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Subadditivity Inequalities for Compact Operators

Jean-Christophe Bourin, Tetsuo Harada, and Eun-Young Lee

Abstract. Some subadditivity inequalities for matrices and concave functions also hold for Hilbert space operators, but (unfortunately!) with an additional ε term. It does not seem possible to erase this residual term. However, in case of compact operators we show that the ε term is unnecessary. Further, these inequalities are strict in a certain sense when some natural assumptions are satisfied. The discussion also emphasizes matrices and their compressions and several open questions or conjectures are considered, both in the matrix and operator settings.

1 From Matrices to Operators

Given two positive Hilbert space operators A, B and an arbitrary small fixed $\varepsilon > 0$, the following inequality (see [6]) holds for non-negative concave functions $f(t)$ on the positive half-line,

$$f(A+B) \le Uf(A)U^* + Vf(B)V^* + \varepsilon I \tag{1.1}$$

where U, V are some unitary operators and I denotes the identity. In the fundamental finite-dimensional case, proved in [1], we may of course take $\varepsilon = 0$. Thus, denoting by $\mathbb{M}_n$ the algebra of n-by-n matrices and by $\mathbb{M}_n^+$ the positive (semi-definite) part, we have the following.

Theorem 1.1 *Let $f(t)$ be a monotone concave function on $[0, \infty)$ with $f(0) \ge 0$ and let $A, B \in \mathbb{M}_n^+$. Then, for some unitaries $U, V \in \mathbb{M}_n$,*

$$f(A+B) \le Uf(A)U^* + Vf(B)V^*.$$

The first motivation of this article is to find conditions allowing, likewise in the matrix case, to drop the ε term in (1.1). In the course of settling this question, we noted that, in case of compact operators with dense ranges, we may not only drop the ε term, but also obtain a certain strict type inequality, stronger than (1.1) without the ε term, when $f(t)$ is strictly concave, *i.e.*,

$$f\Big(\frac{a+b}{2}\Big) > \frac{f(a)+f(b)}{2}$$

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for all $a \neq b$ in the positive half-line.

The proofs in the next sections depend on the well-known matrix case and belong to the scope of matrix analysis. However we will completely adapt it and this article is self-contained. A good account on Theorem 1.1 and (1.1), related matrix/operator inequalities and applications can be found in the survey [6]. Identifying $A \in \mathbb{M}_n$ as an operator on $\mathcal{F} = \mathbb{C}^n$, Theorem 1.1 yields results for compressions $A_{\mathcal{S}}$ of A onto subspaces $\mathcal{S} \subset \mathcal{F}$. The following is implicit in the proofs of [6, Corollaries 3.5–3.7].

Corollary 1.2 *Let $A \in \mathbb{M}_n^+$ and let $\mathcal{S}$ be a subspace of $\mathcal{F}$. If $f(t)$ is a monotone, concave function on $[0, \infty)$ such that $f(0) \geq 0$, then*

$$f(A) \leq Jf(A_{\mathcal{S}})J^* + Kf(A_{\mathcal{S}^\perp})K^*$$

for some isometries $J\colon \mathcal{S} \to \mathcal{F}$ and $K\colon \mathcal{S}^\perp \to \mathcal{F}$.

We will give in Corollary 2.3 below a version for compact operators with a detailed proof. Let us close this introduction by mentioning some questions still open in the matrix case.

Question 1.3 Theorem 1.1 implies the Rotfel'd trace inequality [8]:

$$\operatorname{Tr} h(A+B) \leq \operatorname{Tr}\big(h(A) + h(B)\big)$$

for all $A, B \in \mathbb{M}_n^+$ and all concave functions on $[0, \infty)$ with $h(0) \geq 0$. Indeed, we may approach $h(t)$ by $f(t) + at$ where $f(t)$ is monotone and concave, $f(0) = h(0)$, $a \in \mathbb{R}$, and then apply the theorem to $f(t)$. Does Theorem 1.1 hold for a non-monotone concave function $f(t)$ with $f(0) \geq 0$?

Question 1.4 It is not possible to take $U = V$ in Theorem 1.1, even for a simple operator monotone function such as $\sqrt{t}$. Writing the inequality as $f(A+B) - Vf(B)V^* \leq Uf(A)U^*$ it is natural to ask whether

$$0 \leq f(A+B) - Vf(B)V^* \leq Uf(A)U^*$$

could hold too, for some suitable unitaries U, V. In particular, does it hold when $f(t)$ is operator monotone?

