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# ABELIAN GROUPS IN WHICH EVERY $\alpha$-PURE SUBGROUP IS $\beta$-PURE

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**1. Introduction.** The determination of the abelian groups in which every neat subgroup is pure is a relatively routine exercise (see [**6**]). There are numerous problems of this type; for example, the determination of the groups in which every pure subgroup is isotype or the groups in which every subgroup is isotype. These are all special cases of the general problem of determining the abelian groups in which every $\alpha$-pure subgroup is $\beta$-pure for arbitrary ordinal numbers $\alpha$ and $\beta$. The solution of this general problem is the object of this paper.

The abelian groups in which every pure subgroup is a direct summand have been characterized (see [**3**]). An application of our main theorem will yield a characterization of the abelian groups in which every $\alpha$-pure subgroup is a direct summand, where $\alpha$ is an arbitrary ordinal number.

**2. Preliminaries.** We shall use the word group to mean abelian group in this paper and the terminology and notation of [**2**] will generally be followed. Let $G$ be a group, $p$ a prime, and $\alpha$ an ordinal number. The subgroup $p^\alpha G$ of $G$ is defined inductively to be $p(p^{\alpha-1}G)$ if $\alpha - 1$ exists and $\bigcap_{\sigma<\alpha} p^\sigma G$ otherwise. A subgroup $H$ of $G$ is *$p^\alpha$-pure* (sometimes called weakly $p^\alpha$-pure) in $G$ provided $H \cap p^\sigma G = p^\sigma H$ for every $\sigma \leqq \alpha$ and $H$ is *$\alpha$-pure* provided $H$ is $p^\alpha$-pure for each prime $p$. $H$ is *isotype* (*$p$-isotype*) in $G$ if $H$ is $\alpha$-pure ($p^\alpha$-pure) in $G$ for every ordinal $\alpha$. It is convenient to consider the class of extended ordinal numbers, which is obtained by adjoining the element $\infty$ to the ordinal numbers as a last element (that is, $\alpha < \infty$ for every ordinal $\alpha$). If we let $p^\infty G = \bigcap_{\alpha<\infty} p^\alpha G$, then $H$ is $p$-isotype in $G$ provided that $H \cap p^\alpha G = p^\alpha H$ for every $\alpha \leqq \infty$ (that is, $H$ is $p^\infty$-pure). Similarly, an isotype subgroup is now an $\infty$-pure subgroup. A 1-pure ($p^1$-pure) subgroup is commonly called a *neat* (*$p$-neat*) subgroup, and an $\omega$-pure ($p^\omega$-pure) subgroup is called a *pure* (*$p$-pure*) subgroup.

Let $p$ and $q$ be primes and $\alpha$ and $\beta$ extended ordinal numbers. We shall call $G$ a *$[q^\alpha, p^\beta]$-group* if every $q^\alpha$-pure subgroup of $G$ is also $p^\beta$-pure. (In the case $q = p$, we are concerned only with $\alpha < \beta$.) $G$ will be an *$[\alpha, \beta]$-group* if every $\alpha$-pure subgroup of $G$ is $\beta$-pure. The characterization of the $[\alpha, \beta]$-groups (Theorem 4.3) is our main result. We also determine the $[q^\alpha, p^\beta]$-groups (Theorems 4.1 and 4.2). If $G$ has the property that every $\alpha$-pure subgroup is a direct summand, then $G$ will be called an *$[\alpha, S]$-group*. An application of Theorem 4.3 will lead to a characterization of the $[\alpha, S]$-groups (Theorem 5.3).

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If $x \in p^\alpha G$ and $x \notin p^{\alpha+1}G$, then $\alpha$ is called the *p-height*, $h_p(x)$, of $x$ in $G$. If $x \in p^\infty G$, then $h_p(x) = \infty$. The Ulm $p$-function, $f_p$, of $G$ is defined by

$$f_p(\alpha) = \operatorname{rank} \frac{p^\alpha G[p]}{p^{\alpha+1}G[p]}$$

for each ordinal $\alpha$. Thus $f_p(\alpha) \neq 0$ if and only if $G[p]$ contains an element $x$ with $h_p(x) = \alpha$. The following elementary result, which is used in Section 4, exhibits a relationship between the Ulm function and the purity properties of a group.

