TULLIO LEVI-CIVITA.

By H. S. Ruse.

Tullio Levi-Civita, Honorary Member of the Edinburgh Mathematical Society, was born at Padua on March 29th, 1873, and died in Rome on December 29th, 1941.

He gained his doctorate at the University of Padua in 1892, and six years later became Extra Professor of Mechanics at the same University. Appointed full Professor in 1902, he remained at Padua until his election in 1918 to the Chair of Higher Analysis in the University of Rome, which he exchanged in 1922 for the Chair of Theoretical Mechanics. This he occupied until his retirement in 1938. His wife, whom he married in 1914, and who survives him, was a former pupil, Libera Trevisani.[1]

Levi-Civita's work as a mathematician is notable for its quality, quantity and range. He wrote some hundred and fifty to two hundred papers and memoirs, and was the author of several treatises.[2] There are few modern branches of mathematical physics to which he did not at one time or another contribute—classical mechanics, hydromechanics, thermodynamics, elasticity, the strength of materials, astronomy, electromagnetism, optics, relativity and quantum mechanics—yet some of his greatest work was in pure mathematics. One thinks of him, in fact, primarily as a mathematician rather than as a theoretical physicist, for most of his work in the applications of mathematics to physical problems was written more from the mathematical than from the physical angle. His writings are characterised by lucidity, simplicity of style and a pleasing economy of symbolism.

His early work included contributions to the theory of numbers, to the theory of functions of two complex variables, to partial differential equations, and, most notably of all, to the absolute differential calculus. At Padua, one of his Professors had been Gregorio Ricci, who in 1892 had published the first work on the

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[1] I am indebted to Mr L. Roth for a few of the personal items appearing in this notice.

[2] A complete list of his works to 1937 is given in *Annuario della Pontificia Accademia delle Scienze*, I (1937), pp. 498-511.

subject now often known as tensor analysis. Ricci and Levi-Civita together developed this theory, and in 1901 published a joint memoir, " Méthodes du calcul différentiel absolu et leurs applications " in the *Mathematische Annalen* (vol. 54). Their work gained relatively little attention until, fifteen years later, it was used by Einstein in the formulation of the general theory of relativity. Originally it was a technique rather than a separate branch of mathematics, providing as it did a way of writing theorems of differential geometry and the calculus in a form at once concise and general, and it was not until after the development of relativity, followed shortly afterwards by Levi-Civita's definition of parallelism in Riemannian geometry, that it assumed the full place it now holds as one of the main branches of modern mathematics.

It is impossible here to make more than brief mention of a few of his manifold contributions to mechanics and mathematical physics. Among the best known of these is the theorem of analytical dynamics called after him,[1] namely that, to any set of m invariant relations of a Hamiltonian system, which are in involution, there corresponds a family of ∞^m particular solutions of the Hamiltonian system, whose determination depends upon the integration of a system of order $(m-1)$. In celestial mechanics he obtained[2] a reduction of the famous problem of three bodies. More recently, in lectures[3] delivered in 1937 at the Harvard Tercentenary Conference, he tackled the relativistic problem of n bodies, but confined himself to seeking first-order relativistic corrections to the differential equations of motion on the basis of certain approximation hypotheses. He considered[4] the case $n=2$ in more detail, treating the relativistic corrections as first-order perturbations in the Newtonian motion, and was able to relate his results to actual astronomical observations.

Among other contributions made by him to mathematical

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[1] *Rendiconti del R. Acc. dei Lincei*, 10 (1901), 3. The enunciation of the theorem here given is taken from Whittaker, *Analytical Dynamics* (3rd ed., 1927), p. 328.

[2] *Atti del R. Ist. Veneto*, 74 (1915), 907.

[3] *American Journal of Math.*, 59 (1937), 9.

[4] *Ibid.*, p. 225. An earlier discussion of the relativistic 2-body problem appears in *L'enseignement mathématique*, 1935, p. 149.

physics may be mentioned his investigation[1] of the Newtonian attraction of thin tubes with applications to the theory of Saturn's rings; his general method[2] in hydrodynamics of determining the flow past an obstacle, and also his rigorous determination of waves in a canal; in geometrical optics his reciprocal of the Malus-Dupin theorem, and in relativistic optics his determination[3] of the laws of refraction and reflection in a general space-time. He made numerous other contributions to relativity, and in 1933 interested himself in the problem of reconciling Dirac's equations in quantum mechanics with the relativistic principle of covariance. Regarding the last, it is interesting to note that, in his own words,[4] " Dirac's equations must, in my opinion, be abandoned, without, it is understood, giving up the main progress realised by them in the domain of restricted relativity "—this because their transformation-laws seemed to him to require the introduction into space-time of a lattice (a system of four mutually perpendicular congruences of lines) which, having no intrinsic physical significance, should, but in fact did not, disappear from the equations in their final relativistic form.

Reference has already been made to his definition of parallelism in Riemannian geometry. This was published in 1917 in a paper[5] " Nozione di parallelismo in una varietà qualunque e conseguente specificazione della curvatura Riemanniana." The basic idea was simple, but its influence on the development of differential geometry was profound. Consider an ordinary surface S in three-dimensional Euclidean space. Let P, P' be two points upon it and let $\mathbf{u}, \mathbf{u}'$ be two unit vectors, localised at P, P' respectively, tangential to S. If S is developable, then $\mathbf{u}, \mathbf{u}'$ are defined to be parallel with respect to S if they are parallel in the ordinary sense when the surface is developed into a plane. If S is not developable, suppose P, P' to be joined by a curve C lying in S; then if $\mathbf{u}, \mathbf{u}'$ are parallel with respect to the developable

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[1] *Atti del R. Acc. dei Lincei*, 18 (2) (1908), pp. 1, 413, 553 ; *Atti del R. Ist. Veneto*, 68 (1909), 557 ; *Rendiconti del Circ. Math. di Palermo*, 33 (1912), 354.

