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Free Groups Generated by Two Heisenberg Translations

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Abstract. In this paper, we will discuss the groups generated by two Heisenberg translations of $\mathbf{PU}(2,1)$ and determine when they are free.

1 Introduction

Many authors have studied a subgroup G of $\mathbf{PSL}(2,\mathbf{C})$ generated by two non-commuting, parabolic, linear fractional transformations

$$A=\begin{pmatrix}1&\mu\\0&1\end{pmatrix},\quad B=\begin{pmatrix}1&0\\\mu&1\end{pmatrix}$$

in an attempt to determine when G is free. As is easily seen, the elements A, B can also be simultaneously transformed into the forms

$$A=\begin{pmatrix}1&2\\0&1\end{pmatrix},\quad B=\begin{pmatrix}1&0\\\lambda&1\end{pmatrix}.$$

The problem then becomes to find the set of complex numbers μ or λ such that the corresponding group G is free. It is a well-known result of Sanov [8] that G is free when $\lambda=2$. Brenner [1] showed that G is free provided $|\mu|\geq 2$. This is equivalent to the condition $|\lambda|\geq 2$. By using a variant of the Klein combination theorem, Chang, Jennings, and Ree [2] improved this to the weaker condition that all of $|\lambda|$, $|\lambda-1|$, and $|\lambda+1|$ are at least 1. By a more detailed analysis, Lyndon and Ullman [5] showed that the exterior of an "eye" formed by the circle $|z|=1$ and two of its tangents from the point ± 2 consist entirely of free points.

In [10], we studied the group $\langle f,g\rangle$ generated by two elliptic elements f and g in $\mathbf{PU}(2,1)$ and gave a condition to guarantee that the group $\langle f,g\rangle$ is discrete, non-elementary, and isomorphic to the free product $\langle f\rangle * \langle g\rangle$. In this paper, we will discuss the groups generated by two non-commuting parabolic transformations of $\mathbf{PU}(2,1)$ and determine when they are free. First, we study the groups generated by two Heisenberg vertical translations.

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Theorem 1.1 *Suppose the subgroup $G \subset \mathbf{PU}(2,1)$ is generated by*

$$A = \begin{pmatrix} 1 & 0 & ti \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ti & 0 & 1 \end{pmatrix},$$

where $t \in \mathbf{R}$. If $|t| \geq 2$, then G is free.

Second, we use Schwartz's result concerning ideal triangle groups that proved a conjecture of Goldman and Parker [4] and some results of a free group to show the following theorem.

Theorem 1.2 *Suppose the subgroup $G \subset \mathbf{PU}(2,1)$ is generated by*

$$A = \begin{pmatrix} 1 & 2\sqrt{2} & -4 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2\sqrt{2}\mu & 1 & 0 \\ -4|\mu|^2 & -2\sqrt{2}\bar{\mu} & 1 \end{pmatrix},$$

where $\mu \in \mathbf{C}$. If

$$|\mu| \geq \frac{1}{\sqrt{1 + (\tan^{-1}(\sqrt{125/3}))^2}},$$

then G is freely generated by A and B.

2 Complex Hyperbolic Space

First, we recall some terminologies. More details can be found in [3]. Let $\mathbf{C}^{2,1}$ denote the complex vector space of dimension 3, equipped with a non-degenerate Hermitian form of signature $(2,1)$. There are several such forms. We use the following form, called the *second Hermitian form*,

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z},$$

where $\mathbf{z}, \mathbf{w}$ are column vectors in $\mathbf{C}^{2,1}$, the Hermitian transpose is denoted by $*$ and H is the Hermitian matrix

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Consider the following subsets of $\mathbf{C}^{2,1}$:

$$V_- = \left\{ \mathbf{v} \in \mathbf{C}^{2,1} : \langle \mathbf{v}, \mathbf{v} \rangle < 0 \right\},$$
$$V_0 = \left\{ \mathbf{v} \in \mathbf{C}^{2,1} : \langle \mathbf{v}, \mathbf{v} \rangle = 0 \right\}.$$

Let $\mathbf{P} \colon \mathbf{C}^{2,1} - \{0\} \to \mathbf{CP}^{2,1}$ be the canonical projection onto complex projective space. Then $\mathbf{H}^2_{\mathbf{C}} = \mathbf{P}(V_-)$, associated with the Bergman metric, is a complex hyperbolic space. The biholomorphic isometry group of $\mathbf{H}^2_{\mathbf{C}}$ is $\mathbf{PU}(2,1)$ acting by linear projective transformations. Here $\mathbf{PU}(2,1)$ is the projective unitary group with respect to the Hermitian form defined on $\mathbf{C}^{2,1}$. As in real hyperbolic geometry, a holomorphic complex hyperbolic isometry g is said to be:

(a) *loxodromic* if it fixes no point in $\mathbf{H}^2_{\mathbf{C}}$ but exactly two points of $\partial\mathbf{H}^2_{\mathbf{C}}$;
(b) *parabolic* if it fixes fixes no point in $\mathbf{H}^2_{\mathbf{C}}$ but exactly one point of $\partial\mathbf{H}^2_{\mathbf{C}}$;
(c) *elliptic* if it fixes at least one point of $\mathbf{H}^2_{\mathbf{C}}$.

A finite point z lies in the boundary of the Siegel domain if its standard lift to $\mathbf{C}^{2,1}$ is

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}, \quad \text{where} \quad z_1 + \overline{z_1} + |z_2|^2 = 0.$$

We write $\xi = z_2/\sqrt{2} \in \mathbf{C}$, and this condition becomes $2\Re(z_1) = -2|\xi|^2$. Hence we may write $z_1 = -|\xi|^2 + i\nu$ for $\nu \in \mathbf{R}$. That is, for $\xi \in \mathbf{C}$ and $\nu \in \mathbf{R}$,

$$\mathbf{z} = \begin{pmatrix} -|\xi|^2 + i\nu \\ \sqrt{2}\xi \\ 1 \end{pmatrix}.$$

Thus we may identify the boundary of the Siegel domain with the one point compactification of $\mathbf{C} \times \mathbf{R}$.

Consider the map T from $\mathbf{C} \times \mathbf{R}$ to $GL(3, \mathbf{C})$ given by

$$T(\xi, \nu) = \begin{pmatrix} 1 & -\sqrt{2}\,\overline{\xi} & -|\xi|^2 + i\nu \\ 0 & 1 & \sqrt{2}\,\xi \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that $T(\xi, \nu)$ sends the origin to the point (ξ, ν). It is also easy to see that $T(\xi, \nu)$ is in $\mathbf{PU}(2, 1)$. Multiplying two matrices of $T(\xi, \nu)$ and $T(\zeta, \tau)$, we have

$$T(\xi, \nu)T(\zeta, \tau) = T\big(\xi + \zeta, \nu + \tau + 2\Im(\overline{\xi}\zeta)\big).$$

This means that T is a group homomorphism from $\mathbf{C} \times \mathbf{R}$ to $\mathbf{PU}(2, 1)$ with the group law

$$(\xi, \nu) * (\zeta, \tau) = \big(\xi + \zeta, \nu + \tau + 2\Im(\overline{\xi}\zeta)\big).$$

This group law gives $\mathbf{C} \times \mathbf{R}$ the structure of the 3 dimensional Heisenberg group $\mathcal{N}$.

Geometrically, we think of the $\mathbf{C}$ factor of $\mathcal{N}$ as being horizontal and the $\mathbf{R}$ factor as being vertical. We refer to $T(\xi, \nu)$ as a Heisenberg translation by (ξ, ν). A Heisenberg translation by $(0, t)$ is called a vertical translation by t. The Heisenberg translations are ordinary translations in the horizontal direction and shears in the vertical direction.

Next, we define a metric on the Heisenberg group called the Cygan metric. The Heisenberg norm assigns to (ξ, ν) the non-negative real number

$$|(\xi, \nu)|_0 = \big(\|\xi\|^4 + \nu^2\big)^{\frac{1}{4}} = \big|\,\|\xi\|^2 - i\nu\big|^{\frac{1}{2}},$$

where $\|\xi\|^2 = \langle\langle \xi, \xi \rangle\rangle = \sum |\xi_i|^2$. This enables us to define the *Cygan metric* on the Heisenberg group:

$$\rho_0\big((\xi_1, \nu_1), (\xi_2, \nu_2)\big) = \Big|\big(\xi_1 - \xi_2, \nu_1 - \nu_2 + 2\Im\langle\langle \xi_1, \xi_2 \rangle\rangle\big)\Big|_0 = \big|(\xi_1, \nu_1)^{-1} * (\xi_2, \nu_2)\big|_0.$$