LEMMA 2.1. *Let $G$ be a group, $p$ a prime, and $\alpha$ and $\beta$ extended ordinals satisfying $1 \leqq \alpha < \beta \leqq \infty$. Suppose that $f_p(\gamma) = 0$ for $\alpha \leqq \gamma + 1 < \beta$ and that $\alpha - 1$ exists. If $H$ is a $p^\alpha$-pure subgroup of $G$, then $H$ is also $p^\beta$-pure.*

*Proof*. It will suffice to show that if $\alpha \leqq \sigma < \beta$ and $H$ is $p^\sigma$-pure, then $H$ is $p^{\sigma+1}$-pure. Thus assume $H$ is $p^\sigma$-pure and let $x \in H \cap p^{\sigma+1}G$. Then $x \in p^\sigma H \subset p^\alpha H$. Thus $x = ph$, $h \in p^{\alpha-1}H$. But also $x = py$, $y \in p^\sigma G$. Therefore $h = y + z$, where $z \in p^{\alpha-1}G[p]$. The hypothesis on $f_p(\gamma)$ requires that $p^{\alpha-1}G[p] \subset p^\sigma G[p]$. Thus

$$h = y + z \in p^\sigma G \cap H = p^\sigma H.$$

Therefore $x = ph \in p^{\sigma+1}H$, as desired.

The *p-length*, $\lambda_p(G)$, of $G$ is the least ordinal $\lambda$ satisfying $p^\lambda G = p^{\lambda+1}G$. The following lemma will be useful. (The proof is routine; see [**5**].)

LEMMA 2.2 [**5**]. *Let $\alpha < \lambda_p(G_p)$.* ($G_p$ is the $p$-primary component of $G$.) *Then there exists a finite ordinal $n$ such that $\alpha + n < \lambda_p(G_p)$ and $f_p(\alpha + n) \neq 0$.*

We need to define two parameters involving the Ulm $p$-function for use in the next section.

*Definition* 2.3. Let $G$ be a group and $p$ a prime. Let $\gamma$ be an ordinal satisfying $1 \leqq \gamma \leqq \lambda_p(G)$ and let $\delta$ be any ordinal. We define $n_\gamma$ and $s(\delta)$ by

$$n_\gamma = \begin{cases} \inf\{n \geqq 0 \mid f_p(\gamma - 1 + n) \neq 0\} \text{ if } \gamma - 1 \text{ exists} \\ 0 \text{ if } \gamma \text{ is a limit ordinal,} \end{cases}$$

and

$$s(\delta) = \sup\{\sigma + 1 \mid \sigma + 1 < \delta \text{ and } f_p(\sigma) \neq 0\}.$$

Note that Lemma 2.2 implies that $n_\gamma$ is a finite ordinal. Note also that $\gamma \leqq s(\delta)$ implies $\gamma + n_\gamma \leqq \delta$, with strict inequality holding when $\gamma$ is not a limit ordinal.

Let $H$ and $K$ be subgroups of $G$. We say that $H$ is *K-high* in $G$ provided $H \cap K = 0$ and $L \cap K \neq 0$ for every subgroup $L$ that properly contains $H$. The following lemma will be used in the next section. Refer to [**4**] for the proof.