Question 1.5 Let $f\colon [0, \infty) \to [0, \infty)$ be concave and $A, B \in \mathbb{M}_n^+$. Theorem 1.1 entails that

$$\|f(A+B)\| \leq \|f(A)\| + \|f(B)\|$$

for all symmetric norms $\|\cdot\|$, those norms such that $\|UTV\| = \|T\|$ for all $T \in \mathbb{M}_n$ and all unitaries U, V. In fact, much stronger norm inequalities hold, see [5] and references therein. A semi-norm $\|\cdot\|_w$ on $\mathbb{M}_n$ is weakly symmetric if $\|T\|_w = \|UTU^*\|_w$ for all T and any unitary U. Is it possible to extend the previous symmetric norm inequality to

$$\|f(A+B)\|_w \leq \|f(A)\|_w + \|f(B)\|_w$$

for all weakly symmetric semi-norms? The following is known [2]: This holds in the special case of $\|T\|_w = \operatorname{diam} W(T)$, the diameter of the numerical range. This holds too for any weakly symmetric semi-norms on 2×2 matrices.

2 Compact Operators and Concave Functions

Let $\mathbb{B}$ be the space of all bounded linear operators on a separable Hilbert space $\mathcal{H}$, let $\mathbb{B}^+$ its positive part and $\mathbb{B}_a^+$ its absolutely positive part: $A \in \mathbb{B}_a^+$ iff $\langle h, Ah \rangle > 0$ for all non-zero $h \in \mathcal{H}$. We write $A \geq_a B$ when $A - B \in \mathbb{B}_a^+$. We denote by $\mathbb{K}$ the ideal of compact operators on $\mathcal{H}$, and by $\mathbb{K}^+$ and $\mathbb{K}_a^+$ the positive and absolutely positive parts. We have the following results.

Theorem 2.1 *Let $f(t)$ be a monotone concave function on $[0, \infty)$ such that $f(0) \geq 0$.*

(i) *If $A, B \in \mathbb{K}^+$, then for some partial isometries $J, K \in \mathbb{B}$,*

$$f(A+B) \leq J f(A) J^* + K f(B) K^*. \tag{2.1}$$

Moreover, if $f(t)$ is strictly concave and $f(0) = 0$, we can take $\operatorname{supp} J = \operatorname{supp} A$ *and* $\operatorname{supp} K = \operatorname{supp} B$.

(ii) *If $A, B \in \mathbb{K}_a^+$, then for some unitary $U, V \in \mathbb{B}$,*

$$f(A+B) \leq U f(A) U^* + V f(B) V^*.$$

Moreover, if $f(t)$ is strictly concave, this can be improved as

$$f(A+B) \leq_a U f(A) U^* + V f(B) V^*. \tag{2.2}$$

Here the notation $\operatorname{supp} T$ means the support of $T \in \mathbb{B}$, that is, the orthogonal of its kernel. Recall that a partial isometry J is an operator such that J^*J and JJ^* are projections.

Note that deleting the ε part in (1.1) is quite meaningful when considering ideals of compact operators. We refer to [9] for the Calkin theory of ideals of compact operators. The next corollary follows from (2.1) and the property that $A \geq B \geq 0$ and $A \in \mathbb{J}$ ensures that $B \in \mathbb{J}$ in any ideal $\mathbb{J}$.

Corollary 2.2 *Let $\mathbb{J}$ be an ideal of $\mathbb{K}$. Let $A, B \in \mathbb{K}^+$ and let $f(t)$ be a non-negative monotone, concave function on $[0, \infty)$. If both $f(A) \in \mathbb{J}$ and $f(B) \in \mathbb{J}$, then we also have $f(A+B) \in \mathbb{J}$.*

Corollary 2.3 *Let $A \in \mathbb{K}_a^+$ and let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. If $f(t)$ is a monotone, strictly concave function on $[0, \infty)$ such that $f(0) \geq 0$, then*

$$f(A) \leq J f(A_{\mathcal{S}}) J^* + K f(A_{\mathcal{S}^\perp}) K^*$$

for some isometries $J \colon \mathcal{S} \to \mathcal{H}$ and $K \colon \mathcal{S}^\perp \to \mathcal{H}$.

We first state a lemma which is the adaption to the matrix case [6, Lemma 3.4].

Lemma 2.4 *Let $A \in \mathbb{K}_a^+$ and let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then,*

$$A = J A_{\mathcal{S}} J^* + K A_{\mathcal{S}^\perp} K^*$$

for some isometries $J \colon \mathcal{S} \to \mathcal{H}$ and $K \colon \mathcal{S}^\perp \to \mathcal{H}$.