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The Heisenberg group acts on itself by Heisenberg translation. For $(\xi_0,\nu_0) \in \mathcal{N}$, this is

$$T(\xi_0,\nu_0)\colon (\xi,\nu) \longmapsto \big(\xi+\xi_0, \nu+\nu_0+2\Im\langle\langle \xi_0,\xi\rangle\rangle\big) = (\xi_0,\nu_0) * (\xi,\nu).$$

We now discuss the Cygan sphere. The Cygan sphere of centre $z_0 = (\xi_0,\nu_0) \in \partial\mathbf{H}^2_{\mathbf{C}}$ with radius r is defined as

$$S_r(z_0) = \left\{ z \in \partial\mathbf{H}^2_{\mathbf{C}} : \rho_0(z,z_0) = r \right\}.$$

In terms of coordinates, $S_r(z_0)$ is given by

$$S_r(z_0) = \left\{ z = (\xi,\nu) : \big| |\xi-\xi_0|^2 + i\nu - i\nu_0 - 2i\Im(\xi\overline{\xi_0}) \big| = r^2 \right\}.$$

Suppose that J be the element in $\mathbf{PU}(2,1)$, then

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We investigate the effect of J on Heisenberg coordinates:

$$J\begin{pmatrix} -|\xi|^2+i\nu \\ \sqrt{2}\xi \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{2}\xi \\ |\xi|^2-i\nu \end{pmatrix} \approx \begin{pmatrix} \frac{-|\xi|^2-i\nu}{|\xi|^4+\nu^2} \\ \frac{-\sqrt{2}\xi}{|\xi|^2-i\nu} \\ 1 \end{pmatrix}.$$

Thus the map J carries the point (ξ,ν) of $\partial\mathbf{H}^2_{\mathbf{C}}$ with coordinates

$$J(\xi,\nu) = \left(\frac{-\xi}{|\xi|^2-i\nu}, \frac{-\nu}{|\xi|^4+\nu^2} \right).$$

Similarly, elements of $\mathbf{PU}(2,1)$ fixing 0 may be obtained from those fixing ∞ by conjugating by J. Thus we may speak of Heisenberg translation by (τ,t) fixing 0. This is just the conjugation by J of the translation by (τ,t) fixing ∞. It has the following matrix representation:

$$\begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2}\tau & 1 & 0 \\ -|\tau|^2+it & -\sqrt{2}\overline{\tau} & 1 \end{pmatrix}.$$

3 The Proof of Theorem 1.1

We first consider a simple case, that is, the group G generated by two Heisenberg vertical translations. For convenience, we need the following normalizations

$$A = \begin{pmatrix} 1 & 0 & it \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ it & 0 & 1 \end{pmatrix},$$

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where $t \in \mathbf{R}$.

It is easy to get $J^2 = \mathrm{Id}$ and $JBJ = A$. We prove this result by the similar argument in [2]. We define two Heisenberg balls

$$S_1 = \rho_0\big((\xi,\nu),(0,1)\big) \leq 1$$
$$S_2 = \rho_0\big((\xi,\nu),(0,-1)\big) \leq 1$$

and two unbound sets

$$\Delta_1 = \big\{(\xi,\nu) : |\nu| \leq 1/2\big\}$$
$$\Delta_2 = \big\{(\xi,\nu) : |\nu| \geq 1/2\big\}.$$

Lemma 3.1 *If $Z = (\xi,\nu) \in \Delta_2$, then $JZ \in S_1$ or $JZ \in S_2$. If, on the other hand, $Z = (\xi,\nu)$ is not in $S_1 \cup S_2$, then $JZ \in \Delta_1$.*

Proof Suppose that $Z = (\xi,\nu) \in \Delta_2$, then $|\nu| \geq 1/2$. By a simple calculation, the Cygan distance between JZ and $(0,1)$ is

$$\begin{aligned}\rho_0^2(JZ,(0,1)) &= \left|\left|\frac{-\xi}{|\xi|^2 - i\nu}\right|^2 + i\Big(\frac{-\nu}{|\xi|^4+\nu^2} - 1\Big)\right| \\ &= \sqrt{\Big(\frac{|\xi|^2}{|\xi|^4+\nu^2}\Big)^2 + \Big(1 + \frac{\nu}{|\xi|^4+\nu^2}\Big)^2} \\ &= \frac{\sqrt{|\xi|^4 + (|\xi|^4+\nu^2+\nu)^2}}{|\xi|^4+\nu^2}.\end{aligned}$$