LEMMA 2.4 [**4**]. *Let $G$ be a group, $p$ a prime, $\alpha$ an ordinal, and $K$ a subgroup of $G$ with $K \subset p^\alpha G$. If $H$ is $K$-high in $G$, then $H$ is $p^{\alpha+1}$-pure in $G$.*

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In the sequel $T(p)$ will denote an arbitrary torsion group whose $p$-primary component is zero. $B_k^p$ will denote an arbitrary direct sum of cyclic $p$-groups of orders $p^k$. The subgroup generated by the subset $S$ of $G$ will be denoted by $\langle S\rangle$ and $n^{-1}H$ will denote the subgroup $\{x \in G | nx \in H\}$, where $H$ is a subgroup of $G$ and $n$ an integer.

**3. Two existence lemmas.** The messy part of the proofs of our main theorems lies in the construction of $p^\alpha$-pure subgroups satisfying various specified conditions. We have isolated this work in the two lemmas of this section.

LEMMA 3.1. *Let $G$ be a group, $H$ a subgroup of $G$ and $p$ a prime. There exists a subgroup $H^*$ of $G$ that contains $H$ and satisfies the following properties:*

(1) *$H^*$ is $q$-isotype in $G$ for each prime $q \neq p$.*

(2) *$H$ is $p$-isotype in $H^*$.*

*In particular, $H^*$ is $p^\alpha$-pure in $G$ if and only if $H$ is $p^\alpha$-pure in $G$, where $\alpha$ is arbitrary extended ordinal.*

*Proof.* The construction of $H^*$ is similar to the proof of Proposition 26.2 in [**2**]. We let $H_0 = H$ and proceed inductively. Having defined $H_n$ we let $H_{n+1} = \langle \bigcup_{q\neq p} q^{-1}H_n\rangle$. Then we let $H^* = \bigcup_{n<\omega} H_n$.

To show that $H^*$ is $q$-isotype it suffices to show that if $H^*$ is $q^\alpha$-pure for an arbitrary ordinal $\alpha$, then $H^*$ is $q^{\alpha+1}$-pure. Thus assume $H^*$ is $q^\alpha$-pure and let $x \in H^* \cap q^{\alpha+1}G$. Then $x = qg$ with $g \in q^\alpha G$. Now $x \in H_k$ for some $k$, so $qg \in H_k$. Thus $g \in H_{k+1}$. Therefore, $g \in H^* \cap q^\alpha G = q^\alpha H^*$ and $x = qg \in q^{\alpha+1}H^*$, as desired.

The same type of proof shows that $H$ is $p$-isotype in $H^*$. Assume $H$ is $p^\alpha$-pure in $H^*$ and let $x \in H \cap p^{\alpha+1}H^*$. Then $x = pg$ with $g \in p^\alpha H^*$. Since $g \in H^*$ there exists an integer $m > 0$ such that $mg \in H$ and $m$ and $p$ are relatively prime. Thus $1 = rm + sp$ for integers $r$ and $s$, so $g = r(mg) + s(pg) \in H$. Therefore, $g \in H \cap p^\alpha H^* = p^\alpha H$. Hence $x = pg \in p^{\alpha+1}H$, as desired.

Clearly $H$ is $p^\alpha$-pure if $H^*$ is $p^\alpha$-pure in $G$. Conversely, suppose $H \cap p^\beta G = p^\beta H$ for all $\beta \leqq \alpha$. Let $x \in H^* \cap p^\beta G$. As above, $mx \in H$. Thus

$$mx \in H \cap p^\beta G = p^\beta H \subset p^\beta H^*.$$

Since $m$ is relatively prime to $p$, a routine transfinite induction argument shows that $mx \in p^\beta H^*$ implies $x \in p^\beta H^*$. Thus $H^*$ is $p^\alpha$-pure. This completes the proof.