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Proof With respect to the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp}$, we have a block matrix representation

$$A = \begin{bmatrix} A_{\mathcal{S}} & X \\ X^* & A_{\mathcal{S}^{\perp}} \end{bmatrix}$$

which can be factorized as a square of a positive block matrix,

$$A = \begin{bmatrix} C & Y \\ Y^* & D \end{bmatrix} \begin{bmatrix} C & Y \\ Y^* & D \end{bmatrix}$$

and observe that it can be written as

$$\begin{bmatrix} C & 0 \\ Y^* & 0 \end{bmatrix} \begin{bmatrix} C & Y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & D \end{bmatrix} \begin{bmatrix} 0 & 0 \\ Y^* & D \end{bmatrix} = TT^* + SS^*. \tag{2.3}$$

Then, note that $C \geq_a 0$ on $\mathcal{S}$ so that T has polar decomposition $T = J|T|$ where $\operatorname{supp} T = \operatorname{supp} J = \mathcal{S}$, hence $TT^* = JT^*TJ^*$ where J is a partial isometry with support $\mathcal{S}$ and

$$T^*T = \begin{bmatrix} C^2 + YY^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{\mathcal{S}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly $S^*S = KSS^*K^*$ where K is partial isometry with support $\mathcal{S}^{\perp}$ and

$$SS^* = \begin{bmatrix} 0 & 0 \\ 0 & A_{\mathcal{S}^{\perp}} \end{bmatrix}.$$

Therefore, from (2.3) we have

$$A = J \begin{bmatrix} A_{\mathcal{S}} & 0 \\ 0 & 0 \end{bmatrix} J^* + K \begin{bmatrix} 0 & 0 \\ 0 & A_{\mathcal{S}^{\perp}} \end{bmatrix} K^*. \tag{2.4}$$

Regarding J as an isometry on $\mathcal{S}$ and K as an isometry on $\mathcal{S}^{\perp}$, (2.4) is equivalent to the statement of the lemma. ■

We may now give the proof of Corollary 2.3.

Proof Since $A \geq_a 0$ we have $f(A) = f_0(A)$ where $f_0(t) = f(t)$ for $t > 0$ and $f_0(0) = 0$. As $f_0(t)$ is monotone and strictly concave, we may suppose that $f(t) = f_0(t)$; *i.e.*, that $f(0) = 0$. In the decomposition 2.4, note that

$$\operatorname{supp} J \begin{bmatrix} A_{\mathcal{S}} & 0 \\ 0 & 0 \end{bmatrix} J^* = J(\mathcal{S})$$

and

$$\operatorname{supp} K \begin{bmatrix} 0 & 0 \\ 0 & A_{\mathcal{S}^{\perp}} \end{bmatrix} K^* = K(\mathcal{S}^{\perp}).$$

Thus, applying (2.1) to (2.4), we have some partial isometries J_0 and K_0 with supports $J(\mathcal{S})$ and $K(\mathcal{S}^\perp)$ respectively, such that

$$\begin{aligned} f(A) &= f\left(J\begin{bmatrix} A_{\mathcal{S}} & 0 \\ 0 & 0\end{bmatrix}J^* + K\begin{bmatrix} 0 & 0 \\ 0 & A_{\mathcal{S}^\perp}\end{bmatrix}K^*\right) \\ &\le J_0 f\left(J\begin{bmatrix} A_{\mathcal{S}} & 0 \\ 0 & 0\end{bmatrix}J^*\right)J_0^* + K_0 f\left(K\begin{bmatrix} 0 & 0 \\ 0 & A_{\mathcal{S}^\perp}\end{bmatrix}K^*\right)K_0^* \\ &= J_0 J\begin{bmatrix} f(A_{\mathcal{S}}) & 0 \\ 0 & 0\end{bmatrix}J^* J_0^* + K_0 K\begin{bmatrix} 0 & 0 \\ 0 & f(A_{\mathcal{S}^\perp})\end{bmatrix}K^* K_0^* \end{aligned}$$

where we used the condition $f(0) = 0$ for the last equality. Regarding $J_0 J$ as an isometry on $\mathcal{S}$ and $K_0 K$ as an isometry on $\mathcal{S}^\perp$, this completes the proof. ■

Remark 2.5. If the condition $A \in \mathbb{K}_a^+$ in Corollary 2.3 is relaxed to $A \in \mathbb{K}^+$ and the strictly concave condition is relaxed to a merely concave, then Corollary 2.3 holds for some partial isometries $J\colon \mathcal{S} \to \mathcal{H}$ and $K\colon \mathcal{S}^\perp \to \mathcal{H}$. Indeed (2.3)–(2.4) still hold for $A \in \mathbb{K}^+$, with partial isometries J, K such that $\operatorname{supp} J \subset \mathcal{S}$ and $\operatorname{supp} K \subset \mathcal{S}^\perp$.