[2] An account of this is given by L. M. Milne-Thomson, *Theoretical Hydrodynamics* (1938), p. 305.

[3] *Atti della Pont. Acc. Nuovi Lincei*, LXXXIV. (1931), 332.

[4] *Bull. American Math. Soc.* (1933), p. 535, in particular p. 546 ; see also *Berliner Sitzb.* (Phys.-Math.), 1933, p. 3.

[5] *Rendiconti del Circ. Math. di Palermo*, 42 (1917), 173.

circumscribed to S along C, they are said to be parallel with respect to S along the curve C, or, alternatively, $\mathbf{u}$ is said to be obtained from $\mathbf{u}$ by the parallel displacement of the latter vector along C. It follows at once that, for a general surface, there is no unique direction at P' parallel to that of the vector $\mathbf{u}$ at P, $\mathbf{u}'$ being dependent upon the choice of the curve C. But if P, P' are neighbouring points, and if infinitesimals of order higher than the first are neglected, then there *is* at P' a unique vector parallel to $\mathbf{u}$ at P. From this arises the concept of infinitesimal parallelism, readily generalised to n-dimensional Riemannian space, and a geometrical interpretation of the vanishing of the covariant differential of a vector. This discovery by Levi-Civita, together with the contemporary development of general relativity and the search for a unified theory of gravitation and electromagnetism by Weyl, Eddington, Einstein and others, quickly led to generalisations of Riemannian geometry. On the purely mathematical side perhaps the most interesting consequence of these generalisations was the rapprochement, if not the complete reconciliation, which they brought about between differential geometry and the geometries of the Erlanger Programm. In 1870 Felix Klein had defined a geometry to be the invariant theory of a transformation group, a definition which included such geometries as Euclidean, affine and projective, but did not include Riemannian. In the light of Levi-Civita's definition of parallelism it was seen that the spaces of differential geometry could, so to speak, be regarded as an assemblage of isomorphic Klein spaces, each associated with a point of an "underlying space," and in this way there grew an extensive literature directly inspired by the work of Levi-Civita. So extensive, indeed, did the literature become, that differential geometry seemed in the late 1920's and early 1930's to be almost in danger of self-suffocation: a Pelion of detail, painstakingly worked out by research students the world over, was piled upon an Ossa of greater and greater generalisations. Yet among the mass of notes, papers, memoirs and treatises there was a large kernel of genuine mathematical importance. Levi-Civita himself, it is true, made no special contribution to these developments, and it seems not unlikely that many of them could have held little interest for him, being too far removed from the simple directness of his own work. Certainly he himself never fell a victim to the fascinations of notation, which, among some tensorists, led to a tangled undergrowth of symbolism which rendered their work all but unreadable. Nevertheless, his interest in geometry as such remained with him until the end, some of his last papers, published between 1934 and 1938, being concerned with such topics as the trigonometry of curvilinear triangles on surfaces, and families of isoparametric surfaces in ordinary Euclidean space.

By his early work with Ricci on tensor analysis and by his later discovery of infinitesimal parallelism, Levi-Civita laid the foundations both for relativity and for the establishment of differential geometry as one of the great branches of modern mathematics. These two achievements alone, without his wide contributions to mathematical physics, might have earned him a lasting place among the great men of mathematics.

In person he was unassuming and unfailingly kind, qualities which inspired admiration and affection in his pupils and colleagues. Small of stature and given to vigorous gesticulation when speaking, one thinks of him as intensely Italian, and as possessing all the graces of that once happy people. All the more shabby, therefore, was the treatment he received in 1938, when the introduction of racial legislation by the Fascist Government led to his expulsion from the Royal Academy of the Lincei, and the consequent clouding of his few years of retirement. That his death was ignored by official Italy was not his dishonour, and that he should be commemorated by learned societies in countries at war with Italy is symbolic. Happily, however, the Vatican City formed a cultural oasis where, as a member of the Papal Academy, he was held in full honour until his death. It is of historical interest to note that, when in 1938 an anti-Semitic policy was forced upon the owners of the *Zentralblatt für Mathematik* in Germany, he was one of those whose names were removed from the editorial board. It was as a protest against this action that the rival *Mathematical Reviews* was founded in the United States of America.

He lectured extensively, not only in Italy, but also in Spain, Austria, Germany, France and America. He was a Foreign Member of the Royal Society, an honorary doctor of several European Universities, and an honorary member of learned societies in various countries. He paid a visit to the St Andrews

Colloquium in 1930, and was shortly afterwards elected to his Honorary Membership of the Edinburgh Mathematical Society. Members of the Society who knew him will mourn him as a friend: they and others will honour his memory as a great man and a great mathematician.

Publications (treatises only):

Lezioni di meccanica razionale (with U. Amaldi), 1923-28.

Questioni di meccanica classica e relativistica, 1924.

Lezioni di calcolo differenziale assoluto, 1925 (English translation with additional chapters, *The absolute differential calculus*, 1927).

Caratteristiche dei sistemi differenziali e propogazione ondosa, 1931.

Nozioni di balistica esterna (with U. Amaldi), 1935.