An Interesting Problem in Spherical Trigonometry

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This paper provides an example of the versatility of mathematics and how it can be applied to the formulation of navigation methods which are more elegant and efficient than those traditionally used. For example, there is little use currently made of the calculus or the techniques of differential geometry. A difficulty in applying general mathematical methods is the inconsistency of navigation coordinate systems, which could usefully be changed so that longitude and GHA, for instance, could be measured eastwards in the range $0°–360°$.

The problem considered below is of little importance in itself, either in terms of mathematics or navigation, but it illustrates how alternatives to traditional approaches to navigational requirements can lead to more attractive solutions. This particular problem has been inspired by the book *Theory and Problems of Differential and Integral Calculus in SI Units*.[1]

1. PROBLEM. A ship at the point P_1 ($\psi_1 = \psi$ north, $\lambda_1 = \lambda$) is starting to move from north to south on the meridian λ with a constant velocity $v/\cos\psi$. At the same moment another ship at point P_2 ($\psi_2 = \psi$ north, $\lambda_2 = \lambda + \psi$) is starting to move from east to west on parallel of latitude ψ with constant velocity v. What is the minimum distance between these ships? (In order to make the problem significant, the angular velocities of the ships have been selected to be equal. Here $\lambda_2 > \lambda_1$ and $0° \leqslant \psi \leqslant 90°$.)

2. SOLUTION. The angular velocity of the ships is,

$$\omega = \frac{v \cdot \rho}{R \cdot \cos\psi}. \tag{1}$$

At a time t, the longitude difference $\Delta\lambda$ between the coordinates of points $P_1'(\psi - \omega t, \lambda)$ and $P_2'(\psi, \lambda + \psi - \omega t)$ is

$$\Delta\lambda = \psi - \omega t, \tag{2}$$

(Figure 1). If we use the edge cosine theorem for the triangle $NP_1' P_2'$, it can be written as follows[2]:

$$\cos S = \sin(\psi - \omega t) \cdot \sin\psi + \cos^2(\psi - \omega t) \cdot \cos\psi. \tag{3}$$

The minimum value of S is obtained by setting, the derivative of equation (3), equal to zero. Thus from (3), we get equation (4)

$$\cos(\psi - \omega t) \cdot [2 \cdot \cos\psi \cdot \sin(\psi - \omega t) - \sin\psi] = 0. \tag{4}$$

If the first multiplier $\cos(\psi - \omega t)$ of the equation is set equal to zero then, from equation (3) we obtain,

$$S_{\min} = 90° - \psi. \tag{5}$$

In that case a minimum of equation (3) arises when the ship is at the North Pole. Another minimum of equation (3) is obtained when the second multiplier of equation (4) is set equal to zero. Thus we obtain,

$$\sin(\psi - \omega t) = \tfrac{1}{2}\tan\psi. \tag{6}$$
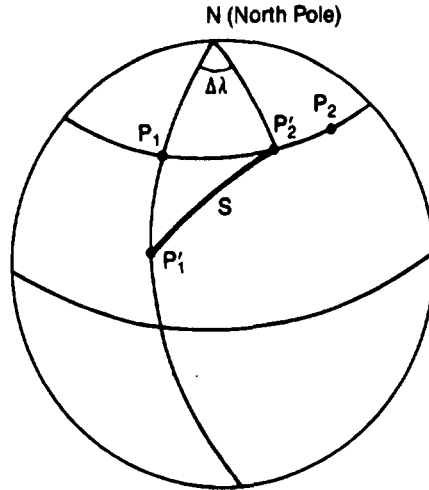
N (North Pole)

Fig. 1.

From equation (6)

$$\cos^2 (\psi - \omega t) = 1 - \tfrac{1}{4} \tan^2 \psi \tag{7}$$

is obtained. If the equalities (6) and (7) are substituted into equation (3), we obtain

$$\cos S_{\min} = \frac{1 + 3 \cos^2 \psi}{4 \cos \psi} = 0.25 \sec \psi + 0.75 \cos \psi. \tag{8}$$

The minimum distance between the ships is the minimum value obtained from (5) and (8).

