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DERIVATIONS AND INVARIANT FORMS OF LIE ALGEBRAS GRADED BY FINITE ROOT SYSTEMS

GEORGIA BENKART

ABSTRACT. Lie algebras graded by finite reduced root systems have been classified up to isomorphism. In this paper we describe the derivation algebras of these Lie algebras and determine when they possess invariant bilinear forms. The results which we develop to do this are much more general and apply to Lie algebras that are completely reducible with respect to the adjoint action of a finite-dimensional subalgebra.

1. **Introduction.**

1.1. Throughout this work **F** will denote a field of characteristic zero, and all tensor products will be over **F**. Unless specified otherwise, all algebras except Lie algebras will be assumed to be unital.

1.2. Let Δ be a finite irreducible reduced root system, and assume Λ is the integer lattice generated by Δ. Following Berman and Moody [BeM], we say that a Lie algebra L over **F** is *graded by* Δ or is Δ*-graded* if

(i) L has a Λ-gradation $L = \bigoplus_{\lambda\in\Lambda} L_\lambda$ in which $L_\lambda \neq (0)$ if and only if $\lambda \in \Delta \cup \{0\}$;

(ii) the split simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda\in\Delta} \mathfrak{g}_\lambda$ whose root system is Δ relative to the split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ is a subalgebra of L, and $L_\lambda \supseteq \mathfrak{g}_\lambda$ for all $\lambda \in \Delta \cup \{0\}$;

(iii) for all $h \in \mathfrak{h}$ the operator $\operatorname{ad} h$ acts diagonally on L_λ with eigenvalue $\lambda(h)$; and

(iv) L is generated by its root spaces L_λ where $\lambda \in \Delta$.

The conditions for L to be a Δ-graded Lie algebra imply that it is a direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules whose highest weights are roots, hence are either the highest long or highest short root or are zero. Thus, condition (iii) in the definition of a Δ-graded Lie algebra can be replaced by:

(iii)$'$ As a $\mathfrak{g}$-module, L is a direct sum of adjoint modules (modules isomorphic to $\mathfrak{g}$), *little adjoint modules* (modules isomorphic to the irreducible $\mathfrak{g}$-module V whose highest weight is the highest short root), or one-dimensional $\mathfrak{g}$-modules; the latter being contained in L_0.

By collecting isomorphic summands, we may suppose that

$$L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus D,$$

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where D is the sum of the trivial $\mathfrak{g}$-modules, and where we may identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes A$. Classifying the Δ-graded Lie algebras has necessitated determining the possibilities for the spaces A, B, and D and the multiplication between the various summands.

1.3. It is easy to see that a Δ-graded Lie algebra L is perfect (*i.e.* $L = [L, L]$), and in particular, $L_0 = \sum_{\lambda \in \Delta} [L_\lambda, L_{-\lambda}]$. If a central extension of a Δ-graded Lie algebra is perfect, then it is a Δ-graded Lie algebra relative to the same root system. For that reason Berman and Moody [BeM] in the simply-laced case and Benkart and Zelmanov ([BZ1], [BZ2]) in the doubly-laced case (see also [N]) classified the Lie algebras graded by finite root systems up to central extensions. Any perfect Lie algebra L has a unique (up to isomorphism) universal central extension which is also perfect, called its *universal covering algebra* (see [Ga] or [MP, Section 1.9]). Recently, Allison, Benkart, and Gao [ABG1] have described the universal covering algebra of an arbitrary Δ-graded Lie algebra L, hence its central extensions and its homology $\mathrm{H}_2(L, \mathbf{F})$ with trivial coefficients. As a consequence, the Lie algebras graded by finite reduced root systems are now completely determined up to isomorphism.

1.4. Among the Lie algebras graded by finite root systems are many important examples. The non-twisted affine algebras (or more accurately their derived algebras) have a realization, $L \cong (\mathfrak{g} \otimes \mathbf{F}[t^{\pm 1}]) \oplus \mathbf{F}c$, where $\mathbf{F}[t^{\pm 1}]$ is the algebra of Laurent polynomials in the variable t over $\mathbf{F}$ and c is a central element, and so they are Δ-graded. The twisted affine algebras of type $D_{r+1}^{(2)}$, $A_{2r-1}^{(2)}$, or $E_6^{(2)}$ can be realized as $L \cong (\mathfrak{g} \otimes \mathbf{F}[t^{\pm 2}]) \oplus (V \otimes t\mathbf{F}[t^{\pm 2}]) \oplus \mathbf{F}c$, where c is central and $\mathfrak{g}$ is a split simple Lie algebra of type B_r, C_r, or F_4, respectively. The twisted affine algebra $D_4^{(3)}$ has a realization as $L \cong (\mathfrak{g} \otimes \mathbf{F}[t^{\pm 3}]) \oplus \big(V \otimes (t\mathbf{F}[t^{\pm 3}] + t^2\mathbf{F}[t^{\pm 3}])\big) \oplus \mathbf{F}c$ where $\mathfrak{g}$ is of type G_2 and c is central (see [K, Chapter 8]). Consequently, these twisted affine algebras are graded by the doubly-laced root systems. The toroidal Lie algebras are the universal covering algebras of the Lie algebras $\mathfrak{T}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{F}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$, $n = 2, 3, \dots$, so they too are Δ-graded. More generally, any perfect Lie algebra which is a central extension of one of the form $\mathfrak{g} \otimes A$, where A is a commutative associative algebra, is graded by the root system of $\mathfrak{g}$. Other examples of Δ-graded Lie algebras include the cores of the extended affine (previously termed quasisimple) Lie algebras of reduced type (see [HT], [BGK], [BGKN], [AABGP]), certain of the intersection matrix algebras of Slodowy (see [Sl]), and all the finite-dimensional simple Lie algebras containing a split maximal toral subalgebra with a reduced root system (see [S]). Thus, the notion of a Δ-graded Lie algebra provides a unifying concept which encompasses many important families of Lie algebras.

1.5. After the finite-dimensional split simple Lie algebras, the most studied of the Δ-graded Lie algebras are the affine algebras because of their significant role in statistical mechanics, conformal field theory, and string theory. The characters of their irreducible highest weight representations give interesting combinatorial identities, and the string functions and generalized string functions of these representations are modular functions related to theta functions. The universal covering algebra of the loop algebra

$\mathfrak{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{F}[t^{\pm 1}]$ is the one-dimensional central extension $L = (\mathfrak{g} \otimes \mathbf{F}[t^{\pm 1}]) \oplus \mathbf{F}c$ (see [W]). The Lie algebra L has infinite root spaces relative to the Cartan subalgebra $\mathfrak{h} \oplus \mathbf{F}c$, hence infinite weight spaces in its representations. Creating a useful representation theory and character theory for the affine algebra L has necessitated having finite-dimensional weight spaces, and this has been accomplished by enlarging L by adjoining derivations. The Casimir operator, a critical tool in proving the character formula, is constructed from a nondegenerate symmetric invariant form on L. Developing a parallel representation theory for arbitrary Lie algebras graded by finite root systems requires determining their derivations and knowing when such an algebra possesses an invariant bilinear form. That is the goal of this present paper.

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2. **The structure of Lie algebras graded by finite root systems.**

2.1. If L is a Lie algebra graded by finite root system, then we can decompose L as a $\mathfrak{g}$-module and collect isomorphic summands to get

$$L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus D.$$

The spaces $\mathfrak{g} \otimes A$, $V \otimes B$, and D are just the isotypic $\mathfrak{g}$-module components. The space $\mathfrak{a} \stackrel{\text{def}}{=} A \oplus B$ is an algebra over A whose properties are summarized in (2.2) below, and D acts as derivations on $\mathfrak{a}$ which map A to A and B to B. When Δ is simply-laced, then $V = (0) = B$ and $\mathfrak{a} = A$. The coordinate algebras $\mathfrak{a}$ listed in (2.2) have inner derivations $D_{\alpha,\beta}$ involving certain expressions in the left multiplication and right multiplication operators L_α, L_β and R_α, R_β and $\operatorname{ad} z = L_z - R_z$ (in the associative case). The exact expressions for the derivations $D_{\alpha,\beta}$ have been determined in the classification results of [BeM], [BZ2], [N], [S], and [ABG2], and this information is displayed below (compare (2.39) of [ABG1]).