Of course, using a strictly concave assumption is not a too severe restriction, and this implies some strict Rotfel'd trace inequality. Here is an application to the determinant type functional on trace class positive operators,

$$A \mapsto \det(I + A) := \prod_{k=1}^{\infty} \lambda_k^{\downarrow}[I + A],$$

where $\lambda_k^{\downarrow}[\,\cdot\,]$, $k = 1, \dots,$ stand for the eigenvalues arranged in descending order.

Corollary 2.6 *Let $A, B \in \mathbb{B}$ be two absolutely positive trace class operators. Then,*

$$\det(I + A + B) < \det(I + A)\det(I + B).$$

Proof Note that $\det(I + X) = \exp\{\operatorname{Tr}\log(I + X)\}$ and apply (2.2) to the strictly concave function $\log(1 + t)$. ■

Obviously, Theorem 2.1 is equivalent to the next statement for convex functions.

Corollary 2.7 *Let $g(t)$ be a monotone, strictly convex function on $[0, \infty)$ with $g(0) \le 0$ and let $A, B \in \mathbb{K}_a^+$. Then, for some unitaries $U, V \in \mathbb{B}$,*

$$g(A + B) \ge_a Ug(A)U^* + Vg(B)V^*. \tag{2.5}$$

For $p \ge 1$, it is well-known that the functional $\mathbb{K}^+ \to [0, \infty]$, $A \mapsto \{\operatorname{Tr} A^{1/p}\}^p$ is concave/superadditive. Applying (2.5) we infer the following alternative.

Corollary 2.8 *Let $g(t)$ be a non-negative, strictly convex function on $[0, \infty)$ such that $g(0) = 0$. Let $A, B \in \mathbb{K}_a^+$ and $p \ge 1$. Then, either*

$$\{\operatorname{Tr} g^{1/p}(A + B)\}^p = \{\operatorname{Tr} g^{1/p}(A)\}^p + \{\operatorname{Tr} g^{1/p}(B)\}^p = \infty$$

or

$$\{\operatorname{Tr} g^{1/p}(A + B)\}^p > \{\operatorname{Tr} g^{1/p}(A)\}^p + \{\operatorname{Tr} g^{1/p}(B)\}^p.$$

Proof We start with (2.5): there exist two unitary operators U, V satisfying

$$g(A+B) \geq_a Ug(A)U^* + Vg(B)V^*.$$

Let $h(t)$ be a strictly increasing function on $[0, \infty)$, with $h(0) = 0$ and let X, Y be two positive compact operators such that $Y \geq_a X \geq_a 0$. Then, by the min-max principle, there exists a unitary W such that $h(X) \leq_a Wh(Y)W^*$. Therefore, from (2.5) and taking $h(t) = t^{1/p}$, we get

$$g^{1/p}(A+B) \geq_a \left(U_0 g(A) U_0^* + V_0 g(B) V_0^*\right)^{1/p}$$

for some unitary operators U_0, V_0. This implies that either

$$\{\operatorname{Tr} g^{1/p}(A+B)\}^p > \left\{\operatorname{Tr}\left(U_0 g(A) U_0^* + V_0 g(B) V_0^*\right)^{1/p}\right\}^p$$

in case of the right hand side is finite, or

$$\{\operatorname{Tr} g^{1/p}(A+B)\}^p = \left\{\operatorname{Tr}\left(U_0 g(A) U_0^* + V_0 g(B) V_0^*\right)^{1/p}\right\}^p = \infty$$

in case of the right hand side is infinite. Using the superadditivity of $A \mapsto \{\operatorname{Tr} A^{1/p}\}^p$ on $\mathbb{K}^+$ completes the proof. ∎

Remark 2.9. The functional $A \mapsto \{\operatorname{Tr} A^{1/p}\}^p$ on $\mathbb{M}_n^+$ is a special case of a family of concave superadditive functionals on positive definite matrices, called symmetric anti-norms and studied in [3], [4].

The most attractive part of Theorem 2.1 is (2.2) involving the absolute order $\leq_a$. We conjecture that (2.2) can be extended to positive operators in $\mathbb{B}$ with dense ranges.

Conjecture 2.10 Inequality (2.2) also holds for all $A, B \in B_a^+$.

This conjecture is proved for some special cases [2], for instance when A, B are in a type II-1 factor and $f(t)$ is concave on $[0, \infty)$, and further C^2 on $(0, \infty)$ with $f''(t) < 0$ for all $t > 0$. It seems also natural to propose the following conjecture.