Then $JZ \in S_1$ is equal to

$$\frac{\sqrt{|\xi|^4 + (|\xi|^4+\nu^2+\nu)^2}}{|\xi|^4+\nu^2} \leq 1.$$

Thus we get $(|\xi|^4+\nu^2)(1+2\nu) \leq 0$. That is $1+2\nu \leq 0$. So if $\nu \leq -\frac{1}{2}$, then we have $\rho_0(JZ,(0,1)) \leq 1$. Similarly, if $\nu \geq \frac{1}{2}$, then we have $\rho_0(JZ,(0,-1)) \leq 1$. If $Z \notin S_1$, then

$$\rho_0^2\big(Z,(0,1)\big) = \big||\xi|^2 + i(\nu-1)\big| = \sqrt{|\xi|^4 + (\nu-1)^2} \geq 1.$$

That is,

$$\frac{-\nu}{|\xi|^4+\nu^2} \geq -\frac{1}{2}.$$

At the same time, $Z \notin S_2$ implies

$$\rho_0^2\big(Z,(0,-1)\big) = \big||\xi|^2 + i(\nu+1)\big| = \sqrt{|\xi|^4 + (\nu+1)^2} \geq 1.$$

This means that

$$\frac{-\nu}{|\xi|^4+\nu^2} \leq \frac{1}{2}.$$

Therefore, $JZ \in \Delta_1$. ■

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Lemma 3.2 *If $t \geq 2$, then $Z \in \Delta_2$ implies $B^n(Z) \in \Delta_1$ for a non-zero integer n.*

Proof We first note that

$$B^n = J \begin{pmatrix} 1 & 0 & nti \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} J.$$

We write

$$Z_1 = JZ, \quad Z_2 = Z_1 * (0, nt), \quad B^n(Z) = JZ_2.$$

By Lemma 3.1, $Z_1 \in S_1$ or $Z_1 \in S_2$. Suppose that $Z_1 = (\xi_1, \nu_1)$. If $Z_1 \in S_1$, then $|\xi_1|^4 + (\nu_1 - 1)^2 \leq 1$. It is easy to see that $|\xi_1|^4 + (\nu_1 + nt - 1)^2 \geq 1$ for $t \geq 2$. Thus $Z_2 = Z_1 * (0, nt) = (\xi_1, \nu_1 + nt)$ is not in S_1. Similarly, if $Z_1 \in S_2$, then we have $Z_2 \notin S_2$. Then, again by Lemma 3.1, we have $B^n(Z) = JZ \in \Delta_1$. ∎

Proof of Theorem 1.1 If group $\langle A, B\rangle$ is not free, there must exist a non-trivial word g such that

$$g = B^{n_r} A^{m_r} \cdots B^{n_1} A^{m_1} = \mathrm{Id},$$

where we can always assume that the integers $n_r, m_r, \dots, n_1, m_1$ are all not zero.

Define

$$Z_1 = A^{m_1}(0), Z_1' = B^{n_1}(Z_1), \dots, Z_k = A^{m_k}(Z_{k-1}'), Z_k' = B^{n_k}(Z_k) (k \leq r-1);$$

and $Z_r = A^{-m_r} B^{-n_r}(0)$. Then we have $Z_{r-1}' = Z_r$.

Since $Z_1 = (0, m_1) \in \Delta_2, |m_1| \geq 1$. By Lemma 3.1, we get $B^{n_1}(Z_1) \in \Delta_1$. We may write $B^{n_1}(Z_1) = (\xi_1, \nu_1)$. Now we have $Z_2 = Z_1 * (0, m_2) = (\xi_1, \nu_1 + m_2)$. Then $B^{n_1}(Z_1) \in \Delta_1$ implies $|\nu_1| \leq \frac{1}{2}$. Since

$$|\nu_1 + m_1| \geq |m_1| - |\nu_1| \geq 1 - \frac{1}{2},$$

we have $Z_2 \in \Delta_2$. Thus $Z_2' = B^{n_2}(Z_2) \in \Delta_1$. Repeating this argument, we find $Z_{r-1}' \in \Delta_1$.

