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ON UNIQUENESS AND CONTINUABILITY OF THE EMDEN–FOWLER EQUATION

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Abstract

The Emden–Fowler equation $x''(t)+a(t)|x|^{\gamma}\operatorname{sgn} x=0$, $t \geqq 0$, is said to be in the sublinear or superlinear case according to whether $\gamma<1$ or $\gamma>1$. Conditions on $a(t)$ are given to ensure local uniqueness of solutions in the sublinear case and continuability of solutions in the superlinear case. Boundedness of solutions is also studied.

1. Introduction

We are interested in the generalized Emden–Fowler equation

$$x''(t)+a(t)|x|^{\gamma}\operatorname{sgn} x=0 \qquad t \geqq 0 \tag{1}$$

where $a(t)$ is a non-negative continuous function on $[0,\infty)$. The more general second order non-linear differential equation

$$(p(t)u'(t))'+q(t)u^{\gamma}=0 \qquad t \geqq 0$$

can be transformed into (1) under suitable change of variables. For a thorough survey on results concerning solutions of (1), please consult Wong (1973).

We classify equation (1) into sublinear and superlinear cases according to $\gamma>1$ or $\gamma<1$ respectively. Problems arising from these cases are different, though in a sense dually related.

Let us first consider the sublinear case. The function $a(t)|x|^{\gamma}$ is not Lipschitz continuous in a neighbourhood of $x=0$. Hence local uniqueness of solutions of initial value problems of (1) is not guaranteed by the classical theorem. In other words, there might exist more than one solution of (1) in an interval $[0,\varepsilon]$ such that

$$x(0)=0 \qquad x'(0)=x_1.$$

Coffman and Wong (1972) have shown that if $x_1 \neq 0$, the solution is necessarily unique. So the whole problem reduces to that of uniqueness of the initially stationary solution, namely

$$x(0) = x'(0) = 0. \tag{2}$$

In other words, one tries to determine whether the trivial solution $x(t) \equiv 0$ is the only initially stationary solution of (1) in a neighbourhood of $t = 0$.

No conclusion can be drawn unless some conditions are imposed on the function $a(t)$. Heidel (1970) obtains the first result in this direction. He proves that if $a(t) > 0$ and is of bounded variation in a neighbourhood of 0, then the local solution of (1) satisfying (2) is unique.

Coffman and Wong (1972) obtain the weaker condition that $\log t^{\gamma+3} a(t)$ be of finite lower variation in a neighbourhood of $t = 0$. They obtain another related condition that $\log a(t)$ be of finite upper variation in a neighbourhood of $t = 0$. This latter condition automatically implies that $a(0) > 0$, and hence cannot be applied when $a(t)$ has an isolated zero at 0.

In the superlinear case, uniqueness is no more a problem. The difficulty is now continuability. We say that "all solutions of (1) are continuable through a point $b < \infty$" if for some $a < b$ every local solution of (1) in (a, b) has a C^2 extension on an open interval containing b. The crucial criterion is whether all solutions of (1) in (a, b) are bounded as $t \to b$. With a shift of the t-axis followed by a reflection, the problem of continuability through b is equivalent to the following one:

What conditions should be imposed on $a(t)$ so that in a neighbourhood $(0, \varepsilon]$ of $t = 0$, there exists no solution of (1) such that $\limsup_{t\to 0} x(t) = \infty$? (We shall call such solutions initially unbounded.)

Interesting enough, the two conditions discovered by Coffman and Wong are exactly those they found for the uniqueness problem in the sublinear case.

In both cases they arrive at their results by studying some Liapunov-like functions associated with (1) and differential inequalities derived from them.

In this paper we employ a more elementary approach which furnishes us with a condition similar to Coffman and Wong's first condition. When $\sqrt{5} > \gamma > \sqrt{5} - 2$ our result improves theirs. But unfortunately for other values of γ, it is the other way round. The same method also enables us to improve their second condition.

Our method is also applicable to the study of boundedness of solutions of (1).

2. Some inequalities

Results in this section hold for both $\gamma > 1$ and $\gamma < 1$.

Let τ be a zero of $x(t)$, and T an adjacent (either to the left or to the right of τ) extreme point at which $x(t)$ attains its maximum or minimum. $\tau, T \neq 0$. Suppose that $a(t) > 0$, for $t > 0$. Without loss of generality, we may assume that $\tau < T$. We write $X = x(T)$, $V = x'(\tau)$.

$$M = \max\{a(t) : t \in [\tau, T]\}$$

$$m = \min\{a(t) : t \in [\tau, T]\}$$

LEMMA 1.