Conjecture 2.11 Let $h(t)$ be a continuous, (strictly) increasing function on the real line and let $A, B \in \mathbb{B}$ be Hermitian and such that $A \leq_a B$. Then, for some unitary $U \in \mathbb{B}$, we have $h(A) \leq_a Uh(B)U^*$.

3 Proof of Theorem 2.1

We first prove (2.2), in fact its equivalent convex version (2.5). We need two lemmas. The first one is adapted from [7].

Lemma 3.1 *Let $g(t)$ be a strictly convex function on $[0, \infty)$ with $g(0) = 0$, let $\{a_k\}_{k=1}^\infty$ be a bounded sequence in $(0, \infty)$, and let $\{w_k\}_{k=1}^\infty$ be a sequence in $[0, \infty)$ such that $0 < \sum_{k=1}^\infty w_k < 1$. Then,*

$$g\left(\sum_{k=1}^\infty w_k a_k\right) < \sum_{k=1}^\infty w_k g(a_k).$$

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Proof Let $w_0 := 1 - \sum_{k=1}^{\infty} w_k$ and $a_0 := 0$. We then have to show

$$g\Big(\sum_{k=0}^{\infty} w_k a_k\Big) < \sum_{k=0}^{\infty} w_k g(a_k).$$

This holds for the $\leq$ sign. We claim that equality cannot hold. Indeed, the equality would imply $w_{k_0} = 1$ for some index k_0, hence a contradiction. This follows from the following general fact for a probability measure w on a compact interval $[\alpha, \beta]$:

$$g\Big(\int_{[\alpha,\beta]} t\, dw(t)\Big) = \int_{[\alpha,\beta]} g(t)\, dw(t) \Longrightarrow \text{ the support of } w \text{ is singleton.}$$

To see this, let us recall the standard proof for the Jensen inequality based on supporting lines. Set $t_0 = \int_{[\alpha,\beta]} t\, dw(t)$ and choose real numbers a, b satisfying

$$g(t) \geq at + b, \quad g(t_0) = at_0 + b.$$

Integration of both sides of this inequality gives rise to

$$\int_{[\alpha,\beta]} g(t)\, dw(t) \geq at_0 + b \int_{[\alpha,\beta]} dw(t) = at_0 + b = g(t_0).$$

When $g(t)$ is strictly convex, we have $g(t) > at + b$ for $t \neq t_0$. Thus, if the equality occurs in the above, then the support of w must be $\{t_0\}$. ■

Lemma 3.2 *Let $g(t)$ be a monotone, strictly convex function on $[0, \infty)$ with $g(0) \leq 0$, let $A \in \mathbb{K}_a^+$, and let $Z \in \mathbb{B}$ such that $0 \leq_a Z^*Z \leq_a I$ and $0 \leq_a ZZ^* \leq_a I$. Then, for some unitary $U \in \mathbb{B}$,*

$$g(Z^*AZ) \leq_a UZ^*g(A)ZU^*.$$

Proof Note that if the inequality holds for $g(t)$ then it also holds for $g(t) - c$ for any $c > 0$ since Z is a contraction. Thus we may and do assume $g(0) = 0$. The following fact follows from Lemma 3.1: for $w \in \mathcal{H}$ with $0 < \|w\| < 1$,

$$g(\langle w, Aw\rangle) < \langle w, g(A)w\rangle. \tag{3.1}$$

We then consider two cases.

1. $g(t)$ is (strictly) decreasing. Since the assumptions ensure that Z is a one-to-one contraction with a dense range it follows that $Z^*AZ \in \mathbb{K}_a^+$ as a compact positive operator with a zero null space. Thus $-g(Z^*AZ) \in \mathbb{K}_a^+$ too, hence there exists an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and a non-decreasing sequence of eigenvalues, $\{\lambda_k^{\uparrow}[g(Z^*AZ)]\}_{k=1}^{\infty}$ with $\lim\limits_{k\to\infty} \lambda_k^{\uparrow}[g(Z^*AZ)] = 0$, such that

$$g(Z^*AZ) = \sum_{k=1}^{\infty} \lambda_k^{\uparrow}[g(Z^*AZ)] e_k \otimes e_k.$$

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Similarly $-g(A) \in \mathbb{K}_a^+$, thus $-Z^*g(A)Z \in \mathbb{K}^+$ too. Hence there is an orthonormal basis $\{e'_k\}_{k=1}^{\infty}$ and a non-decreasing sequence $\{\lambda_k^{\uparrow}[Z^*g(A)Z]\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \lambda_k^{\uparrow}[Z^*g(A)Z] = 0$ such that

$$Z^*g(A)Z = \sum_{k=1}^{\infty} \lambda_k^{\uparrow}[Z^*g(A)Z] e'_k \otimes e'_k.$$