2.2. (a) A_r, $(r \geq 3)$: $\mathfrak{a}$ is an associative algebra and

$$D_{\alpha,\beta} = \frac{1}{r+1} \operatorname{ad}[\alpha, \beta]$$

(b) A_2: $\mathfrak{a}$ is an alternative algebra and

$$D_{\alpha,\beta} = \frac{1}{3}(L_{[\alpha,\beta]} - R_{[\alpha,\beta]} - 3[L_\alpha, R_\beta])$$

(c) A_1: $\mathfrak{a}$ is a Jordan algebra and

$$D_{\alpha,\beta} = \frac{1}{2}[L_\alpha, L_\beta]$$

(d) D_r, $(r \geq 4)$, E_6, E_7, E_8: $\mathfrak{a}$ is a commutative, associative algebra and

$$D_{\alpha,\beta} = 0$$

(e) C_r $(r \geq 4)$: $\mathfrak{a}$ is an associative algebra with involution σ, A (resp. B) is the set of symmetric (resp. skew-symmetric) elements relative to σ, and

$$D_{\alpha,\beta} = \frac{1}{4r}(\mathrm{ad}[\alpha,\beta] + \mathrm{ad}[\alpha^\sigma,\beta^\sigma])$$

(f) C_3: $\mathfrak{a}$ is an alternative with involution σ, A (resp. B) is the set of symmetric (resp. skew-symmetric) elements relative to σ, A lies in the nucleus (associative center of $\mathfrak{a}$) and

$$D_{\alpha,\beta} = \frac{1}{12}(L_{[\alpha,\beta]} - R_{[\alpha,\beta]} - 3[L_\alpha, R_\beta] + L_{[\alpha^\sigma,\beta^\sigma]} - R_{[\alpha^\sigma,\beta^\sigma]} - 3[L_{\alpha^\sigma}, R_{\beta^\sigma}])$$

(g) C_2: $\mathfrak{a}$ is the Peirce half space of a unital Jordan algebra containing a triangle $\mathcal{T}$; σ is the restriction of the connection involution determined by $\mathcal{T}$ to $\mathfrak{a}$; A (resp. B) is the set of symmetric (resp. skew-symmetric) elements in $\mathfrak{a}$ with respect to σ; the product is given in (2.47) of [ABG1] and

$$D_{\alpha,\beta} = \frac{1}{2}([M_\alpha, M_\beta] + [M_{\alpha^\sigma}, M_{\beta^\sigma}]),$$

where $M_\alpha\gamma = \frac{1}{2}(\alpha\gamma + \gamma\alpha)$ for all $\gamma \in \mathfrak{a}$ (see 2.43 of [ABG1] for unexplained terminology).

(h) B_r $(r \geq 3)$: $\mathfrak{a} = A \oplus B$ is the Jordan algebra over A of a symmetric bilinear form and

$$D_{\alpha,\beta} = -[L_\alpha, L_\beta]$$

(i) F_4: $\mathfrak{a} = A \oplus B$ is an alternative algebra over A with a normalized trace mapping satisfying ch_2, B is the set of elements of trace zero, and

$$D_{\alpha,\beta} = \frac{1}{4}(L_{[\alpha,\beta]} - R_{[\alpha,\beta]} - 3[L_\alpha, R_\beta])$$

(j) G_2: $\mathfrak{a} = A \oplus B$ is a Jordan algebra over A with a normalized trace mapping satisfying ch_3, B is the set of elements of trace zero, and

$$D_{\alpha,\beta} = [L_\alpha, L_\beta].$$

2.3. For types F_4, G_2, $B_r, r \geq 3$, $\mathfrak{a} = A \oplus B$ where A is a commutative associative algebra. There is a *normalized trace* on the algebra $\mathfrak{a}$; that is, an A-linear functional $\mathfrak{t}\colon \mathfrak{a} \longrightarrow A$ such that

$$\mathfrak{t}(\alpha\alpha') = \mathfrak{t}(\alpha'\alpha)$$
$$\mathfrak{t}\big((\alpha\alpha')\alpha''\big) = \mathfrak{t}\big(\alpha(\alpha'\alpha'')\big)$$
$$\mathfrak{t}(1) = 1$$

for all $\alpha, \alpha', \alpha'' \in \mathfrak{a}$. The space B, which is the set of elements of trace zero,

$$B = \{b \in \mathfrak{a} \mid \mathfrak{t}(b) = 0\},$$

is an A-module. Relative to the multiplication,

$$b * b' = bb' - \mathfrak{t}(bb')1 \in B, \quad \text{for } b, b' \in B, \tag{2.4}$$

B is a (not necessarily unital) algebra. Moreover, the product on $\mathfrak{a}$ can be expressed as follows:

$$(a+b)(a'+b') = aa' + (b,b') + ab' + a'b + b * b', \tag{2.5}$$

where $(b,b') = t(bb')$ is the associated symmetric A-bilinear form. The trace satisfies the Cayley-Hamilton equation $\mathrm{ch}_2(x) = 0$ of 2×2 matrices when Δ is of type B_r or F_4 and the Cayley-Hamilton equation $\mathrm{ch}_3(x) = 0$ of 3×3 matrices when Δ is of type G_2.

2.6. It is useful to know that the following properties hold for the inner derivations $D_{\alpha,\beta}$:

$$\begin{gathered} D_{\alpha,\beta} + D_{\beta,\alpha} = 0, \\ D_{\alpha\beta,\gamma} + D_{\beta\gamma,\alpha} + D_{\gamma\alpha,\beta} = 0, \\ [E, D_{\alpha,\beta}] = D_{E\alpha,\beta} + D_{\alpha,E\beta}, \\ D_{\alpha,\beta}(A) \subseteq A, \quad D_{\alpha,\beta}(B) \subseteq B, \\ D_{a,b} = 0, \end{gathered} \tag{2.7}$$

for all $\alpha, \beta, \gamma \in \mathfrak{a}$, $a \in A$, $b \in B$, and $E \in \mathrm{Der}(\mathfrak{a})$. For any algebra $\mathfrak{a} = A \oplus B$ satisfying (2.2) and (2.3) and having inner derivations as in (2.2), the space $L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$ can be given the structure of a Δ-graded Lie algebra with trivial center using the multiplication in (2.15) or (2.16) below with $\{\alpha, \beta\} = D_{\alpha,\beta}$ for all $\alpha, \beta \in \mathfrak{a}$.

2.8. Let $\mathfrak{a} = A \oplus B$ be a coordinate algebra of an arbitrary Δ-graded Lie algebra, and assume $\mathfrak{s}$ is the subspace of $\mathfrak{a} \otimes \mathfrak{a}$ spanned by the elements

$$\begin{gathered} \alpha \otimes \beta + \beta \otimes \alpha \\ \alpha\beta \otimes \gamma + \beta\gamma \otimes \alpha + \gamma\alpha \otimes \beta \\ a \otimes b \end{gathered} \tag{2.9}$$

where α, β, γ are arbitrary elements of $\mathfrak{a}$, and $a \in A$, $b \in B$. Let

$$\{\mathfrak{a}, \mathfrak{a}\} \stackrel{\text{def}}{=\!=} (\mathfrak{a} \otimes \mathfrak{a}) / \mathfrak{s} \tag{2.10}$$

be the factor space, and for $\alpha, \beta \in \mathfrak{a}$, let $\{\alpha, \beta\}$ denote the coset $\alpha \otimes \beta + \mathfrak{s}$ in $\{\mathfrak{a}, \mathfrak{a}\}$.

2.11. The space $D_{\mathfrak{a},\mathfrak{a}} = D_{A,A} + D_{B,B}$ is a Lie subalgebra of $\mathrm{Der}(\mathfrak{a})$ which leaves invariant A and B. Since $\mathfrak{a}$ is a $D_{\mathfrak{a},\mathfrak{a}}$-module, so is the tensor product $\mathfrak{a} \otimes \mathfrak{a}$. The space $\mathfrak{s}$ is invariant under $D_{\mathfrak{a},\mathfrak{a}}$, and so $\{\mathfrak{a}, \mathfrak{a}\}$ is a $D_{\mathfrak{a},\mathfrak{a}}$-module under the induced action:

$$D_{\alpha,\alpha'}\{\beta, \beta'\} = \{D_{\alpha,\alpha'}\beta, \beta'\} + \{\beta, D_{\alpha,\alpha'}\beta'\}.$$

This allows us to define a multiplication on $\{\mathfrak{a}, \mathfrak{a}\}$ by

$$[\{\alpha, \alpha'\}, \{\beta, \beta'\}] = D_{\alpha,\alpha'}\{\beta, \beta'\} = \{D_{\alpha,\alpha'}\beta, \beta'\} + \{\beta, D_{\alpha,\alpha'}\beta'\}, \tag{2.12}$$

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giving it the structure of a Lie algebra (see [ABG1, Section 4]).