(i) $$\frac{2m}{1+\gamma}|X|^{1+\gamma} \leqq V^2 \leqq \frac{2M}{1+\gamma}|X|^{1+\gamma}.$$

(ii) $$\frac{C_1}{\sqrt{M}}|X|^{(1-\gamma)/2} \leqq |T-\tau| \leqq \frac{C_1}{\sqrt{m}}|X|^{(1-\gamma)/2},$$

where

$$C_1 = \sqrt{(1+\gamma)/2}\int_0^1 \frac{dy}{(1-y^{1+\gamma})^{1/2}}.$$

$$\frac{C_2}{M^{1/(1+\gamma)}}|V|^{(1-\gamma)/(1+\gamma)} \leqq |T-\tau| \leqq \frac{C_2}{m^{1/(1+\gamma)}}|V|^{(1-\gamma)/(1+\gamma)}$$

where

$$C_2 = \left(\frac{1+\gamma}{2}\right)^{1/(1+\gamma)}\int_0^1 \frac{dy}{(1-y^{1+\gamma})^{1/2}}.$$

PROOF. We prove only the inequalities relating to m, the proofs of those relating to M being similar. Without loss of generality we assume that $x(t) \geqq 0$ in $[\tau, T]$.

Equation (1) can be written as

$$v\frac{dv}{dx} = -a(t)x^{\gamma}$$

where $v = dx/dt$. Integrating both sides with respect to x over $[t, T]$, we obtain

$$\frac{v^2(t)}{2} = \int_{x(t)}^{X} a(t)x^{\gamma}\,dx$$

(3) $$\geqq \frac{m}{1+\gamma}(X^{1+\gamma} - x^{1+\gamma}(t)).$$

Taking $t = \tau$, we obtain the required inequality in (i).

(3) gives

$$\frac{dx}{dt} \geqq \left[\frac{2m}{1+\gamma} (X^{1+\gamma} - x^{1+\gamma}) \right]^{1/2}.$$

Integrating this inequality, we obtain

$$T - \tau \leqq \left(\frac{1+\gamma}{2m} \right)^{1/2} \int_0^X \frac{dx}{(X^{1+\gamma} - x^{1+\gamma})^{1/2}}.$$

The substitution $x(t) = Xy(t)$ now gives the required inequality in (ii).

COROLLARY 2. *There exists* $s \in [\tau, T]$ (*or* $[T, \tau]$) *such that*

$$\frac{2a(s)}{1+\gamma} |X|^{1+\gamma} = V^2.$$

COROLLARY 3. *Let* $\tau_{n+1} < T_{n+1} < \tau_n < T_n$ *be such that* τ_{n+1}, τ_n *are two successive zeros of* $x(t)$ *and* T_{n+1}, T_n *are two successive extreme points of* $x(t)$. *Then there exist*

$$s_n \in [\tau_n, T_n], \quad s_{n'} \in [T_{n+1}, \tau_n]$$

and

$$s_{n+1} \in [\tau_{n+1}, T_{n+1}]$$

such that

$$|V_n / V_{n+1}| = |x'(\tau_n)/x'(\tau_{n+1})| = (a(s_{n'})/a(s_{n+1}))^{1/2}$$

$$|X_n / X_{n+1}| = |x(T_n)/x(T_{n+1})| = (a(s_{n'})/a(s_n))^{1/(1+\gamma)}.$$

3. Rate of decay (growth) of initially stationary (unbounded) solutions as $t \to 0$

Coffman and Wong (1972) have proved the necessity part of the following theorem. The sufficiency part is obvious.

THEOREM 4. $x(t)$ *is an initially stationary* (*when* $\gamma < 1$) *or an initially unbounded* (*when* $\gamma > 1$) *solution if and only if* $t = 0$ *is an accumulation point of the zeros of* $x(t)$.

THEOREM 5. *Suppose* $\gamma < 1$. *Let* $x(t)$ *be an initially stationary solution of* (1), *then*

$$|X| = o(T^{2/(1-\gamma)})$$

$$|V| = o(\tau^{(1+\gamma)/(1-\gamma)})$$

as $T, \tau \to 0$.

PROOF. We only give the proof for the rate of decay of X, that for V being similar.

Case 1. $a(0) = 0$.