Fix k and let $\mathcal{S}$ be the k-dimensional subspace $\mathcal{S} = \operatorname{span}\{e_1, \dots, e_k\}$. Then

$$\lambda_k^{\uparrow}[g(Z^*AZ)] = \max_{h\in\mathcal{S};\|h\|=1} \langle h, g(Z^*AZ)h\rangle.$$

Similarly,

$$\lambda_k^{\uparrow}[Z^*g(A)Z] = \max_{h\in\mathcal{T};\|h\|=1} \langle h, Z^*g(A)Zh\rangle$$

where $\mathcal{T} = \operatorname{span}\{e'_1, \dots, e'_k\}$. We have, using (3.1),

$$\begin{aligned} \max_{h\in\mathcal{T};\|h\|=1} \langle h, Z^*g(A)Zh\rangle &= \max_{h\in\mathcal{T};\|h\|=1} \langle Zh, g(A)Zh\rangle \\ &> \max_{h\in\mathcal{T};\|h\|=1} g(\langle Zh, AZh\rangle) \\ &= \max_{h\in\mathcal{T};\|h\|=1} g(\langle h, Z^*AZh\rangle) \\ &= g\big(\min_{h\in\mathcal{T};\|h\|=1} \langle h, Z^*AZh\rangle\big) \end{aligned} \tag{3.2}$$

where we use that $g(t)$ is decreasing for the last equality. Next, from the min-max principle and since $\mathcal{S}$ is a spectral subspace of Z^*AZ corresponding to the k largest eingenvalues, we have

$$\min_{h\in\mathcal{T};\|h\|=1} \langle h, Z^*AZh\rangle \le \min_{h\in\mathcal{S};\|h\|=1} \langle h, Z^*AZh\rangle.$$

Thus, since $g(t)$ is decreasing,

$$\begin{aligned} g\big(\min_{h\in\mathcal{T};\|h\|=1} \langle h, Z^*AZh\rangle\big) &\ge g\big(\min_{h\in\mathcal{S};\ \|h\|=1} \langle h, Z^*AZh\rangle\big) \\ &= \max_{h\in\mathcal{S};\|h\|=1} \langle h, g(Z^*AZ)h\rangle \\ &= \lambda_k^{\uparrow}[g(Z^*AZ)]. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) we have strict inequalities between the list of eigenvalues of $Z^*g(A)Z$ and those of $g(Z^*AZ)$. The fact that the two lists of eigenvectors are bases then ensure the existence of a unitary U, defined by $Ue'_k = e_k$, such that

$$g(Z^*AZ) \le_a UZ^*g(A)ZU^*.$$

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2. $g(t)$ is strictly increasing. The proof is similar to, but simpler than, that of the previous case; we deal with operators in $\mathbb{K}_a^+$ and the usual list of eigenvalues arranged in decreasing order $\lambda_k^{\downarrow}[\,\cdot\,]$. We have, for each integer $k \geq 1$, a subspace $\mathcal{F} \subset \mathcal{H}$ of dimension k (a spectral subspace of Z^*AZ corresponding to its k-largest eigenvalues) such that

$$\begin{aligned}\lambda_k^{\downarrow}[g(Z^*AZ)] &= \min_{h\in\mathcal{F};\|h\|=1} \langle h, g(Z^*AZ)h\rangle \\ &= \min_{h\in\mathcal{F};\|h\|=1} g(\langle h, Z^*AZh\rangle) \\ &= \min_{h\in\mathcal{F};\|h\|=1} g(\langle Zh, AZh\rangle),\end{aligned}$$

where we have used the monotonicity of $g(t)$. Then, using (3.1) with $w = Zh$ and the min-max principle,

$$\begin{aligned}\lambda_k^{\downarrow}[g(Z^*AZ)] &< \min_{h\in\mathcal{F};\|h\|=1} \langle Zh, g(A)Zh\rangle \\ &\leq \lambda_k^{\downarrow}[Z^*g(A)Z].\end{aligned}$$

The existence of U follows as in the decreasing case. ∎

We turn to the proof (2.5), the convex version of Theorem 2.1 (2.2).

Proof of (2.5) We may confine the proof to the case $g(0) = 0$ as if (2.5) holds for a function $g(t)$ then it also holds for $g(t)-\alpha$ for any $\alpha > 0$. This assumption combined with the monotonicity of $g(t)$ entails that $g(t)$ has a constant sign $\varepsilon \in \{-1, 1\}$, hence $g(t) = \varepsilon|g|(t)$.