The mapping $\mathfrak{a} \otimes \mathfrak{a} \longrightarrow \mathrm{Der}(\mathfrak{a})$, $\alpha \otimes \beta \longmapsto D_{\alpha,\beta}$ has $\mathfrak{S}$ in its kernel because of (2.4), and the induced mapping $\rho\colon \{\mathfrak{a}, \mathfrak{a}\} \longrightarrow \mathrm{Der}(\mathfrak{a})$ given by

$$\rho\colon \{\alpha, \alpha'\} \longmapsto D_{\alpha,\alpha'} \tag{2.13}$$

is a surjective Lie algebra homomorphism. The kernel of ρ is the *full skew-dihedral homology group* of $\mathfrak{a}$,

$$\mathrm{HF}(\mathfrak{a}) = \ker \rho = \Big\{ \sum_i \{\alpha_i, \beta_i\} \in \{\mathfrak{a}, \mathfrak{a}\} \;\Big|\; \sum_i D_{\alpha_i,\beta_i} = 0 \Big\}.$$

THEOREM 2.14([ABG1, THEOREMS 4.13 AND 4.20]). *Let $\hat{L} = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \{\mathfrak{a}, \mathfrak{a}\}$, where $\mathfrak{a} = A \oplus B$ is as in (2.2), and define a multiplication on $\hat{L}$ by the following: for $\Delta = A_r (r \geq 2)$, $C_r (r \geq 2)$,*

$$\begin{aligned} [x \otimes a, y \otimes a'] &= [x, y] \otimes \frac{1}{2}(a \circ a') + (x \circ y) \otimes \frac{1}{2}[a, a'] + (x|y)\{a, a'\} \\ [x \otimes a, u \otimes b] &= (x \circ u) \otimes \frac{1}{2}[a, b] + [x, u] \otimes \frac{1}{2} a \circ b = -[u \otimes b, x \otimes a] \\ [u \otimes b, v \otimes b'] &= [u, v] \otimes \frac{1}{2}(b \circ b') + (u \circ v) \otimes \frac{1}{2}[b, b'] + (u|v)\{b, b'\}, \end{aligned} \tag{2.15}$$

for $\Delta = A_1, B_r, (r \geq 3), D_r, (r \geq 4), E_6, E_7, E_8, F_4, G_2$,

$$\begin{aligned} [x \otimes a, y \otimes a'] &= [x, y] \otimes aa' + (x|y)\{a, a'\} \\ [x \otimes a, u \otimes b] &= xu \otimes ab = -[u \otimes b, x \otimes a] \\ [u \otimes b, v \otimes b'] &= \partial_{u,v} \otimes (b, b') + (u * v) \otimes (b * b') + (u|v)\{b, b'\}, \end{aligned} \tag{2.16}$$

and for all Δ,

$$\begin{aligned} [\{\alpha, \alpha'\}, x \otimes a] &= x \otimes D_{\alpha,\alpha'} a = -[x \otimes a, \{\alpha, \alpha'\}] \\ [\{\alpha, \alpha'\}, u \otimes b] &= u \otimes D_{\alpha,\alpha'} b = -[u \otimes b, \{\alpha, \alpha'\}] \\ [\{\alpha, \alpha'\}, \{\beta, \beta'\}] &= \{D_{\alpha,\alpha'}\beta, \beta'\} + \{\beta, D_{\alpha,\alpha'}\beta'\}, \end{aligned} \tag{2.17}$$

for all $a, a' \in A$, $b, b' \in B$, $\alpha, \alpha' \in \mathfrak{a} = A \oplus B$, $x, y \in \mathfrak{g}$, $u, v \in V$. (See (2.18) for an explanation of the notation used in (2.15) and (2.16).) Then $\hat{L}$ is a Lie algebra graded by the root system Δ of $\mathfrak{g}$ whose center is the full skew dihedral homology group

$$\mathrm{HF}(\mathfrak{a}) = \Big\{ \sum_i \{\alpha_i, \beta_i\} \in \{\mathfrak{a}, \mathfrak{a}\} \;\Big|\; \sum_i D_{\alpha_i,\beta_i} = 0 \Big\}.$$

Let $L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$ be a Lie algebra graded by the finite reduced root system Δ and having the centralizer of $\mathfrak{g}$ in L given by the inner derivations $D_{\mathfrak{a},\mathfrak{a}}$ and having multiplication given by (2.15) and (2.16) with $D_{\alpha,\beta}$ in place of $\{\alpha, \beta\}$. Then $(\hat{L}, \hat{\pi})$, where $\hat{\pi}\colon \hat{L} \longrightarrow L$ is given by $\hat{\pi}\colon x \otimes a \longmapsto x \otimes a$; $\hat{\pi}\colon u \otimes b \longmapsto u \otimes b$; $\hat{\pi}\colon \{\alpha, \alpha'\} \longmapsto D_{\alpha,\alpha'}$, is the universal covering algebra of L.

If $\mathcal{S}$ is a subspace of $\mathrm{HF}(\mathfrak{a})$, *then*

$$L(\mathcal{S}) \stackrel{\text{def}}{=\!=} \hat{L}/\mathcal{S} = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \{\mathfrak{a}, \mathfrak{a}\}/\mathcal{S}$$

is a Δ*-graded Lie algebra. Every* Δ*-graded Lie algebra is isomorphic to* $L(\mathcal{S})$ *for some subspace* $\mathcal{S}$ *of* $\mathrm{HF}(\mathfrak{a})$.

2.18 *Explanation of the terms in (2.15) and (2.16).* In Theorem 2.14 $(x|y)$ is the Killing form on $\mathfrak{g}$. When $\Delta = B_r$, F_4, and G_2, the module V can be regarded as the space of elements of trace zero with respect to a normalized trace τ on an algebra $X = \mathbf{F}1 \oplus V$ (see [BZ2, Section 3]). In the B_r-case, X is the Jordan algebra of a nondegenerate symmetric bilinear form; in the F_4-case, X is the 27-dimensional split exceptional Jordan algebra over $\mathbf{F}$; and in the G_2-case, X is the alternative algebra of split octonions over $\mathbf{F}$. In all these cases $\partial_{u,v} \in \mathfrak{g}$ is a certain multiple of the inner derivation determined by $u, v \in V$, and $u * v$ and $(u|v)$ are respectively the V and $\mathbf{F}1$ components of the product uv in X. The mappings $\partial \colon V \otimes V \longrightarrow \mathfrak{g}$, $u \otimes v \longmapsto \partial_{u,v}$, and $V \otimes V \longrightarrow V$, $u \otimes v \longmapsto u * v$, are $\mathfrak{g}$-module homomorphisms, (in [BZ2] these are denoted by π and ρ respectively), and $(u|v)$ is the unique (up to scalar multiple) $\mathfrak{g}$-invariant bilinear form on V with values in $\mathbf{F}$. Multiplication in X is given by $(\zeta 1 + u)(\theta 1 + v) = \zeta\theta 1 + (u|v) + \zeta v + \theta u + u * v$, where $u * v = uv - \tau(uv)1$ and $(u|v) = \tau(uv)$ is the associated bilinear form.

When $\mathfrak{g}$ is of type A_r, $r \geq 2$, we view $\mathfrak{g}$ as the Lie algebra $\mathrm{sl}_{r+1}(\mathbf{F})$ of $(r+1) \times (r+1)$ matrices of trace zero. For C_r, $r \geq 2$, we regard $\mathfrak{g}$ as the Lie algebra $sp_{2r}(\mathbf{F})$ of $(2r) \times (2r)$ matrices which are skew-symmetric relative to the bilinear form whose matrix is $E = \sum_{i=1}^{r} e_{i,2r+1-i} - \sum_{i=1}^{r} e_{2r+1-i,i}$. (Here $e_{i,j}$ denotes the standard matrix unit.) We identify V with the $(2r) \times (2r)$ matrices of trace zero that are symmetric relative to that same form. In these cases there is a symmetric product $w \circ z$ on $\mathfrak{g} \oplus V$ specified by

$$w \circ z = \begin{cases} wz + zw - \frac{2}{r+1}\operatorname{tr}(wz) I_{r+1} & \text{if } \Delta = A_r \text{ and } w, z \in \mathfrak{g} \\ wz + zw - \frac{1}{r}\operatorname{tr}(wz) I_{2r} & \text{if } \Delta = C_r \text{ and } w, z \in \mathfrak{g} \text{ or } V, \end{cases}$$

which gives a $\mathfrak{g}$-module homomorphism. We let $[w, z] = wz - zw$ denote the usual matrix commutator. Since we have matrix realizations, it is customary in these cases to assume the forms are given by the matrix trace, $(x|y) = \operatorname{tr}(xy)$ and $(u|v) = \operatorname{tr}(uv)$. The Killing form is just a scalar multiple of the trace, so this change is inconsequential.

