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# REFLEXIVITY INDEX OF THE PRODUCT OF SOME TOPOLOGICAL SPACES AND LATTICES

**BINGZHANG MA**

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**Abstract**

We introduce a technique that is helpful in evaluating the reflexivity index of several classes of topological spaces and lattices. The main results are related to products: we give a sufficient condition for the product of a topological space and a nest of balls to have low reflexivity index and determine the reflexivity index of all compact connected 2-manifolds.

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## 1. Introduction

Reflexivity was introduced by Halmos in his study of the invariant subspace problem [4] and has received a lot of attention (see [2]). The notion of reflexivity index, which in some sense measures the complexity of a lattice, was introduced in [14] in the context of arbitrary subset lattices. These lattices can be regarded as closed set lattices equipped with the discrete topology (see [5, 7, 13]). The reflexivity index of various types of closed subspace lattices has been calculated (see [6, 10]).

In [11], the authors studied the reflexivity index of closed set lattices of topological spaces equipped with the usual topology and evaluated the index of some examples, such as spheres and nests of balls with a common centre. However, a direct calculation of the reflexivity index appears to be difficult in general. By focusing on the product of lattices and topological spaces, we determine the index of some new examples, such as closed balls, some new nests and compact connected 2-manifolds.

Let us start with some basic notions. For a topological space $X$, let $\mathcal{S}(X)$ denote the set of all closed subsets of $X$ and let $C(X)$ denote the set of all continuous endomorphisms on $X$. For any $\mathcal{L} \subseteq \mathcal{S}(X)$ and any $\mathcal{F} \subseteq C(X)$, define

$$\operatorname{Alg}\mathcal{L} = \{f \in C(X) : f(A) \subseteq A \text{ for all } A \in \mathcal{L}\},$$
$$\operatorname{Lat}\mathcal{F} = \{A \in \mathcal{S}(X) : f(A) \subseteq A \text{ for all } f \in \mathcal{F}\}.$$

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We say that a subset $\mathcal{L}$ of $\mathcal{S}(X)$ is *reflexive* if $\operatorname{Lat}\operatorname{Alg}\mathcal{L} = \mathcal{L}$. Since $\operatorname{Lat}\mathcal{F} = \operatorname{Lat}\operatorname{Alg}\operatorname{Lat}\mathcal{F}$ for any $\mathcal{F} \subseteq C(X)$, $\mathcal{L}$ is reflexive if and only if $\mathcal{L} = \operatorname{Lat}\mathcal{F}$ for some $\mathcal{F} \subseteq C(X)$.

The set $C(X)$ is a semigroup under the operation of map composition, with identity id, where $\mathrm{id}(x) = x$ for all $x \in X$. The topology on $X$ induces a topology on $C(X)$, whose sub-basic open neighbourhoods of $\varphi \in C(X)$ are subsets of $C(X)$ of the form $\mathcal{N}(x, \varphi, U) = \{\psi \in C(X) : \psi(x) \in U\}$, where $U$ is any open neighbourhood of $\varphi(x)$ in $X$. It can be verified that for any $\mathcal{L} \subseteq \mathcal{S}(X)$, $\operatorname{Alg}\mathcal{L}$ is a closed sub-semigroup of $C(X)$. From [11], we see that for any $\mathcal{F} \subseteq C(X)$, $\operatorname{Lat}\mathcal{F} = \operatorname{Lat}\widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}}$ is the closure in the induced topology of the semigroup of all finite products of elements in $\mathcal{F}$. It follows that the set $\widehat{\mathcal{F}}$ is also a closed sub-semigroup of $C(X)$.

The meet and join of any collection $\{A_\omega : \omega \in \Omega\}$ of closed subsets of $X$ are defined by $\bigwedge_{\omega\in\Omega} A_\omega = \bigcap_{\omega\in\Omega} A_\omega$ and $\bigvee_{\omega\in\Omega} A_\omega = \overline{\bigcup_{\omega\in\Omega} A_\omega}$, where $\overline{A}$ denotes the closure of the set $A$. With these operations, $\mathcal{S}(X)$ is a complete lattice. We call any complete sublattice of $\mathcal{S}(X)$ containing $\varnothing$ and $X$ a *closed set lattice*. For any $\mathcal{F} \subseteq C(X)$, $\operatorname{Lat}\mathcal{F}$ is a closed set lattice, and so any reflexive family of closed subsets is a closed set lattice.