By assumption, $(A + B)$ has a dense range and its unbounded inverse has a dense domain and is such that the operators $X := A^{1/2}(A + B)^{-1/2}$ and $Y := B^{1/2}(A + B)^{-1/2}$, at first defined on the domain of $(A + B)^{-1/2}$, may be extended as one-to-one contractions with dense ranges, more precisely,

$$0 \leq_a X^*X \leq_a I, \quad 0 \leq_a XX^* \leq_a I \quad \text{and} \quad 0 \leq_a Y^*Y \leq_a I, \quad 0 \leq_a YY^* \leq_a I.$$

Note also that

$$A = X(A+B)X^* \quad \text{and} \quad B = Y(A+B)Y^*.$$

For any one to one operator T with a dense range, the polar decomposition shows that T^*T and TT^* are unitarily equivalent. Hence, using Lemma 3.2 and the previous observation with $T = \big(|g|(A + B)\big)^{1/2}X^*$ we have two unitary operators U_0 and U such that

$$\begin{aligned}g(A) &= g\big(X(A+B)X^*\big) \\ &\leq_a U_0Xg(A+B)X^*U_0^* \\ &= \varepsilon U^*\big(|g|(A+B)\big)^{1/2}X^*X\big(|g|(A+B)\big)^{1/2}U,\end{aligned}$$

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so,

$$Ug(A)U^* \le_a \varepsilon\big(|g|(A+B)\big)^{1/2}X^*X\big(|g|(A+B)\big)^{1/2}. \tag{3.4}$$

Similarly there exists a unitary operator V such that

$$Vg(B)V^* \le_a \varepsilon\big(|g|(A+B)\big)^{1/2}Y^*Y\big(|g|(A+B)\big)^{1/2}. \tag{3.5}$$

Adding (3.4) and (3.5) we get

$$Ug(A)U^* + Vg(B)V^* \le_a g(A+B)$$

since $X^*X + Y^*Y = I$. ■

Inspecting the above proof shows that Conjecture 2.10 would follow from the next one, stating that Lemma 3.2 might still hold in $\mathbb{B}_a^+$.

Conjecture 3.3 Lemma 3.2 still holds for all $A \in \mathbb{B}_a^+$.

Now, we turn to the proof of the remaining parts of Theorem 2.1. As before we consider the convex versions and thus have to show the following:

Let $g(t)$ be a monotone convex function on $[0,\infty)$ such that $g(0) \le 0$.

(i) *If $A, B \in \mathbb{K}^+$, then for some partial isometries $J, K \in \mathbb{B}$,*

$$g(A+B) \ge Jg(A)J^* + Kg(B)K^*. \tag{3.6}$$

Moreover, if $g(t)$ is strictly convex and $g(0) = 0$, we can take supp $J =$ supp A *and* supp $K =$ supp B.

(ii) *If $A, B \in \mathbb{K}_a^+$, then for some unitary $U, V \in \mathbb{B}$,*

$$g(A+B) \le Ug(A)U^* + Vg(B)V^*. \tag{3.7}$$

We indicate below how to adapt the proof of (2.5) in order to get (3.6) and (3.7).

The proof of (3.7) is quite similar to the previous proof of (2.5), though simpler; we do not need Lemma 3.1 and we replace the strict inequality (3.1) by the standard Jensen inequality,

$$g(\langle w, Aw\rangle) \le \langle w, g(A)w\rangle$$

for all vectors $w \in \mathcal{H}$ with $\|w\| \le 1$. This yields a version of Lemma 3.2 for convex functions where the $\le_a$ sign is replaced by the usual order $\le$, and we may then repeat the proof of (2.5), but with the $\le$ sign, and thus obtain (3.7).

The weaker statement (3.6) is obtained by first establishing the following variation of Lemma 3.2.