Suppose the contrary. Then there exists a constant α such that no matter how small $\varepsilon > 0$ is, we can find a $T < \varepsilon$ such that

$$|X| > \alpha T^{2/(1-\gamma)}.$$

Since $a(t)$ is continuous, we can choose ε so near 0 that

$$N = \max\{a(t) : t \in [0, \varepsilon]\}$$

is small enough to satisfy

$$\alpha^{(1-\gamma)/2} C_1 > \sqrt{N}.$$

By Lemma 1 (ii),

$$T - \tau \geqq \frac{C_1}{\sqrt{M}} |X|^{(1-\gamma)/2} \geqq \frac{C_1}{\sqrt{N}} \alpha^{(1-\gamma)/2} T > T.$$

Hence $\tau < 0$. This means that $x(t)$ has no other zeros in $[0, T]$, contradicting Theorem 4.

Case 2. $a(0) > 0$.

Similar method as in case 1 shows that $X = O(T^{2/(1-\gamma)})$. Suppose that $\limsup_{T \to 0} |X|/T^{2(1-\gamma)} = \alpha \neq 0$. Since $a(t)$ is continuous and $a(0) \neq 0$,

$$\lim_{t \to 0} k(t) = \lim_{t \to 0} \frac{\min\{a(s) : 0 \leqq s \leqq t\}}{\max\{a(s) : 0 \leqq s \leqq t\}} = 1.$$

Let $N = \max\{a(t) : t \in [0, 1]\}$. From the definition of α, for any $\delta > 0$, there exist two successive extreme values of $x(t)$ at, say, T_n and $T_{n+1} (< T_n < 1)$, as near 0 as we wish such that

$$\alpha - \delta < |X_n|/T_n^{2/(1-\gamma)} < \alpha + \delta$$

and

$$|X_{n+1}|/T_{n+1}^{2/(1-\gamma)} < \alpha + \delta.$$

By Lemma 1 (ii)

$$T_n - T_{n+1} > \frac{C_1}{\sqrt{M_n}} [(\alpha - \delta) T_n^{2/(1-\gamma)}]^{(1-\gamma)/2}$$

where

$$M_n = \max\{a(t) : t \in [\tau_n, T_n]\}.$$

Therefore

$$T_{n+1} < \left[1 - \frac{C_1}{\sqrt{M_n}}(\alpha - \delta)^{(1-\gamma)/2}\right] T_n$$

$$\leqq \left[1 - \frac{C_1}{\sqrt{N}}(\alpha - \delta)^{(1-\gamma)/2}\right] T_n = C_3 T_n$$

and

$$|X_{n+1}| < (\alpha + \delta) C_3^{2/(1-\gamma)} T_n^{2/(1-\gamma)}$$

from which

$$|X_{n+1}/X_n| < C_3^{2/(1-\gamma)} \frac{\alpha + \delta}{\alpha - \delta}.$$

However Corollary 3 implies that

$$[k(T_n)]^{1/(1+\gamma)} < C_3^{2/(1-\gamma)} \frac{\alpha + \delta}{\alpha - \delta}.$$

The right hand side is definitely less than 1 if δ is suitably chosen. However since T_n can be as near zero as we wish, $k(T_n)$ can be made very near to 1. This is a contradiction.

THEOREM 6. *Suppose* $\gamma > 1$. *Let* $x(t)$ *be an initially unbounded solution of* (1). *Then*

$$|X|^{-1} = o(T^{2/(\gamma-1)})$$

$$|V|^{-1} = o(\tau^{(\gamma+1)/(\gamma-1)})$$

as $T, \tau \to 0$.

PROOF. The proof is exactly the same as that of Theorem 5.

4. Main theorems

THEOREM 7. *Let* $\gamma < 1$. *Suppose that* $\log t^{2(1+\gamma)/(1-\gamma)} a(t)$ *has finite lower variation in* $[0, \varepsilon]$. *Then the trivial solution is the only initially stationary solution of* (1) *in* $[0, \varepsilon]$.