On the other hand, $Z_r = A^{-m_r} B^{-n_r}(0) = A^{-m_r}(0) = (0, -m_r)$, $|m_r| \geq \frac{1}{2}$, so $Z_r \in \Delta_2$. Therefore we get a contradiction. Since G is also generated by A^{-1} and $JA^{-1}J$, we can see that G is also free for $-t < 2$. ∎

Remark 3.3 In fact, G is also free in other values of t. By an elementary calculation, any word of G has the form

$$g_t = \begin{pmatrix} P_1(t) & 0 & P_2(t)i \\ 0 & 1 & 0 \\ P_3(t)i & 0 & P_4(t) \end{pmatrix},$$

where $P_1(t), P_2(t), P_3(t), P_4(t)$ are integer polynomials of t. Then one can obtain the following results as in [2, 5]:

(a) If t is a transcendental number, then $G = \langle A, B\rangle$ is free.
(b) Any point in the real line is a limit of algebraic free points.

4 The Proof of Theorem 1.2

All of the results in this section about free groups can be found in [7]. We include them here for completeness, as they will be used in the proof of our second main result. The subgroups of a free product group have a particularly simple structure as described in the following lemma.

Lemma 4.1 (Kurosh Subgroup Theorem) *Let $G = *_{i\in I} H_i$ be the free product of a collection of group H_i. If A is a subgroup of G, then A decomposes as a free product of the form*

$$A = F * \big(*_{i\in I} \big(*_{j\in J(i)} A \cap u_j H_i u_j^{-1} \big) \big),$$

where F is a free group. That is, A is the free product of a free group and of various subgroups that are the intersections of A with conjugates of the H_i.

For a very simple "topological proof" in terms of fundamental group of graphs and of coverings, see Massey's text [6]. Note that applying the Kurosh Subgroup Theorem to the free product of infinite cyclic groups implies that subgroups of free groups are free. Here are a few other special cases.

Lemma 4.2 *Let $G = \langle a, b, c | a^2, b^2, c^2\rangle$ and $H = \langle ab, bc\rangle$. If $G = \langle a\rangle * \langle b\rangle * \langle c\rangle$, then H is free.*

Proof Since G is the free product of the three cyclic groups of order 2 and H is a subgroup of G, the Kurosh Subgroup Theorem tells us that H is a free product of conjugates of these three cyclic groups and a free group. We claim that $H = \langle ab, bc\rangle$ is to be the free group in the letters ab and bc. We know that an element in a free product of groups has finite order if and only if it is conjugated to some element of finite order in some of the free factors. This is not the case for ab and bc, and thus H is a torsion-free subgroup. It follows that H is free by the Kurosh Subgroup Theorem. ■

Next, we recall a few basic facts of complex ideal triangle group that we shall need in the proof of Theorem 1.2. We refer to a triple of distinct points (p_1, p_2, p_3) in $\partial\mathbf{H}^2_{\mathbf{C}}$ as an *ideal triangle* and define its Cartan's angular invariant

$$\mathbb{A}(p_1, p_2, p_3) = \arg\big(-\langle \mathbf{p_1}, \mathbf{p_2}\rangle\langle \mathbf{p_2}, \mathbf{p_3}\rangle\langle \mathbf{p_3}, \mathbf{p_1}\rangle\big) \in [-\pi/\mathbf{2}, \pi/\mathbf{2}].$$

This invariant characterizes the triple (p_1, p_2, p_3) up to the equivalence

$$(p_1, p_2, p_3) \sim g(p_1, p_2, p_3), g \in \mathbf{PU}(2, 1).$$

The points p_1, p_2, p_3 determine complex geodesics C_1, C_2, C_3 that are fixed by the inversions l_1, l_2, l_3, respectively. In [4], Goldman and Parker used $\mathbb{A}$ to parametrize the homomorphism representation

$$\rho_{\mathbb{A}}\colon \mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2 \longrightarrow \mathbf{PU}(2, 1),$$

$$(a, b, c) \longmapsto (\iota_1, \iota_2, \iota_3).$$

Their main result is that for $|\mathbb{A}| < \tan^{-1}(35)$, $\rho_{\mathbb{A}}$ is a discrete embedding and not a discrete embedding for $|\mathbb{A}| > \tan^{-1}(\sqrt{125/3})$. The latter necessary condition was conjectured to be sufficient. Schwartz [9] proved this conjecture.