LEMMA 3.2. *Let $G$ be a group, $p$ a prime, and $\delta$ an ordinal such that $p^\delta G$ contains a non-zero element, $g_0$, whose order is infinite or a power of $p$. For each ordinal $\gamma$ let $m_\gamma = -1$ if $\gamma - 1$ exists and let $m_\gamma = 0$ otherwise.*

*For each ordinal $\gamma$ satisfying $1 \leqq \gamma \leqq s(\delta)$ there exists a subgroup $H_\gamma$ of $G$ satisfying the following properties:*

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(1) *$H_\gamma$ is $p^{\gamma+n_\gamma}$-pure in $G$.*
(2) *$H_\sigma \subset H_\gamma$ if $\sigma < \gamma$.*
(3) *$H_\gamma \cap p^{\gamma+n_\gamma+m_\gamma}G[p] = p^\delta G[p]$.*
(4) *$g_0 \in H_\gamma$.*
(5) *$g_0 \notin p^{\gamma+n_\gamma+1}H_\gamma$.*
*In particular, $H_\gamma$ is not a $p^{\delta+1}$-pure subgroup of $G$ in general and is not $p^\delta$-pure if $\delta$ is not a limit ordinal.*

*Proof.* We use transfinite induction on $\gamma$. Starting first with the induction step, we assume that for each ordinal $\sigma < \gamma$ the subgroup $H_\sigma$ exists and satisfies (1) through (5). If $\gamma$ is a limit ordinal, then we let

$$H_\gamma = \bigcup_{\sigma<\gamma} H_\sigma.$$

Routine computations will verify that $H_\gamma$ satisfies (1) through (5). If $\gamma - 1$ exists and $n_{\gamma-1} > 0$, then we let $H_\gamma = H_{\gamma-1}$. Properties (1) through (5) are trivially satisfied (here we have $n_\gamma = n_{\gamma-1} - 1$). If $\gamma - 1$ exists and $n_{\gamma-1} = 0$, then we must construct $H_\gamma$ from $H_{\gamma-1}$. Since $f_p(\gamma + n_\gamma - 1) \neq 0$ there exists an element $x \in G[p]$ with $h_p(x) = \gamma + n_\gamma - 1$. Note that $\gamma + n_\gamma < \delta$, since $\gamma \leqq s(\delta)$. Thus we may write

$$p^{\gamma+n_\gamma-1}G[p] = p^\delta G[p] \oplus K$$

with $x \in K$ and $g_0 = py$ with $y \in p^{\gamma+n_\gamma}G$. Let $z = x + y$ and $L = \langle H_{\gamma-1}, z\rangle$. We assert that $K \cap L = 0$. If not, then there exist elements $v \in K$ and $w \in H_{\gamma-1}$ and an integer $t$ such that $v = w + p^t z \neq 0$. If $t = 0$, then $g_0 = pz = -pw$. But

$$w \in H_{\gamma-1} \cap p^{\gamma+n_\gamma-1}G \subset p^{\gamma-1}H_{\gamma-1}.$$

Thus $g_0 \in p^\gamma H_{\gamma-1}$, contradicting that $H_{\gamma-1}$ satisfies (5). On the other hand if $t > 0$, then $v = p^{t-1}g_0 + w \in H_{\gamma-1}$; that is, $v \in K \cap H_{\gamma-1}$. But $K \cap H_{\gamma-1} = 0$ because $H_{\gamma-1}$ satisfies (3). Thus we conclude that $K \cap L = 0$, as desired. Now we let $H_\gamma$ be a $K$-high subgroup of $G$ containing $L$. Then $H_\gamma \supset H_{\gamma-1}$, satisfying (2). $H_\gamma$ satisfies (1) as a consequence of Lemma 2.4. Using the fact that $p^\delta G[p] \subset H_{\gamma-1}$, one easily verifies (3) for $H_\gamma$. $H_\gamma$ satisfies (4) because $z \in H_\gamma$ and $pz = g_0$. In order to see that (5) is valid, let us suppose that $g_0 \in p^{\gamma+n_\gamma+1}H_\gamma$. Then $g_0 = ph$ with $h \in p^{\gamma+n_\gamma}H$. Thus $z = h + u$, where $u \in H_\gamma \cap p^{\gamma+n_\gamma-1}G[p]$ because $h_p(z) = \gamma + n_\gamma - 1$. Thus $u \in p^\delta G$ because of (3). But then $z = h + u \in p^{\gamma+n_\gamma}G$, a contradiction. Thus $H_\gamma$ must satisfy (5). This completes the induction part of the proof.