PROOF. Let $x(t)$ be a non-trivial solution of (1) in $[0, \varepsilon]$. We attempt to show that it cannot be initially stationary. We can settle right away the case of non-oscillatory $x(t)$ by Theorem 4. Hence assume that $x(t)$ has infinitely many zeros in $[0, \varepsilon]$. Take any $t_0 \in [0, \varepsilon]$ such that $x(t_0) \neq 0$. Consider the first zero of $x(t)$ we meet when we move from t_0 to 0. This must be an isolated zero in view of Theorem 4. Call it τ_1. Continue moving to the left and we shall meet the second zero of $x(t)$, which must also be an isolated zero in view of Theorem 4. As we continue the process, we obtain a decreasing sequence of zeros of $x(t)$, namely $\{\tau_n\}$. Let $x(t)$ attain its extreme values at $T_n \in (\tau_n, \tau_{n-1})$. The sequence $\{\tau_n\}$ must converge, say to $\tau \in [0, t_0]$. According to Corollary 3,

$$|X_n/X_{n+1}| = (a(s_{n'})/a(s_n))^{1/(1+\gamma)}.$$

Hence,

$$|X_{n+1}/X_1| = \left(\prod_{i=1}^{n} \frac{a(s_i)}{a(s_{i'})}\right)^{1/(1+\gamma)}.$$

But

$$\log\left(\prod_{i=1}^{n} \frac{s_i^{2(1+\gamma)/(1-\gamma)}}{s_{i'}^{2(1+\gamma)/(1-\gamma)}} \frac{a(s_i)}{a(s_{i'})}\right) \geqq -\begin{Bmatrix}\text{lower variation of}\\ \log t^{2(1+\gamma)/(1-\gamma)} a(t)\end{Bmatrix} = \beta.$$

Therefore

$$\begin{aligned} e^{\beta} &\leqq \prod_{i=1}^{n} \frac{s_i^{2(1+\gamma)/(1-\gamma)}}{s_{i'}^{2(1+\gamma)/(1-\gamma)}} \frac{a(s_i)}{a(s_{i'})} \\ &\leqq \prod_{i=1}^{n} \frac{T_i^{2(1+\gamma)/(1-\gamma)}}{T_{i+1}^{2(1+\gamma)/(1-\gamma)}} \frac{a(s_i)}{a(s_{i'})} \\ &= \left(\prod_{i=1}^{n} \frac{a(s_i)}{a(s_{i'})}\right)\left(\frac{T_1}{T_{n+1}}\right)^{2(1+\gamma)/(1-\gamma)} \end{aligned}$$

from which,

$$|X_{n+1}/X_1| \geqq (e^{\beta} T_{n+1}/T_1)^{2/(1-\gamma)}.$$

If $\tau \neq 0$, this inequality contradicts the fact that $X_n \to 0$ as $n \to \infty$ (Theorem 4). If $\tau = 0$, this inequality contradicts Theorem 5.

THEOREM 8. *Let* $\gamma < 1$. *Suppose that* $\log t^{2(1+\gamma)/(\gamma-1)} a(t)$ *has finite upper variation in* $[0, \varepsilon]$. *Then the trivial solution is the only initially stationary solution of* (1) *in* $[0, \varepsilon]$.

PROOF. The proof is the same as that of Theorem 7. However this time we consider $|V_{n+1}/V_n|$ instead of $|X_n/X_{n+1}|$.

The same arguments apply also to the superlinear case if we take Theorem 6 into account. We just state the theorems without proof.

THEOREM 9. *Let* $\gamma > 1$. *Suppose that* $\log t^{2(1+\gamma)/(\gamma-1)} a(t)$ *has finite lower variation in* $[0, \varepsilon]$. *Then there exists no initially unbounded solution in* $[0, \varepsilon]$.

THEOREM 10. *Let* $\gamma > 1$. *Suppose that* $\log t^{2(1+\gamma)/(1-\gamma)} a(t)$ *has finite upper variation in* $[0, \varepsilon]$. *Then there exists no initially unbounded solution in* $[0, \varepsilon]$.

These two theorems can easily be reformulated in a form using continuability.

5. Boundedness of solutions

The following theorem holds in both cases $\gamma < 1$ and $\gamma > 1$.

THEOREM 11. *If either*

(i) *Lower variation of* $\log a(t) < \infty$, *or*

(ii) *Upper variation of* $\log a(t) < \infty$ *and* $a(t) \geqq m > 0$ *for all sufficiently large* t,

then all solutions of (1) *are bounded.*

Wong proves this theorem for a somewhat more general equation. In the case of equation (1), our approach gives a simple proof.

PROOF. When (i) is satisfied, the same argument as in the proof of Theorem 7 shows that the amplitude of the oscillations are bounded. Similarly when (ii) is satisfied, the V_n's are bounded from above, say

$$|V_n| \leqq k < \infty.$$

Corollary 2 then gives

$$|X_n|^{(1+\gamma)} \leqq \frac{1+\gamma}{2m} k^2.$$

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