In the C_r-case, $\mathfrak{a} = A \oplus B$ is an algebra with involution σ which is associative when $r \geq 4$ and alternative when $r = 3$, and A is the set of symmetric elements and B is the set of skew-symmetric elements relative to σ. When $r = 3$, the set of symmetric elements A must lie in the nucleus (associative center) of the algebra $\mathfrak{a}$. Now when $r = 2$, there is a unital Jordan algebra $(J, \cdot)$ and a triangle $\mathcal{T} = (p_1, p_2, q)$ of elements in J such that

$$p_1^2 = p_1, \quad p_2^2 = p_2, \quad p_1 \cdot p_2 = 0,$$
$$p_1 \cdot q = \frac{1}{2} q, \quad p_2 \cdot q = \frac{1}{2} q \quad \text{and} \quad q^2 = p_1 + p_2 = 1.$$

The algebra J has the Peirce decomposition $J = J_{11} \oplus J_{12} \oplus J_{22}$, where J_{11}, J_{12}, and J_{22} are the $1, 0$, and $1/2$ eigenspaces with respect to the left multiplication operator L_{p_1}

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determined by the idempotent p_1. The connection involution $\sigma: J \longrightarrow J$, given by $\sigma(x) = 2(q \cdot x) \cdot q - x$, maps the Peirce half space $\mathfrak{a} = J_{12}$ into itself, and $\mathfrak{a} = A \oplus B$ where A and B are the 1 and -1-eigenspaces respectively of σ on $\mathfrak{a}$. The mapping $J_{11} \longrightarrow A$ given by $x_{11} \longmapsto x_{11} \cdot q$ is a bijection. Using that fact, we can define a product on $\mathfrak{a}$, as in [ABG2], by

$$\begin{aligned} aa' &= x_{11} \cdot a', \\ ab &= x_{11} \cdot b, \\ ba &= x_{11}^{\sigma} \cdot b, \quad \text{and} \\ bb' &= -(b \cdot b') \cdot q, \end{aligned} \tag{2.19}$$

for all $a = x_{11} \cdot q$, $a' \in A$, and $b, b' \in B$. The mapping σ is an involution with respect to this multiplication. In [ABG2] it is shown that the coordinate algebra $\mathfrak{a} = A \oplus B$ of a Lie algebra graded by the root system C_2 is such a Peirce half space with multiplication as in (2.19), and A and B are the symmetric and skew-symmetric elements relative to the connection involution restricted to $\mathfrak{a}$.

2.20. The classification results of [ABG1], [BeM], [BZ2], [N] determine the Δ-graded Lie algebras up to central isogeny, that is, they state that the universal covering algebra of any Δ-graded Lie algebra is isomorphic to the universal covering algebra of $L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$ (so isomorphic to $\hat{L}$). Theorem 2.14 describes all the Δ-graded Lie algebras up to isomorphism. Because each Δ-graded Lie algebra is isomorphic to a covering algebra of L, we will write $\tilde{L}$ henceforth for an arbitrary Δ-graded Lie algebra.

3. **The derivations of Lie algebras graded by finite root systems.**

3.1. We begin by developing some very general results concerning derivations of Lie algebras having the property that they are completely reducible relative to some subalgebra $\mathfrak{g}$. These results apply to many different Lie algebras, but we content ourselves here to use them to study Δ-graded Lie algebras.

PROPOSITION 3.2. *Suppose L is a Lie algebra over a field $\mathbf{F}$ and $\mathfrak{g}$ is a finite-dimensional subalgebra of L. Relative to the adjoint action of $\mathfrak{g}$ on L assume L decomposes into a sum of $\mathfrak{g}$-modules, say $L = \bigoplus_{i \in I \cup \{0\}} V_i$, where $V_0 = \mathfrak{g}$. Assume $\mathrm{H}^1(\mathfrak{g}, V_i) = (0)$ for all $i \in I \cup \{0\}$. If $d \in \operatorname{Der} L$, then there exists an element $v \in L$ so that if $d' = d + \operatorname{ad} v$, then $d'(\mathfrak{g}) = 0$.*

PROOF. Let $\pi_i: L \longrightarrow V_i$ denote the projection onto the $\mathfrak{g}$-module V_i and assume $d_i = \pi_i d$. Then for $x, y \in \mathfrak{g}$,

$$\begin{aligned} d_i([x,y]) &= \pi_i d([x,y]) = \pi_i\big([d(x),y]\big) + \pi_i\big([x,d(y)]\big) \\ &= \big[\pi_i\big(d(x)\big), y\big] + \big[x, \pi_i\big(d(y)\big)\big] = [d_i(x), y] + [x, d_i(y)]. \end{aligned}$$

Consequently, $d_i \in \operatorname{Der}(\mathfrak{g}, V_i)$ for each i. Since $\mathrm{H}^1(\mathfrak{g}, V_i) = (0)$, we have $\operatorname{Der}(\mathfrak{g}, V_i) = \operatorname{Inder}(\mathfrak{g}, V_i)$ for all $i \in I \cup \{0\}$. Thus, there exists a $v_i \in V_i$ such that $d_i(x) = x \cdot v_i =$

$[x, v_i]$ for all $x \in \mathfrak{g}$. Only finitely many v_i are nonzero since $\mathfrak{g}$ is assumed to be finite-dimensional. Set $v = \sum_{i \in I \cup \{0\}} v_i$. Then clearly $(d + \operatorname{ad} v)(x) = d(x) - \sum_i [x, v_i] = 0$ for all $x \in \mathfrak{g}$. ∎

3.3. By collecting isomorphic summands in the decomposition of such a Lie algebra L, we may assume there is some subset J of I and a vector space A_j over $\mathbf{F}$ for each $j \in J \cup \{0\}$ with basis $\{a_\ell^{(j)}\}$ such that

$$L = \bigoplus_{j \in J \cup \{0\}} V_j \otimes A_j,$$

where $V_j \otimes a_\ell^{(j)} \cong V_j$ as $\mathfrak{g}$-modules. We suppose $a_0^{(0)} = 1 \in A_0$ and identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1 = V_0 \otimes 1$. Then $[x \otimes 1, v_j \otimes a_\ell^{(j)}] = x \cdot v_j \otimes a_\ell^{(j)} = [x, v_j] \otimes a_\ell^j$ for all $x \in \mathfrak{g}$.

PROPOSITION 3.4. *With assumptions as in Proposition 3.2, suppose further that* $\dim \operatorname{Hom}_{\mathfrak{g}}(V_k, V_j) = \delta_{k,j}$ *for all* $j, k \in J \cup \{0\}$*. Assume* $d \in \operatorname{Der}(L)$ *and* $d(\mathfrak{g}) = 0$*. Then for each* $k \in J \cup \{0\}$ *there exists a linear transformation* $d^{(k)}: A_k \longrightarrow A_k$ *such that* $d(v_k \otimes a) = v_k \otimes d^{(k)}(a)$ *for all* $a \in A_k$ *and* $v_k \in V_k$*.*

PROOF. Let $\pi_\ell^{(j)}: L \longrightarrow V_j \otimes a_\ell^{(j)}$ denote the projection of L onto the $\mathfrak{g}$-module $V_j \otimes a_\ell^{(j)}$. Fix a basis element $a_m^{(k)}$ of A_k. Since

$$\pi_\ell^{(j)} d([x, v_k] \otimes a_m^{(k)}) = \pi_\ell^{(j)} d([x \otimes 1, v_k \otimes a_m^{(k)}]) = [x \otimes 1, \pi_\ell^{(j)} d(v_k \otimes a_m^{(k)})],$$

for all $x \in \mathfrak{g}$, the map $\pi_\ell^{(j)} d$ when restricted to $V_k \otimes a_m^{(k)}$ can be regarded as a $\mathfrak{g}$-module homomorphism in $\operatorname{Hom}_{\mathfrak{g}}(V_k, V_j)$. By assumption this homomorphism must be zero if $j \neq k$ and a multiple of the identity, say $c_{\ell,m}^{(k)} \operatorname{id}_{V_k}$ if $j = k$. Thus, $d(v_k \otimes a_m^{(k)}) = \sum_\ell \pi_\ell^{(k)} d(v_k \otimes a_m^{(k)}) = \sum_\ell v_k \otimes c_{\ell,m}^{(k)} a_\ell^{(k)}$. Consequently for each $k \in J \cup \{0\}$, the derivation d induces a linear transformation $d^{(k)}: A_k \longrightarrow A_k$ defined by

$$d^{(k)}(a_m^{(k)}) = \sum_\ell c_{\ell,m}^{(k)} a_\ell^{(k)},$$

and $d(v_k \otimes a) = v_k \otimes d^{(k)}(a)$ for all $a \in A_k$ and $v_k \in V_k$ as claimed. ∎