For any reflexive closed set lattice $\mathcal{L}$, $\operatorname{Alg}\mathcal{L}$ is the largest of all subsets $\mathcal{F}$ of $C(X)$ with the property that $\operatorname{Lat}\mathcal{F} = \mathcal{L}$. It is of interest to determine the minimal size of such subsets.

DEFINITION 1.1. The reflexivity index, $\kappa(\mathcal{L})$, of a reflexive closed set lattice $\mathcal{L}$ is

$$\kappa(\mathcal{L}) = \min\{|\mathcal{F}| : \operatorname{Lat}\mathcal{F} = \mathcal{L}\}.$$

If $\mathcal{L} = \{\varnothing, X\}$), we denote the reflexivity index of $\mathcal{L}$ by $\kappa(X)$ for convenience and call it the reflexivity index of the topological space $X$. If $\kappa(X) = 1$, we say that the space $X$ is *minimal* and a map $f$ in $\mathcal{F}$ a minimal map.

For a topological space, the reflexivity index is a topological invariant. Intuitively, a larger reflexivity index suggests a more complicated topological space.

In Section 2 of this paper, we introduce the $A$-property, which is a property of many topological spaces, and give a sufficient condition for the product of a topological space and a nest of balls to have low reflexivity index. By viewing a closed ball as the product of a lower dimensional ball and a nest consisting of a single element, we show that all finite dimensional closed balls have reflexivity index 2. In Section 3, we show that spaces glued together by a finite number of closed balls also have low reflexivity index, and then determine the index of compact connected 2-manifolds. Finally, in Section 4, using similar methods, we generalise our results to some noncompact cases. The main results are Theorems 2.8, 2.9 and 3.2.

## 2. The $A$-property and reflexivity index of related products

We introduce some basic notions first. Let $\mathcal{H}$ denote a separable infinite dimensional real Hilbert space; $\mathcal{H}$ is sometimes written as $\mathbb{R}^\infty$. Note that any Hilbert space can be viewed as a topological space equipped with the norm topology. For

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$1 \leq n < \infty$, let $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. We also write $D^\infty = \{x \in \mathbb{R}^\infty : \|x\| \leq 1\}$ and $S^\infty = \{x \in \mathbb{R}^\infty : \|x\| = 1\}$.

For a topological space $X \subset \mathcal{H}$, if there is an open neighbourhood $O$ of some $x \in X$ with the property that $O \cap X$ is strictly contained in $X$ and homeomorphic to $\mathbb{R}^n$ for some $n$ with $1 \leq n \leq \infty$, then we say that $X$ is an $A_0$-*space* and $O \cap X$ is an $A_0$-*part* of $X$ with dimension $n$. An $A_0$-part $E$ of $X$ with the largest (in the sense of cardinality) dimension $N \leq \infty$ is called an $A_1$-*part* of $X$. Thus, $X$ is an $A_0$-space if and only if $X$ contains an $A_1$-part. The fact that $E$ is homeomorphic to $\mathbb{R}^N$ is sometimes written $\dim(E) = N$. Let $\partial X$ denote the set of points in $X$ that do not admit open neighbourhoods whose intersection with $X$ is homeomorphic to $\mathbb{R}^n$ for some $n$ with $1 \leq n \leq \infty$; we call $\partial X$ the *boundary* of $X$.

Let $\mathcal{F}$ be a set of continuous maps from $X$ to itself and $f = g_m \circ g_{m-1} \circ \cdots \circ g_1$ be a finite composition of elements in $\mathcal{F}$. We write $f = g_m g_{m-1} \cdots g_1$ or $f = \prod_{i=1}^m g_i$ for convenience. Fix some $f_0 \in \mathcal{F}$ and denote by $\Lambda$ the subset of $\{1, 2, \ldots, m\}$ such that $g_i = f_0$ if and only if $i \in \Lambda$. Let $g_0$ be the identity map and write $f = \prod_{i=0}^m g_i$ and $O_{f_0}(f, x) = \{\prod_{j=0}^{i-1} g_j(x) : i \in \Lambda\}$.