To start the induction we construct $H_1$, imitating the method for $H_\gamma$ in the previous paragraph. Thus we choose $x \in G[p]$ with $h_p(x) = n_1$ and, as before, write $p^{n_1}G[p] = p^\delta G[p] \oplus K$ with $x \in K$. Now we let $L = \langle p^\delta G[p], z\rangle$ ($z$ is obtained as before) and we consider $K \cap L$. If $v$ is a non-zero element of $K \cap L$, then $v = w + nz$, where $w \in p^\delta G[p]$ and $n$ is a positive integer. Clearly $p$ does not divide $n$. But $p(nz) = 0$, so $ng_0 = 0$. Now we use the restriction on the order of $g_0$ given in the hypothesis; that is, $ng_0 = 0$ is not possible if the order of $g$ is infinite or a power of $p$. Thus it follows that

$K \cap L = 0$. We let $H_1$ be a $K$-high subgroup and routine computations will verify that $H_1$ satisfies properties (1) through (5). This completes the proof.

**4. The main theorems.** There are actually three main theorems. The characterization of the $[q^\alpha, p^\beta]$-groups, $p \neq q$, is considerably easier than that of the $[p^\alpha, p^\beta]$-groups. The $[\alpha, \beta]$-groups are easily determined once the $[p^\alpha, p^\beta]$-groups are known.

THEOREM 4.1. *Let $G$ be a group, $p$ and $q$ distinct primes and $\alpha$ and $\beta$ extended ordinals with $\beta > 0$. Then $G$ is a $[q^\alpha, p^\beta]$-group if and only if $pG = T(p)$.*

*Proof.* Assume first that $pG = T(p)$. Now $H \cap T(p) \subset p^\gamma H$ for any subgroup $H$ and ordinal $\gamma > 0$. Thus, in particular, every $q^\alpha$-pure subgroup is $p^\beta$-pure.

Conversely assume that $G$ is a $[q^\alpha, p^\beta]$-group. If $pG \neq T(p)$, then $pG$ contains an element $x$ whose order is either infinite or a power of $p$. Let $H = \langle x \rangle$. Then $H$ is not $p^1$-pure, so $H^*$ (see Lemma 3.1) is not $p^\beta$-pure for any $\beta > 0$. But $H^*$ is $q$-isotype. Thus we must have $pG = T(p)$, as desired.

THEOREM 4.2. *Let $G$ be a group, $p$ a prime, and $\alpha$ and $\beta$ extended ordinals with $1 \leqq \alpha < \beta \leqq \infty$.*

(1) *If $\alpha$ is a limit ordinal, then $G$ is a $[p^\alpha, p^\beta]$-group if and only if either* (a) *or* (b) *is valid:*

(a) $\lambda_p(G_p) < \alpha$.

(b) $p^\alpha G = B_1{}^p \oplus T(p)$.

(2) *If $\alpha - 1$ exists, then $G$ is a $[p^\alpha, p^\beta]$-group if and only if either* (a) *or* (b) *is valid:*

(a) $f_p(\gamma) = 0$ *if $\gamma$ satisfies $\alpha \leqq \gamma + 1 < \beta$.*

(b) $p^{\alpha-1}G = B_k{}^p \oplus B_{k+1}{}^p \oplus T(p)$ *for some positive integer $k$.*

*Proof of* (1). For the necessity we use Lemma 3.2. If (b) is not valid, then there exists an element $g_0 \in p^{\alpha+1}G$ whose order is either infinite or a power of $p$. If (a) also is not valid, then $\alpha = s(\alpha) \leqq s(\alpha + 1)$. Taking $\gamma = \alpha$ and $\delta = \alpha + 1$ in Lemma 3.2 we have a $p^\alpha$-pure subgroup $H_\alpha$ which is not $p^{\alpha+1}$-pure. Thus either (a) or (b) must be valid in order for $G$ to be a $[p^\alpha, p^\beta]$-group.