Let $g(t)$ be a monotone, convex function on $[0,\infty)$ with $g(0) \le 0$, let $A \in \mathbb{K}^+$, and let $Z \in \mathbb{B}$ be a contraction. Then

$$g(Z^*AZ) \le JZ^*g(A)ZJ^* \tag{3.9}$$

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for some partial isometry J with

$$\operatorname{supp} J^* = \operatorname{supp} g(Z^*AZ) \quad \text{and} \quad \operatorname{supp} J = \operatorname{supp} Z^*g(A)Z.$$

To prove (3.9), we proceed as in the proof of Lemma 3.2, except that we use $(*)$ in place of (3.1). We may assume that $g(0) = 0$ and we obtain the eigenvalue inequalities

$$\lambda_k^{\uparrow}[g(Z^*AZ)] \leq \lambda_k^{\uparrow}[Z^*g(A)Z], \quad (k \geq 1)$$

in case of $g(t)$ is decreasing, and

$$\lambda_k^{\downarrow}[g(Z^*AZ)] \leq \lambda_k^{\downarrow}[Z^*g(A)Z], \quad (k \geq 1)$$

in case of $g(t)$ is increasing. This ensures that (3.9) holds for some partial isometry J with $\operatorname{supp} J^* = \operatorname{supp} g(Z^*AZ)$ and $\operatorname{supp} J = \operatorname{supp} Z^*g(A)Z$.

Having (3.9) at our disposal, we may infer (3.6) by repeating the proof of (2.5), but for the $\leq$ sign and with partial isometries in place of unitaries: We can suppose that the space $\mathcal{H}$ is the closure of $\operatorname{supp} A + \operatorname{supp} B$. Hence we may then define $(A+B)^{-1/2}$. Next, recall that in the proof of (2.5) we have used the relation

$$g(A) = g\big(X(A+B)X^*\big),$$

where $X := A^{1/2}(A+B)^{-1/2}$ is a contraction on $\mathcal{H}$ and further $\operatorname{supp} X^* = \operatorname{supp} A$. By using (3.9) we obtain a partial isometry U_0 such that

$$g(A) = g\big(X(A+B)X^*\big) \leq U_0 X g(A+B) X^* U_0^* \tag{3.10}$$

and

$$\operatorname{supp} U_0^* = \operatorname{supp} g(A), \quad \operatorname{supp} U_0 = \operatorname{supp} X g(A+B) X^*. \tag{3.11}$$

From (3.10) and (3.11), we also have

$$U_0^* g(A) U_0 \leq X g(A+B) X^*. \tag{3.12}$$

On the other hand, if ε denotes the constant sign of $g(t)$,

$$Xg(A+B)X^* = J_0 \varepsilon |g|^{1/2}(A+B) X^* X |g|^{1/2}(A+B) J_0^*$$

for some partial isometry J_0 with

$$\operatorname{supp} J_0^* = \operatorname{supp} X g(A+B) X^* = \operatorname{supp} U_0 \tag{3.13}$$

so that (3.12) yields

$$J_0^* U_0^* g(A) U_0 J_0 \leq \varepsilon |g|^{1/2}(A+B) X^* X |g|^{1/2}(A+B). \tag{3.14}$$

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Now, observe that the condition (3.13) ensures that $U_0 J_0$ is a partial isometry, thus (3.14) can be written as

$$Jg(A)J^* \le \varepsilon |g|^{1/2}(A+B)X^*X|g|^{1/2}(A+B). \tag{3.15}$$

for some partial isometry J. We have a similar expression

$$Kg(B)K^* \le \varepsilon |g|^{1/2}(A+B)Y^*Y|g|^{1/2}(A+B) \tag{3.16}$$

for some partial isometry K. As in the proof of (2.5), summing (3.15) and (3.16) yields (3.6).

It remains to establish (3.6) with $\operatorname{supp} J = \operatorname{supp} A$ and $\operatorname{supp} K = \operatorname{supp} B$ whenever we assume $g(0) = 0$ and $g(t)$ is monotone, strictly convex. (Such a refinement is necessary to have isometries in Corollary 2.3.) The assumptions entail $g(0) = 0$ and $g(t) \neq 0$ for all $t > 0$ so that $\operatorname{supp} g(A) = \operatorname{supp} A$ and, by still supposing that $\mathcal{H}$ is the closure of $\operatorname{supp} A + \operatorname{supp} B$, we have $g(A+B) \ge_a 0$. Thus, the partial isometry U_0 defined in (3.10) can be chosen such that

$$\operatorname{supp} U_0^* = \operatorname{supp} U_0 = \operatorname{supp} A.$$

It then follows from (3.13) that the partial isometry J in (3.15) satisfies to $\operatorname{supp} J = \operatorname{supp} A$. Similarly, the partial isometry K in (3.16) satisfies to $\operatorname{supp} K = \operatorname{supp} B$ as desired.

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Laboratoire de mathématiques, Université de Franche-Comté, 25000 Besaçon, France
e-mail: jcbourin@univ-fcomte.fr

6-12-28-102 Tamura, Sawaraku, Fukuoka 814-0175, Japan
e-mail: tharada1005@gmail.com

Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
e-mail: eylee89@knu.ac.kr