3.5. The hypotheses of these propositions are satisfied by the split simple Lie algebra $\mathfrak{g}$, the little adjoint module V, the trivial $\mathfrak{g}$-module, and any Δ-graded Lie algebra $\tilde{L} = L(\mathcal{S}) = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \{\mathfrak{a}, \mathfrak{a}\}/\mathcal{S}$. The condition that the first cohomology group vanishes is just the first Whitehead lemma for $\mathfrak{g}$. Therefore, we may apply these results to any Δ-graded Lie algebra. This brings us to the main result (Theorem 3.6) of this section. I am grateful to Yun Gao for providing me with the calculation in (3.9) below.

THEOREM 3.6. *Suppose* $\tilde{L} = L(\mathcal{S}) = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \{\mathfrak{a}, \mathfrak{a}\}/\mathcal{S}$ *is a Lie algebra graded by the finite reduced root system* Δ*. Let* $\mathfrak{a} = A \oplus B$*. Then*

$$\operatorname{Der}(\tilde{L}) = \operatorname{ad} \tilde{L} + \operatorname{Der}_*(\mathfrak{a}, \mathcal{S})$$

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where $\mathrm{Der}_*(\mathfrak{a}) = \{D \in \mathrm{Der}(\mathfrak{a}) \mid D(A) \subseteq A$, *and* $D(B) \subseteq B\}$ *and* $\mathrm{Der}_*(\mathfrak{a}, \mathcal{S}) = \{D \in \mathrm{Der}_*(\mathfrak{a}) \mid D(\mathcal{S}) \subseteq \mathcal{S}\}$.

PROOF. Suppose $\tilde{L} = L(\mathcal{S}) = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \{\mathfrak{a}, \mathfrak{a}\}/\mathcal{S}$ is a Δ-graded Lie algebra. For convenience of notation write $\mathcal{D} = \{\mathfrak{a}, \mathfrak{a}\}/\mathcal{S}$ and set $\langle \alpha, \beta\rangle = \{\alpha, \beta\} + \mathcal{S}$ for all $\alpha, \beta \in \mathfrak{a}$. Then the $\mathcal{D}$-component of $[x \otimes a, y \otimes a']$ is $(x|y)\langle a, a'\rangle$, and analogously the $\mathcal{D}$-component of $[u \otimes b, v \otimes b']$ is $(u|v)\langle b, b'\rangle$ in (2.15) and (2.16).

Assume $d \in \mathrm{Der}(\tilde{L})$. After adjusting by an inner derivation if necessary, we may suppose that $d(\mathfrak{g}) = 0$. Then by Proposition 3.4, d induces a transformation on $\mathfrak{a} = A \oplus B$, say $d_*: \mathfrak{a} \longrightarrow \mathfrak{a}$, such that $d_*: A \longrightarrow A$, $d_*: B \longrightarrow B$, and

(3.7)
$$\begin{aligned} d(x \otimes a) &= x \otimes d_*(a) \\ d(u \otimes b) &= u \otimes d_*(b) \end{aligned}$$

for all $x \in \mathfrak{g}$, $u \in V$. Moreover, since $\mathcal{D}$ is a subalgebra of $\tilde{L}$, which by Proposition 3.4 must be d-invariant, d restricted to $\mathcal{D}$ is a derivation.

Now when Δ is not of type A_r, C_r, $r \geq 2$, we see upon applying d that

$$\begin{aligned} &[x, y] \otimes d_*(aa') + (x|y)d(\langle a, a'\rangle) \\ &\qquad = d([x \otimes a, y \otimes a']) \\ &\qquad = [x \otimes d_*(a), y \otimes a'] + [x \otimes a, y \otimes d_*(a')] \\ &\qquad = [x, y] \otimes \big(d_*(a)a' + ad_*(a')\big) + (x|y)\big(\langle d_*(a), a'\rangle + \langle a, d_*(a')\rangle\big) \end{aligned}$$

for all $x, y \in \mathfrak{g}$ and $a, a' \in A$. From this it follows that d_* is a derivation on A, and $d(\langle a, a'\rangle) = \langle d_*(a), a'\rangle + \langle a, d_*(a')\rangle$ holds. We have similar calculations,

$$\begin{aligned} x \cdot u \otimes d_*(ab) &= d([x \otimes a, u \otimes b]) \\ &= [x \otimes d_*(a), u \otimes b] + [x \otimes a, u \otimes d_*(b)] \\ &= x \cdot u \otimes \big(d_*(a)b + ad_*(b)\big), \end{aligned}$$

and

$$\begin{aligned} &\partial_{u,v} \otimes d_*\big((b, b')\big) + (u * v) \otimes d_*(b * b') + (u|v)d(\langle b, b'\rangle) \\ &\quad = d([u \otimes b, v \otimes b']) \\ &\quad = [u \otimes d_*(b), v \otimes b'] + [u \otimes b, v \otimes d_*(b')] \\ &\quad = \partial_{u,v} \otimes \Big((d_*(b), b') + \big(b, d_*(b')\big)\Big) + (u * v) \otimes \big(d_{\mathfrak{a}}(b) * b' + b * d_*(b')\big) \\ &\qquad + (u|v)\big(\langle d_*(b), b'\rangle + \langle b, d_*(b')\rangle\big). \end{aligned}$$

Whence it follows that d_* is a derivation on $\mathfrak{a}$ belonging to $\mathrm{Der}_*(\mathfrak{a}) = \{D \in \mathrm{Der}(\mathfrak{a}) \mid D(A) \subseteq A$, and $D(B) \subseteq B\}$ and satisfying

(3.8)
$$\begin{aligned} d(\langle a, a'\rangle) &= \langle d_*(a), a'\rangle + \langle a, d_*(a')\rangle \\ d(\langle b, b'\rangle) &= \langle d_*(b), b'\rangle + \langle b, d_*(b')\rangle \end{aligned}$$

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for all $a, a' \in A$, $b, b' \in B$. Every derivation in $\mathrm{Der}_*(\mathfrak{a})$ has a natural action on $\{\mathfrak{a}, \mathfrak{a}\}$. Let η denote the natural homomorphism $\eta\colon \{\mathfrak{a}, \mathfrak{a}\} \longrightarrow \langle \mathfrak{a}, \mathfrak{a}\rangle = \{\mathfrak{a}, \mathfrak{a}\}/\mathcal{S}$. Then by (3.8),

$$\eta d_*(\{\alpha, \beta\}) = \eta(\{d_*\alpha, \beta\} + \{\alpha, d_*\beta\}) \\ = \langle d_*\alpha, \beta\rangle + \langle \alpha, d_*\beta\rangle = d\eta\{\alpha, \beta\} \tag{3.9}$$

for all $\alpha, \beta \in \mathfrak{a}$. This shows that $\eta \circ d_* = d \circ \eta$, and hence that the kernel of η, which is $\mathcal{S}$, is d_*-invariant. Conversely, any derivation $d_* \in \mathrm{Der}_*(\mathfrak{a}, \mathcal{S}) = \{D \in \mathrm{Der}_*(\mathfrak{a}) \mid D(\mathcal{S}) \subseteq \mathcal{S}\}$ gives rise to a corresponding derivation d on L by specifying that (3.7) and (3.8) hold. Thus, we have the desired result for all Δ-graded Lie algebras whose root system Δ is not of types A_r or C_r for $r \geq 2$.