2.1. *Special Conditions.* Three questions are addressed in sections (a), (b) and (c) below:

(a) At which latitude is the initial state at time $t = 0$, at the same time, the minimum distance between the ships? If the edge cosine theorem is applied to the $NP_1 P_2$ spherical triangle, we obtain,

$$\cos S = \sin^2 \psi + \cos^3 \psi. \tag{9}$$

In this special condition, equation (9) should be equal to the equation (8). If equation (9) is set equal to the equation (8), we obtain,

$$\sin^2 \psi . (2 \cos \psi - 1)^2 = 0. \tag{10}$$

From the trigonometric equation (10) we obtain,

$$\psi = 0°, \quad \psi = 60°.$$

In this situation, from equation (8) or (9), we find,

$$S_{\min} = 28° \, 57' \, 18''.09.$$

(b) On which latitude is the minimum distance the greatest?

If the derivative of equation (8) according to ψ is set equal to zero we obtain the equation,

$$4 \sin \psi (1 - 3 \cos^2 \psi) = 0. \tag{11}$$

From the solution of this trigonometric equation, we find,

$$\psi = 54° \, 44' \, 08''.2.$$

This is the latitude which gives the maximum value of the minimum distance. If this ψ value is substituted into equation (8), we find,

$$S_{min} = 30°.$$

This point is indicated by A $(54° \, 44' \, 08''.20, \, 30°)$ in Figure 2.
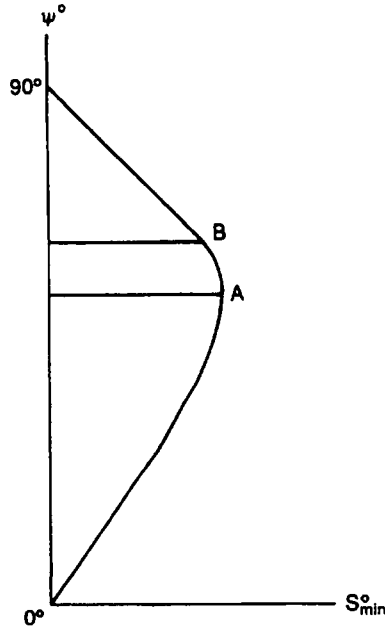

Fig. 2.

(c) Do equations (5) and (8) have a point of intersection?
If equations (5) and (8) are set equal, we obtain the equation,

$$1 + 3 \cos^2 \psi - 4 \cos \psi . \sin \psi = 0. \tag{12}$$

If this trigonometric equation is solved, we obtain,

$$\psi = 63° \, 26' \, 05''.82.$$

This point is indicated by B $(63° \, 26' \, 05''.82, \, 26° \, 33' \, 54''.10)$ in Figure 2.

    3. DISCUSSION. In order to solve trigonometric equation (6):

$$\psi \leqslant \arctan 2 \to \psi \leqslant 63° \, 26' \, 05''.82 \tag{13}$$

and to compute S_{min} from the equality (8)

$$\cos S_{min} \leqslant 1 \to \tfrac{1}{3} \leqslant \cos \psi \leqslant 1 \to 0 \leqslant \psi \leqslant 70° \, 31' \, 43''.6. \tag{14}$$

Equations (13) and (14) should be checked together. Then S_{min}, ψ should be obtained as follows:

$$0 \leqslant \psi \leqslant 63° \, 26' \, 05''.82.$$

From the program arranged for equations (5) and (8) (software) the curve in Figure 2 is obtained.

4. CONCLUSIONS. There are two main conclusions:

(i) The solution of the problem is

$$S_{\min}(\psi) = \begin{bmatrix} \arccos(0.25\sec\psi + 0.75\cos\psi), & 0 \leqslant \psi \leqslant 63° \, 26' \, 05''.82 \\ 90 - \psi, & 63° \, 26' \, 05''.82 \leqslant \psi \leqslant 90° \end{bmatrix}$$

$$\frac{dS_{\min}(\psi)}{d\psi} = \begin{bmatrix} -\dfrac{(3\cos^2\psi - 1)\sin\psi}{4\cos^2\psi \cdot \sin S_{\min}(\psi)}, & 0° \leqslant \psi \leqslant 63° \, 26' \, 05''.82 \\ -1, & 63° \, 26' \, 05''.82 \leqslant \psi \leqslant 90°. \end{bmatrix}$$

Function $S_{\min}(\psi)$ and the first derivative is continuous at every point of the closed interval $[0, \pi]$.

(ii) From the general solution of the problem related to ψ, many numerical examples can be arranged which could be useful for spherical trigonometry education.

REFERENCES

[1] Ayres, F. *Theory and Problems of Differential and Integral Calculus in SI metric Units.*

[2] Sigl, R. (1969). *Ebene und Spharische Trigonometrie.* Akademische Verlagsgesellschaft Frankfurt am Main.

KEY WORDS

1. Spherical trigonometry.   2. Geodesy.   3. Coordinates.