PROPOSITION 2.1. *Suppose that $X \subset \mathcal{H}$ is a topological space containing an $A_1$-part $E$. There is a set of continuous maps $\mathcal{F}$ on $X$ with* $\operatorname{Lat}\mathcal{F} = \{\emptyset, X\}$ *and there is some $f_0 \in \mathcal{F}$ such that for any $x, y \in X$ and $\epsilon > 0$, there is a finite composition $f$ of elements in $\mathcal{F}$ such that $\|f(x) - y\| < \epsilon$ and $O_{f_0}(f, x) \cap E = \emptyset$.*

PROOF. From the definition of $A_0$-space, we may choose a point $q \in X \setminus E$. Let $\mathcal{F} = C(X)$. Choose $p \in E$ and let $f_0 \in \mathcal{F}$ be the map satisfying $f_0(x) = p$ for all $x \in X$. For any $x \in X$ and $y \in X \setminus \{p\}$, there is some constant map in $\mathcal{F}$ sending $x$ to $y$, which is our desired map. For $y = p$, there is $f_1 \in \mathcal{F}$ with $f_1(x) = q$ for all $x$. It follows that $f_0 f_1$ maps $x$ to $y$ and $O_{f_0}(f_0 f_1, x) \cap E = \emptyset$, and $f_0 f_1$ is our desired map. □

We call a set of maps $\mathcal{F}$ with the properties in Proposition 2.1 an $A_1$-*set* of $X$ relative to $E$. We call $f_0$ the $A$- *map* of the $A_1$-set $\mathcal{F}$. It is easy to see that such a set $\mathcal{F}$ contains more than one element for otherwise, there is no $f \in \mathcal{F}$ mapping $x \in E$ to some other point with $O_{f_0}(f, x) \cap E = \emptyset$. This suggests that the minimal cardinality of $A_1$-sets could be interesting and motivates the following definition.

DEFINITION 2.2. For a topological space $X \subset \mathcal{H}$, we say that $X$ has the $A$-*property* if there is an $A_1$-part $E$ of $X$ and the minimal cardinality of $A_1$-sets relative to $E$ is 2. We call $E$ an $A$-*part* of $X$ and a minimal $A_1$-set an $A$-*set* of $X$ relative to $E$.

Note that for any $X \subset \mathcal{H}$, the lattice $\{X, \emptyset\}$ is automatically reflexive [11]. Thus, if $X$ has the $A$-property, then $\kappa(X) \leq 2$. Note also that the $A$-property is a topological invariant. We say that a topological space $Y_1$ not in $\mathcal{H}$ has the $A$-property if it is homeomorphic to some space $Y_2$ in $\mathcal{H}$ with the $A$-property, and $E_1$ is an $A$ (or $A_1$)-part of $Y_1$ if its homeomorphic image $E_2$ in $\mathcal{H}$ is an $A$ (or $A_1$)-part of $Y_2$.

For a real number $s$, let $[s]$ denote the largest integer that is not greater than $s$, and let $\{s\} = s - [s]$. We record a well-known fact.

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PROPOSITION 2.3. *For an irrational number $\alpha$, the set $\{\{n\alpha\} : n \in \mathbb{Z}, n > 0\}$ is dense in* $[0, 1]$.