The sufficiency of (a) is a consequence of Lemma 2.1, where we replace $\alpha$ by $\lambda_p(G_p) + 1$. Thus assume (b) is valid. Let $H$ be a $p^\alpha$-pure subgroup and let $\sigma$ satisfy $\alpha < \sigma \leqq \beta$. Let $x \in H \cap p^\sigma G$. Then $x \in H \cap T(p)$, since $p^\sigma G = T(p)$. Thus the order of $x$ is relatively prime to $p$, which implies that $x \in p^\gamma H$ for every ordinal $\gamma$. In particular, $x \in p^\sigma H$. Thus $H$ is $p^\beta$-pure, as desired.

*Proof of* (2). Assume that $G$ is a $[p^\alpha, p^\beta]$-group and that (a) is not valid. We show that (b) must hold. Our assumption implies that $\alpha \leqq \lambda_p(G_p)$. Thus $n_\alpha$ exists (see Definition 2.3). Let $\bar{f}_p$ be the Ulm $p$-function for $p^{\alpha-1}G$. Then $\bar{f}_p(\gamma) = 0$ if $\gamma < n_\alpha$. If there exists an element $g_0 \in p^{\alpha+n_\alpha+1}G$ whose order is

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infinite or a power of $p$, then by Lemma 3.2 there exists a $p^{\alpha+n_\alpha}$-pure subgroup $H_\alpha$, which is not $p^{\alpha+n_\alpha+1}$-pure. But this is impossible because $\alpha + n_\alpha + 1 \leqq \beta$. Thus

$$p^{\alpha+n_\alpha+1}G = p^{n_\alpha+2}(p^{\alpha-1}G)$$

is a torsion group with zero $p$-primary component. Let $k = n_\alpha + 1$. Then the condition that $\bar{f}_p(\gamma) \neq 0$ only if $k - 1 \leqq \gamma \leqq k$ and the above condition on $p^{k+1}(p^{\alpha-1}G)$ imply that $p^{\alpha-1}G = B_k{}^p \oplus B_{k+1}{}^p \oplus T(p)$, as desired.

Condition (a) is sufficient according to Lemma 2.1. Assume (b) is valid and let $H$ be a $p^\alpha$-pure subgroup of $G$. Then $H$ is $p^{\alpha+k-1}$-pure by Lemma 2.1, since $f_p(\gamma) = 0$ for $\alpha \leqq \gamma + 1 < \alpha + k - 1$. Thus let $\sigma$ satisfy $\alpha + k \leqq \sigma \leqq \beta$ and let $x \in H \cap p^\sigma G$. Since $p^\sigma G = p^{\alpha+k}G = T(p)$, the order of $x$ is relatively prime to $p$. Thus $x \in p^\sigma H$. Therefore $H$ is a $p^\beta$-pure subgroup, as desired.

THEOREM 4.3. *Let $G$ be a group and $\alpha$ and $\beta$ extended ordinals satisfying $1 \leqq \alpha < \beta \leqq \infty$. Then $G$ is an $[\alpha, \beta]$-group if and only if $G$ is a $[p^\alpha, p^\beta]$-group for each prime $p$.*

*Proof.* The sufficiency is obvious. Conversely, assume $G$ is an $[\alpha, \beta]$-group and let $H$ be $p^\alpha$-pure. Then $H^*$ (see Lemma 3.1) is $\alpha$-pure. Thus $H^*$ is $\beta$-pure and, hence, $p^\beta$-pure. This makes $H$ $p^\beta$-pure, as desired.

**5. An application.** The objective of this section is the characterization of the $[\alpha, S]$-groups for $1 \leqq \alpha \leqq \infty$. The proof of the following proposition is routine (see [**1**, p. 201, Exercise 11]). $G_t$ denotes the torsion subgroup of $G$.