In the remaining cases suppose

$$\alpha \circ \beta = \alpha\beta + \beta\alpha$$
$$[\alpha, \beta] = \alpha\beta - \beta\alpha$$

for all $\alpha, \beta \in \mathfrak{a}$. Observe that

$$\begin{aligned} &\frac{1}{2}[x, y] \otimes d_*(a \circ a') + \frac{1}{2}(x \circ y) \otimes d_*([a, a']) + (x|y) d(\langle a, a'\rangle) \\ &\quad = d([x \otimes a, x \otimes a']) \\ &\quad = [x \otimes d_*(a), y \otimes a'] + [x \otimes a, y \otimes d_*(a')] \\ &\quad = \frac{1}{2}[x, y] \otimes \big(d_*(a) \circ a' + a \circ d_*(a')\big) \\ &\qquad + \frac{1}{2}(x \circ y) \otimes \big([d_*(a), a'] + [a, d_*(a')]\big) \\ &\qquad + (x|y)\big(\langle d_*(a), a'\rangle + \langle a, d_*(a')\rangle\big), \end{aligned} \tag{3.10}$$

Now suppose that $x = y = e_{1,1} - e_{2,2}$ when Δ is of type A. (Recall the identifications we have made in this case of $\mathfrak{g}$ with the matrix Lie algebra $\mathrm{sl}_{r+1}(\mathbf{F})$.) Then $[x, y] = 0$, but $x \circ y \neq 0$ since $r \geq 2$, and we may deduce from (3.10) that

$$d_*([a, a']) = [d_*(a), a'] + [a, d_*(a')]. \tag{3.11}$$

Putting that back in (3.10) we see that

$$d_*(a \circ a') = d_*(a) \circ a' + a \circ d_*(a') \tag{3.12}$$

must hold as well, and these relations can be combined to show that d_* is a derivation on $\mathfrak{a} = A$. Note that (3.8) holds also in this case. Consequently, Theorem 3.6 is true for type A by the same arguments as before.

When Δ is type C, then the three summands of $[x \otimes a, y \otimes a']$ lie in different components, so the corresponding coefficients of $[x, y]$, $x \circ y$, and $(x|y)$ on both sides of (3.10) can be equated to give (3.8), (3.11), (3.12). Similarly, since all the summands in the products in (2.15) lie in different components, the derivation d can be applied to those relations to deduce that d_* is a derivation in $\mathrm{Der}_*(\mathfrak{a}, \mathcal{S})$ and (3.8) holds. Once again, any derivation $d_* \in \mathrm{Der}_*(\mathfrak{a}, \mathcal{S})$ gives rise to one of L by (3.7) and (3.8), so we have all the desired conclusions. ∎

3.13. Special cases of Theorem 3.6 have been known. When $L = \mathfrak{g} \otimes A$, where A is a commutative associative algebra, $\mathrm{Der}(L) = \mathrm{ad}\, L \rtimes \mathrm{Der}(A) = (\mathrm{ad}\, \mathfrak{g} \otimes A) \rtimes \mathrm{Der}(A)$ (see for example, [K, Exercise 7.3–7.5]). In particular, when $\mathfrak{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{F}[t^{\pm 1}]$, the loop algebra corresponding to $\mathfrak{g}$, then

$$\mathrm{Der}\big(\mathfrak{L}(\mathfrak{g})\big) = \mathrm{ad}\, \mathfrak{L}(\mathfrak{g}) \rtimes \mathrm{Der}(\mathbf{F}[t^{\pm 1}]) \cong (\mathfrak{g} \otimes \mathbf{F}[t^{\pm 1}]) \rtimes \mathrm{Der}(\mathbf{F}[t^{\pm 1}]),$$

where the Lie algebra of "outer derivations" $\mathrm{Der}(\mathbf{F}[t^{\pm 1}])$ is a Witt algebra (centerless Virasoro algebra) with basis $\{d_i = t^{i+1}\frac{d}{dt} \mid i \in \mathbf{Z}\}$ and multiplication given by $[d_i, d_j] = (j - i)d_{i+j}$. Benkart and Moody [BM] have determined the derivations of the toroidal Lie algebras and their twisted counterparts. Since the derivation algebra of a Lie algebra that is perfect and centerless and the derivation algebra of its universal covering algebra coincide according to [BM, Theorem 2.2], the derivation algebra of a toroidal Lie algebra is the same as the derivation algebra of $\mathfrak{T}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, which is $\mathrm{ad}\, \mathfrak{T}(\mathfrak{g}) \rtimes \mathrm{Der}(\mathbf{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]) \cong (\mathfrak{g} \otimes \mathbf{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]) \rtimes \mathrm{Der}(\mathbf{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}])$.

If A is any nonassociative algebra with 1, and N is the associative center (or what is often called the nucleus) of A, then Benkart and Osborn [BO] have shown for the algebra $L'_{r+1}(A)$ generated by the elements $ae_{i,j}$ under the commutator product that $\mathrm{Der}\big(L'_{r+1}(A)\big) = \mathrm{ad}\, M_{r+1}(N) + \mathrm{Der}(A)$, where $M_{r+1}(N)$ is the $(r+1) \times (r+1)$-matrices over N, and the derivations in $\mathrm{Der}(A)$ are applied to each matrix entry. In the special case that A is taken to satisfy the conditions in (1.2), this result gives Theorem 3.6 for algebras of type A_r, $r \geq 2$. The derivations of Lie algebras graded by root systems of type A_r have also been computed by Berman, Gao, and Krylyuk in [BGK] for $r \geq 3$ and by Berman, Gao, Krylyuk, and Neher [BGKN] for $r = 2$.

3.14. In [BM] it is shown that if L and K are perfect Lie algebras with universal covering algebras $\hat{L}$ and $\hat{K}$ respectively, and if K acts on L by derivations, then the universal covering algebra of $L \rtimes K$ is $\hat{L} \rtimes \hat{K}$. In particular, since the Virasoro algebra $\mathcal{V}$ is the universal covering algebra of $\mathrm{Der}(\mathbf{F}[t^{\pm 1}])$ (see for example, [GF], [BM], or [MP]), the universal covering algebra of $\mathrm{Der}\big(\mathfrak{L}(\mathfrak{g})\big) = (\mathfrak{g} \otimes \mathbf{F}[t^{\pm 1}]) \rtimes \mathrm{Der}(\mathbf{F}[t^{\pm 1}])$ is the semidirect product $\big((\mathfrak{g} \otimes \mathbf{F}[t^{\pm 1}]) \oplus \mathbf{F}c\big) \rtimes \mathcal{V}$ of the affine algebra with the Virasoro algebra. This Lie algebra appears naturally in representation theory due to the fact that any restricted module for the affine algebra is also a module for $\big((\mathfrak{g} \otimes \mathbf{F}[t^{\pm 1}]) \oplus \mathbf{F}c\big) \rtimes \mathcal{V}$ via the Sugawara operators (see for example, [K, Section 12.8]).

More generally, suppose that $L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus D_{\mathfrak{a},\mathfrak{a}}$ is a Δ-graded Lie algebra whose $\mathcal{D}$-component is given by the inner derivations of $\mathfrak{a} = A \oplus B$. Then $L = L(\mathcal{S})$, where $\mathcal{S} = \mathrm{HF}(\mathfrak{a})$ in the notation of Theorem 2.14, and it is easy to see that $\mathrm{Der}_*(\mathfrak{a}, \mathcal{S}) = \mathrm{Der}_*(\mathfrak{a})$ in this case. Assume $\mathrm{Der}_*(\mathfrak{a}) = D_{\mathfrak{a},\mathfrak{a}} \rtimes K$ where K is Lie subalgebra of $\mathrm{Der}_*(\mathfrak{a})$ which is perfect. Then we can identify $\mathrm{Der}(L)$ with $L + \mathrm{Der}_*(\mathfrak{a}) = L \rtimes K$ because L

is centerless, and the universal covering algebra of $\mathrm{Der}(L)$ is $\hat{L} \rtimes \hat{K}$, where $\hat{L}$ is as in Theorem 2.14.

3.15. The general nature of our arguments allow them to be adapted to variety of different settings. For example, the same methods can be used to compute $\mathrm{Der}(L)$ where L is a Lie algebra graded by the nonreduced root system BC_r (see [ABG2] and [BS]).