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Lemma 4.3 *Let $\iota_1, \iota_2, \iota_3$ be as stated above. Then $\iota_1, \iota_2, \iota_3$ generate a discrete group if and only if $|\mathbb{A}| \leq \tan^{-1}(\sqrt{125/3})$. Furthermore, in this case, $\iota_1, \iota_2, \iota_3$ freely generate the free product $\langle \iota_1 \rangle * \langle \iota_2 \rangle * \langle \iota_3 \rangle$ of the three cyclic groups of order 2.*

Proof of Theorem 1.2 The fixed points of A and B are $p_0 = 0$, $p_1 = \infty$, respectively. Next, we choose another point $p_2 = (\xi, \nu) \in \partial \mathbf{H}^2_{\mathbf{C}}$. Then $p = (p_0, p_1, p_2)$ be a triple of points in $\partial \mathbf{H}^2_{\mathbf{C}}$. Let C_0 be the complex geodesic $\overrightarrow{p_1 p_2}$ spanned by p_1 and p_2. Similarly let $C_1 = \overrightarrow{p_0 p_2}$ and $C_2 = \overrightarrow{p_0 p_1}$. We denote inversion in C_j by $\iota_j \in \mathbf{PU}(2,1)$, $j = 0, 1, 2$. We will determine the coordinate of p_2 such that $A = \iota_0 \iota_1$, $B = \iota_1 \iota_2$.

Choose coordinates so that p_0, p_1, p_2 lift to the following null vectors in $\mathbf{C}^{2,1}$

$$\mathbf{p_0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{p_2} = \begin{pmatrix} -|\xi|^2 - i\nu \\ \sqrt{2}\xi \\ 1 \end{pmatrix},$$

and $\overrightarrow{p_1 p_2}$, $\overrightarrow{p_0 p_2}$, $\overrightarrow{p_0 p_1}$ are respectively determined by the polar vectors

$$c_0 = \begin{pmatrix} \sqrt{2}\,\bar{\xi} \\ -1 \\ 0 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 0 \\ \frac{-|\xi|^2 - i\nu}{\sqrt{|\xi|^4 + \nu^2}} \\ \frac{-\sqrt{2}\,\bar{\xi}}{\sqrt{|\xi|^4 + \nu^2}} \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

in the sense that C_j is the image in projective space of the linear subspace comprising $z \in \mathbf{C}^{2,1}$ such that $\langle z, c_j \rangle = 0$. The corresponding inversion is given by

$$\iota_j = -Z + \frac{2\langle Z, c_j \rangle}{\langle c_j, c_j \rangle}.$$

The inversion generators in $\mathbf{PU}(2,1)$ are

$$\iota_0 = \begin{pmatrix} -1 & -2\sqrt{2}\,\bar{\xi} & -4|\xi|^2 \\ 0 & 1 & -2\sqrt{2}\,\xi \\ 0 & 0 & -1 \end{pmatrix}, \quad \iota_1 = \begin{pmatrix} -1 & 0 & 0 \\ \frac{2\sqrt{2}(|\xi|^2 + i\nu)}{|\xi|^4 + \nu^2} & 1 & 0 \\ \frac{4}{|\xi|^4 + \nu^2} & \frac{-2\sqrt{2}(-|\xi|^2 + i\nu)}{|\xi|^4 + \nu^2} & -1 \end{pmatrix},$$

and

$$\iota_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Comparing the entries of $\iota_0 \iota_2$, $\iota_2 \iota_1$ with A,B, we know that

$$\xi = 1, \quad \mu = \frac{1 + i\nu}{1 + \nu^2}.$$

By a simple calculation, we have

$$\tan \mathbb{A}(p) = \tan \mathbb{A}((p_0, p_1, p_2)) = -\tan(\nu).$$

By Lemma 4.3, we get that the group $\langle \iota_0, \iota_1, \iota_2 \rangle$ freely generates the free product $\langle \iota_1 \rangle * \langle \iota_2 \rangle * \langle \iota_3 \rangle$ when $|\tan(\nu)| \leq \sqrt{125/3}$. That is,

$$|\nu| \leq \tan^{-1}(\sqrt{125/3}).$$

By Lemma 4.2, we show that $\langle \iota_0\iota_2, \iota_1\iota_2 \rangle$ is free if $|\nu| \leq \tan^{-1}(\sqrt{125/3})$. Therefore we get

$$\frac{1}{|\mu|^2} - 1 \leq (\tan^{-1} \sqrt{125/3})^2.$$

That is,

$$|\mu| \geq 1/\sqrt{1 + (\tan^{-1}(\sqrt{125/3}))^2}. \qquad \blacksquare$$

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