \bar{b}\alpha r \end{pmatrix} \in L$. Thus, for every $r \in R$, $f\bar{r} + \bar{b}\alpha r = \bar{b}\alpha r_1$ for some $r_1 \in A$. Hence $f\bar{r} = \bar{b}\alpha r_1 - \bar{b}\alpha r \in B/A$. Consequently, $f(R/A) \subset B/A$ for every $f \in \mathrm{End}(\widehat{R/A})$. In particular $f(B/A) \subset B/A$ for every $f \in \mathrm{End}(\widehat{R/A}) = \mathrm{End}(\widehat{B/A})$. It follows that both R/A and B/A are quasi-injective. Consequently, $(B/A\times 0)\oplus(0\times B/A)$ is quasi-continuous.

Suppose ϕ is any idempotent endomorphism in $\mathrm{End}(\hat{K})$. Then

$$
\begin{aligned}
\phi(K) &= \phi\big(L + [(C/A\times 0)\oplus(0\times C/A)]\big) \\
&\subset \phi\big(L + [(B/A\times 0)\oplus(0\times B/A)]\big) \\
&\subset \phi(L) + \phi\big((B/A\times 0)\oplus(0\times B/A)\big) \\
&\subset L + (B/A\times 0)\oplus(0\times B/A)] \subset R/A\times B/A = K.
\end{aligned}
$$

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Thus K is quasi-continuous. Consequently, $B/A \cong R/A$ ([7], Proposition 1.11) and so $B/A = R/A$, because R/A is continuous and uniform, a contradiction to the maximality of B/A. Hence R/I is right artinian for every non-zero right ideal I of R. This proves R is a PCI-ring.

It is known that a PCI-ring is either semisimple artinian or a simple right noetherian, right hereditary domain (*c.g.* [3], [4]). The following corollary is now immediate.

COROLLARY 2.6. *A simple ring satisfying ($*$) is either simple artinian or a right noetherian, right hereditary domain.*

Since a PCQI-ring satisfies ($*$), we obtain the following.

COROLLARY 2.7 ([9], THEOREM B). *A simple* PCQI-*ring is a* PCI-*ring.*

A ring R is said to be a *right* SI-*ring* if all singular right R-module are injective. We have shown that in the proof of Theorem 2.5, that if R is a simple ring satisfying ($*$) then R/I is semisimple for every essential right ideal I of R. It follows that R is right SI-ring ([8], Proposition 3.1). Conversely, a simple right SI-ring satisfies ($*$). We, thus, have the following.

COROLLARY 2.8. *A simple ring satisfies ($*$) if and only if it is simple right* SI-*ring.*

We conclude with examples of rings satisfying condition ($*$). For the nonsimple ring $R = \begin{pmatrix} \Delta & D \\ 0 & D \end{pmatrix}$, where D is a division ring and Δ is a division subring of D, it is shown in ([10], Theorem, p. 141) that every proper cyclic right R-module is continuous. An example of a simple ring R over which every proper cyclic right R-module is continuous, equivalently, every proper cyclic right R-module is injective (by Theorem 2.5), is given in [2].

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