4. **Invariant forms on Lie algebras graded by finite root systems.**

4.1. In this section we derive necessary and sufficient conditions for a Lie algebra $\tilde{L}$ graded by a finite root system to have an $\tilde{L}$-invariant bilinear form. Because $\tilde{L}$ is perfect such a form must be symmetric (see for example, the argument in [BZ2, (2.3)]), so we will assume that from the outset. Our aim is to establish the following

THEOREM 4.2. *Assume $\tilde{L} = L(S) = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \{\mathfrak{a}, \mathfrak{a}\}/S$ is a Δ-graded Lie algebra, where $\mathfrak{a} = A \oplus B$. If $\tilde{L}$ has an $\tilde{L}$-invariant symmetric bilinear form $(\ ,\ )$, then there exists an $\mathfrak{a}$-invariant symmetric bilinear form $(\ ,\ )_*$ on $\mathfrak{a}$ such that $(A, B)_* = 0$ and*

$$\begin{aligned}
&(x \otimes a, y \otimes a') = (x|y)(a, a')_* \\
&(u \otimes b, v \otimes b') = \xi(u|v)(b, b')_* \\
&(d, \langle a, a'\rangle) = (da, a')_* = -(a, da')_* \\
&(d, \langle b, b'\rangle) = \xi(db, b')_* = -\xi(b, db')_* \\
&(x \otimes a, u \otimes b) = 0 \\
&(x \otimes a, d) = 0 \\
&(u \otimes b, d) = 0,
\end{aligned} \tag{4.3}$$

for all $a, a' \in A$, $b, b' \in B$, $x, y \in \mathfrak{g}$, $u, v \in V$, and $d \in D$. When $\Delta = C_r$, $r \geq 2$, then $\xi = 1$, and when $\Delta = B_r$, $r \geq 3$, F_4, or G_2, then ξ is the nonzero scalar such that $(x|\partial_{u,v}) = \xi(x \cdot u|v)$ holds for all $x \in \mathfrak{g}$, $u, v \in V$ where $\partial_{u,v}$ is as in (2.10). The center $Z(\tilde{L})$ of $\tilde{L}$ is contained in the radical of the form. If the form $(\ ,\ )_*$ on $\mathfrak{a}$ is nondegenerate, then the radical of $(\ ,\ )$ is $Z(\tilde{L})$. Conversely, if the form $(\ ,\ )$ on $\tilde{L}$ is nondegenerate, then the form $(\ ,\ )_*$ on $\mathfrak{a}$ is nondegenerate and $Z(\tilde{L}) = (0)$. Any symmetric $\mathfrak{a}$-invariant bilinear form $(\ ,\ )_*$ such that $(A, B)_* = 0$ determines a symmetric $\tilde{L}$-invariant bilinear form on $\tilde{L}$ given by (4.3).

PROOF. For the Δ-graded algebra $\tilde{L} = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus \{\mathfrak{a}, \mathfrak{a}\}/S$, we set $D = \{\mathfrak{a}, \mathfrak{a}\}/S$ and write $\langle \alpha, \beta\rangle$ for $\{\alpha, \beta\} + S \in D$. We suppose that $\{a_i \mid i \in \underline{\mathfrak{i}}\}$ is a basis for A, $\{b_j \mid j \in \underline{\mathfrak{j}}\}$ is a basis for B, and $\{d_k \mid k \in \underline{\mathfrak{k}}\}$ is a basis for D. Then $\tilde{L}$ is the direct sum of the finite-dimensional $\mathfrak{g}$-modules $M \in \mathfrak{M} \stackrel{\text{def}}{=\!=} \{\mathfrak{g} \otimes a_i, V \otimes b_j, \mathbf{F}d_k \mid i \in \underline{\mathfrak{i}}, j \in \underline{\mathfrak{j}}, k \in \underline{\mathfrak{k}}\}$.

Now suppose $(\ ,\ )$ is an $\tilde{L}$-invariant form on $\tilde{L}$, and let M and N be any two (possibly equal) modules in $\mathfrak{M}$. Then the form restricts to a mapping $M \otimes N \longrightarrow \mathbf{F}$, which must be a $\mathfrak{g}$-module homomorphism by the invariance. Since any module in $\mathfrak{M}$ is irreducible and self-dual, $(M, N) \neq 0$ implies $M \cong N$.

Fix indices $i, i' \in \underline{i}$ and consider the map $x \otimes y \longmapsto (x \otimes a_i, y \otimes a_{i'})$. This gives a $\mathfrak{g}$-module map $\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbf{F}$, which must be a multiple of the Killing form, say $(x \otimes a_i, y \otimes a_{i'}) = \theta_{i,i'}(x|y)$. Define a bilinear form $(\,,\,)_\bullet$ on A by first specifying $(a_i, a_{i'})_\bullet = \theta_{i,i'}$ and then extending this bilinearly to all of A. As a result, $(x \otimes a, y \otimes a') = (x|y)(a, a')_\bullet$.

Since there is also a unique (up to scalar multiple) $\mathfrak{g}$-module homomorphism $V \otimes V \longrightarrow \mathbf{F}$, the same argument shows there is a form on B, which we also denote $(\,,\,)_\bullet$, such that $(u \otimes b, v \otimes b') = (u|v)(b, b')_\bullet$. Because $\mathfrak{g} \otimes A$ and $V \otimes B$ are orthogonal, we may extend the form $(\,,\,)_\bullet$ to all of $\mathfrak{a}$ by decreeing $(A, B)_\bullet = 0$.

The rest of the argument amounts to using the invariance $([f, g], h) = (f, [g, h])$ of the form on L and the invariance of the Killing form on $\mathfrak{g}$ to derive various properties of the bilinear form on $\mathfrak{a}$. By first taking $f = x \otimes a$, $g = y \otimes a'$, and $h = z \otimes a''$, we see that when Δ is not of type A_r or C_r for $r \geq 2$ that the relation

$$(4.4) \qquad (aa', a'')_\bullet = (a, a'a'')_\bullet$$

holds for all $a, a', a'' \in A$. Next with $f = x \otimes a$, $g = u \otimes b$, $h = v \otimes b'$ we see that

$$(4.5) \qquad (x \cdot u|v)(ab, b')_\bullet = (x|\partial_{u,v})\big(a, (b, b')\big)_\bullet.$$

Then with $f = u \otimes b$, $g = v \otimes b'$, $h = w \otimes b''$ we get

$$(4.6) \qquad (u * v|w)(b * b', b'')_\bullet = (u|v * w)(b, b' * b'')_\bullet.$$

Recall when Δ is of types B,F,G that $X = \mathbf{F} \oplus V$ is an algebra with a normalized trace $\tau: X \longrightarrow \mathbf{F}$ and V is the space of elements of trace zero. Multiplication in X is given by $(\zeta 1 + u)(\theta 1 + v) = \zeta\theta 1 + (u|v) + \zeta v + \theta u + u * v$, where $u * v = uv - \tau(uv)1$ and $(u|v) = \tau(uv)$ is the associated bilinear form. Then $(u * v|w) = \tau\big((uv)w\big) - \tau(uv)\tau(w) = \tau\big((uv)w\big) = \tau\big(u(vw)\big) = (u|v * w)$ follows from the properties of the trace. As a result, (4.6) implies that $(b * b', b'')_\bullet = (b, b' * b'')_\bullet$. This is equivalent to saying $(bb', b'')_\bullet = (b, b'b'')_\bullet$ because A and B are orthogonal.

The mappings $\mathfrak{g} \otimes V \otimes V \longrightarrow \mathbf{F}$ given by $x \otimes u \otimes v \longmapsto (x|\partial_{u,v})$ and $x \otimes u \otimes v \longmapsto (x \cdot u|v)$ are $\mathfrak{g}$-module homomorphisms. They must be multiples of each other since the space $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V \otimes V, \mathbf{F})$ is one-dimensional. Thus, there exists a scalar $\xi \neq 0$ such that $(x|\partial_{u,v}) = \xi(x \cdot u|v)$ for all $x \in \mathfrak{g}$, $u, v \in V$. Suppose $(a, a')_* = (a, a')_\bullet$, $(a, b)_* = 0$, and $(b, b')_* = \xi^{-1}(b, b')_\bullet$, for all $a, a' \in A$, $b, b' \in B$. Then our calculations in (4.4)–(4.6) allow us to conclude that the form $(\,,\,)_*$ on $\mathfrak{a}$ is invariant. Substituting $f = d \in D$, $g = x \otimes a$, and $h = y \otimes a'$ and then $f = d \in D$, $g = u \otimes b$, and $h = v \otimes b'$ shows that $(d, \langle a, a' \rangle) = (da, a')_* = -(a, da')_*$, $(d, \langle b, b' \rangle) = \xi(db, b')_* = -\xi(b, db')_*$ as claimed.