PROPOSITION 5.1. *Let $G$ be a group and $\omega \leqq \alpha \leqq \infty$. Then $G$ is an $[\alpha, S]$-group if and only if $G$ splits and both $G_t$ and $G/G_t$ are $[\alpha, S]$-groups.*

Examples may be constructed to show that Proposition 5.1 is not valid if $\alpha < \omega$. The characterization of the torsion-free $[\omega, S]$-groups may be found in [**1**, *p*. 166]. It is clear, however, that these are also the torsion-free $[\alpha, S]$-groups for any $\alpha$. We quote the result below.

PROPOSITION 5.2 [**1**]. *Let $\alpha$ be any extended ordinal. A torsion-free group $G$ is an $[\alpha, S]$-group if and only if $G = D \oplus H$, where $D$ is divisible and $H$ is a direct sum of a finite number of pairwise isomorphic rank one groups.*

If the reduced part of each $p$-primary component of a group $G$ is bounded, then we shall say that $G$ is locally bounded. If the reduced part of each $p$-primary component is bounded by $p^n$ for a fixed integer $n$, then $G$ is $n$-locally bounded.

THEOREM 5.3. *Let $G$ be a group.*

(1) *Let $\omega \leqq \alpha \leqq \infty$. Then $G$ is an $[\alpha, S]$-group if and only if $G$ splits, $G/G_t$ is a torsion-free $[\alpha, S]$-group and $G_t$ is a locally bounded torsion group.*

(2) *Let $1 \leqq n < \omega$. Then $G$ is an $[n, S]$-group if and only if $G$ splits, $G/G_t$ is a torsion-free $[n, S]$-group and either* (a) *or* (b) *is valid:*

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(a) *$G_t$ is an $(n-1)$-locally bounded torsion group.*

(b) *$G$ is a torsion group and for each prime $p$, $G_p = H \oplus K$, where $p^{n-1}H = 0$ and either $K$ is divisible or $K = B_k^p \oplus B_{k+1}^p$ for some integer $k \geqq n$.*

*Proof of* (1). It suffices to establish that a torsion group $G$ is an $[\alpha, S]$-group if and only if $G$ is locally bounded. The sufficiency follows fairly directly from the fact that a bounded pure (that is, $\omega$-pure) subgroup is a direct summand. The necessity is a consequence of the fact that whenever the reduced part of $G_p$ is unbounded, there exist basic subgroups of $G_p$ (which are always isotype) that are not summands. The proof of (1) is now completed by appealing to Proposition 5.1.

*Proof of* (2). If $G$ satisfies the conditions in (2), then $G$ is an $[\infty, S]$-group by (1). From Theorem 4.2 and Theorem 4.3 we deduce that either (a) or (b) suffices to make $G$ an $[n, \infty]$-group. Therefore the conditions in (2) are sufficient to make $G$ an $[n, S]$-group. The conditions in (2) are also necessary conditions, since either (a) or (b) is necessary for $G$ to be an $[n, \infty]$-group. This completes the proof.

## References


1. L. Fuchs, *Abelian groups* (Hungarian Academy of Sciences, Budapest, 1958).
2. ——— *Infinite abelian groups*. Vol. 1 (Academic Press, New York, 1970).
3. L. Fuchs, A. Kertész, and T. Szele, *Abelian groups in which every serving subgroup is a direct summand*, Publ. Math. Debrecen *3* (1953), 95–105.
4. J. M. Irwin and E. A. Walker, *On isotype subgroups of abelian groups*, Bull. Soc. Math. France *89* (1961), 451–460.
5. G. Kolettis, *Direct sums of countable groups*, Duke Math. J. *27* (1960), 111–125.
6. K. Simauti, *On abelian groups in which every neat subgroup is a pure subgroup*, Comment. Math. Univ. St. Paul. *17* (1969), 105–110.


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