When Δ is of type A_r, the substitution $f = x \otimes a$, $g = y \otimes a'$, and $h = z \otimes a''$ gives

$$(4.7) \quad ([x, y]|z)\big((a \circ a', a'')_\bullet - (a, a' \circ a'')_\bullet\big) + (x \circ y|z)\big(([a, a'], a'')_\bullet - (a, [a', a'']_\bullet)\big) = 0.$$

Thus, when $x = y = e_{1,1} - e_{2,2}$, $z = e_{2,2} - e_{3,3}$, we see that $([a, a'], a'')_\bullet = (a, [a', a''])_\bullet$. When this is put back into (4.7) and x, y, z are chosen so $([x, y]|z) \neq 0$, the condition $(a \circ a', a'')_\bullet = (a, a' \circ a'')_\bullet$ results. Combined these say (4.4) holds in this case. Since

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$(d, \langle a, a'\rangle) = (da, a')_\bullet = -(a, da')_\bullet$ holds exactly as before, all the same conclusions are valid in the A-case with $(\,,\,)_* = (\,,\,)_\bullet$.

Now when Δ is of type C_r, the same initial substitution $f = x \otimes a$, $g = y \otimes a'$, and $h = z \otimes a''$ yields $([x, y]|z)\big((a \circ a', a'')_\bullet - (a, a' \circ a'')_\bullet\big) = 0$. Since $[A, A] \subseteq B$, we also have $([a, a'], a'')_\bullet = 0 = (a, [a', a''])_\bullet$ for all $a, a', a'' \in A$. Combining these results gives $(aa', a'')_\bullet = (a, a'a'')_\bullet$. Analogous arguments with f, g, h taken from among the elements $x \otimes a, y \otimes a', u \otimes b, v \otimes b', w \otimes b''$, show that the form $(\,,\,)_\bullet$ is invariant on $\mathfrak{a}$. In particular, setting $f = x \otimes a$, $g = u \otimes b$, and $h = v \otimes b'$ shows

$$([x, u]|v)(ab, b')_\bullet = (x|[u, v])(a|b \circ b')_\bullet.$$

Since x, u, v are $(2r) \times (2r)$ matrices and the forms are given by the trace, $(x|[u, v]) = ([x, u]|v)$ by the invariance of the trace. Thus, we may take $\xi = 1$ and $(\,,\,)_* = (\,,\,)_\bullet$ in the C_r-case. The relations $(d, \langle a, a'\rangle) = (da, a')_* = -(a, da')_*$, $(d, \langle b, b'\rangle) = (db, b')_* = -(b, db')_*$ hold exactly as in the other cases.

To verify the statements about the radical, observe that the center $Z(\tilde{L})$ of $\tilde{L}$ is the sum of trivial $\mathfrak{g}$-modules and so must lie in D. Any $d \in Z(\tilde{L})$ must satisfy $da = 0 = db$ for all $a \in A$ and $b \in B$. Therefore, since $D = \langle A, A\rangle + \langle B, B\rangle$, it follows from (4.3) that $Z(\tilde{L})$ is contained in the radical of the form.

Now let $\{x_i\}$ be a basis for $\mathfrak{g}$, and suppose that $\{y_i\}$ is the dual basis with respect to the form $(\,|\,)$ on $\mathfrak{g}$. Similarly, assume $\{u_j\}$ and $\{v_j\}$ are dual bases of V with respect to the form $(\,|\,)$ on V. Assume initially that the form on $\mathfrak{a}$ is nondegenerate, and suppose for $z = \sum_i x_i \otimes a_i + \sum_j u_j \otimes b_j + d \in \tilde{L}$ that $0 = (z, \tilde{L})$. Then $0 = (z, y_k \otimes a) = (x_k|y_k)(a_k, a)_* = (a_k, a)_*$ for all $a \in A$. Since $(A, B)_* = 0$, and the form on $\mathfrak{a}$ is assumed to be nondegenerate, we must have $a_k = 0$ for each k. Hence, $z = \sum_j u_j \otimes b_j + d$. A similar argument with $v_\ell \otimes b$ shows that $b_\ell = 0$ for all ℓ. Finally then $0 = (d, \langle a, a'\rangle) = (da, a')_*$ for all $a, a' \in A$. Since $da \in A$ and $(A, B)_* = 0$, the nondegeneracy of the form on $\mathfrak{a}$ forces $da = 0$ for all $a \in A$. Analogously, $db = 0$ for all $b \in B$, and from this we see $d \in Z(\tilde{L})$. Therefore, when $(\,|\,)_*$ is nondegenerate, the radical of the form on $\tilde{L}$ is $Z(\tilde{L})$. For the converse, note that if $a + b$ is in the radical of the form on $\mathfrak{a}$, then so are a and b. Moreover, then $x \otimes a$ and $u \otimes b$ are in the radical of the form on $\tilde{L}$ for all $x \in \mathfrak{g}$ and $u \in V$. Consequently, if the form on $\tilde{L}$ is nondegenerate, then so is the form on $\mathfrak{a}$, and $Z(\tilde{L})$ must be (0).

Finally, suppose that $(\,|\,)_*$ is an $\mathfrak{a}$-invariant symmetric bilinear form on $\mathfrak{a}$. Then it can be verified using the expressions for the inner derivations in (2.2) that the derivations $D_{\alpha,\alpha'}$ are skew-symmetric relative to the form:

$$(D_{\alpha,\alpha'}\beta, \beta')_* = -(\beta, D_{\alpha,\alpha'}\beta')_*.$$

This allows us to define an invariant form on the Lie algebra $\{\mathfrak{a}, \mathfrak{a}\}$ by specifying

$$(\{\alpha, \alpha'\}, \{\beta, \beta'\}) = (D_{\alpha,\alpha'}\beta, \beta')_*.$$

Clearly, $\mathrm{HF}(\mathfrak{a})$ is in the radical of this form, and so for any subspace S of $\mathrm{HF}(\mathfrak{a})$ there is an induced form on $D = \{\mathfrak{a}, \mathfrak{a}\}/S$ as in (4.3). Using the $\mathfrak{g}$-invariance of the bilinear forms $(\,|\,)$ on $\mathfrak{g}$ and V we see that every form on $\tilde{L}$ which is specified by (4.3), where $(\,|\,)_*$ is an $\mathfrak{a}$-invariant symmetric bilinear form on $\mathfrak{a}$ such that $(A, B)_* = 0$, gives a symmetric $\tilde{L}$-invariant form. ∎

4.8. Because we have chosen $(u|v)$ to be the symmetric bilinear form coming from the normalized trace on $\mathbf{F}1 \oplus V$, and because in [BZ2] a fixed choice of mapping $u \otimes v \longmapsto \partial_{u,v}$ is made, the scalar ξ must be included above. For example, when Δ is of type B_r, $r \geq 3$, then V is just the natural representation of the Lie algebra $\mathfrak{g}$ on a space of dimension $n = 2r + 1$. We can assume that V has a basis v_i, $i = 1, \ldots, n$, such that $(v_i|v_j) = \delta_{j,n+1-i}$. Then for $x = e_{1,1} - e_{n,n}$, $u = v_1$, $v = v_n$ we have $(x \cdot u|v) = 1$. In [BZ2, Theorem 3.53], $\partial_{u,v}(w) = (u, w)v - (v, w)u$ for all $u, v, w \in V$. For our choice of u, v, $\partial_{u,v} = e_{n,n} - e_{1,1}$. Then $(x|\partial_{u,v}) = (4r - 2)\,\mathrm{tr}(x\partial_{u,v}) = -2(4r - 2) = -2(4r - 2)(x \cdot u|v)$ (see [FH, p. 272]).

4.9. Not every possible coordinate algebra $\mathfrak{a}$ has a nontrivial invariant form. For example, consider the Weyl algebra $\mathfrak{a}$ generated by a, b subject to the relation $[a, b] = ab - ba = 1$. Since the Weyl algebra is an associative algebra with 1, it can serve as a coordinate algebra when $\Delta = A_r$, $r \geq 2$. Suppose $(\,,\,)$ is a symmetric $\mathfrak{a}$-invariant form on $\mathfrak{a}$. Then

$$([a^{m+1}b^n, b], 1) = (a^{m+1}b^n, [b, 1]) = 0.$$

But the left side equals $(m+1)(a^m b^n, 1)$, so 1 is orthogonal to all the monomials $a^m b^n$ for all $m, n \geq 0$. Because these monomials determine a basis for $\mathfrak{a}$, 1 is in the radical of the form. The radical of an invariant form is an ideal, which in this case contains 1. Thus, it must be all of $\mathfrak{a}$. Hence, the only invariant form on the Weyl algebra is the trivial one.

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Department of Mathematics
University of Wisconsin
Madison, Wisconsin 53706
U.S.A.