LEMMA 2.4. *$D^1$ has the A-property.*

PROOF. We study the closed interval $X = [0, 1] \subset \mathbb{R}$ instead of $D^1$ for convenience. Note that the open interval $E = (3/4, 1)$ is an $A_1$-part of $X$. Let $\alpha$ be an irrational number with $1/4 < \alpha < 1/2$ and let

$$f_0(x) = \min\{x + \alpha, 1\}, \quad f_1(x) = \max\{x + \alpha - 1, 0\}.$$

We see that $f_0$ and $f_1$ are continuous maps from $X$ to itself. Given any $a, b \in [0, 1]$, let $g_0$ be the identity map and for $n \geq 1$, let inductively $g_n = f_0$ if $\prod_{i=0}^{n-1} g_i(a) \in [0, 1 - \alpha]$ and $g_n = f_1$ if $\prod_{i=0}^{n-1} g_i(a) \in (1 - \alpha, 1]$. From Proposition 2.3, given any $\epsilon > 0$, there exists $N$ such that $|\prod_{i=1}^{N} g_i(a) - b| < \epsilon$. Note that $\prod_{i=1}^{N} g_i$ is a finite composition of $f_0$ and $f_1$ and $O_{f_0}(\prod_{i=1}^{N} g_i, a) \subset [0, 1 - \alpha]$. It follows that $O_{f_0}(\prod_{i=1}^{N} g_i, a) \cap E = \varnothing$. This completes the proof. □

By saying 'there exists a finite composition $f$ of $f_0$ and $f_1$ mapping $a$ to some point close to $b$', written $f_{\{f_0,f_1\}}(a) \sim b$, we mean that 'given any $\epsilon > 0$, we can find a finite composition $F$ of $f_0$ and $f_1$ satisfying $\|F(a) - b\| < \epsilon$, and $f$ is such a finite composition for some given $\epsilon_0 > 0$'. If $X_0, E$ are subsets of $X$ and we replace '$\|F(a) - b\| < \epsilon$' with '$\|F(a) - b\| < \epsilon, F(a) \in X_0$ and $O_{f_0}(F, a) \cap E = \varnothing$', we obtain the definition of $f_{\{f_0,f_1;X_0;E\}}(a) \simeq b$. Note that $f_{\{f_0,f_1;X_0;E\}}(a) \simeq b$ and $f_{\{f_1,f_0;X_0;E\}}(a) \simeq b$ have different meanings: $f_{\{f_0,f_1;X_0;E\}}(a) \simeq b$ implies $O_{f_0}(F, a) \cap E = \varnothing$ and $f_{\{f_1,f_0;X_0;E\}}(a) \simeq b$ implies $O_{f_1}(F, a) \cap E = \varnothing$. Note also that if $X_0 = X$ and there is no confusion about the set $E$, we may write $f_{\{f_0,f_1\}}(a) \simeq b$ instead of $f_{\{f_0,f_1;X_0;E\}}(a) \simeq b$ for convenience. The proof of Lemma 2.4 amounts to 'there exist continuous maps $f_0$ and $f_1$ on $[0, 1]$ such that for any $a, b \in [0, 1]$, we have $f_{\{f_0,f_1;[0,1];(3/4,1)\}}(a) \simeq b$'. It will be seen later that these notions are very convenient.

There is some 'associativity' related to these notions: if $f_0$ and $f_1$ are continuous, then $g_{1\{f_0,f_1;X;E\}}(a) \simeq b$ and $g_{2\{f_0,f_1;X;E\}}(b) \simeq c$ imply $g_2g_{1\{f_0,f_1;X;E\}}(a) \simeq c$.

We introduce a lattice extension before focusing on products in Lemma 2.6. Let $\mathcal{H}_0$ be a nonzero linear subspace of $\mathcal{H}$. For $c \in \mathcal{H}_0$ and $r \geq 0$, let $B_{\mathcal{H}_0}(c, r)$ or simply $B(c, r)$ denote the (closed) ball $\{x \in \mathcal{H}_0 : \|x - c\| \leq r\}$. Let $\mathcal{N}_0 = \{B(c_i, r_i) : i \in \Lambda\}$ be a nonempty set of balls that is totally ordered under the relation of set inclusion. Assume that $\sup\{r_i : i \in \Lambda\} < \infty$. The set containing $\varnothing$, $\mathcal{N}_0$, arbitrary intersections and the closure of unions of elements in $\mathcal{N}_0$ is a closed set lattice; we call it a *bounded nest of balls*.

For a bounded nest of balls $\mathcal{N}$, there exists a maximal element $\mathcal{N}^+ = B(c^+, r^+) \in \mathcal{N}$ that contains any other element in $\mathcal{N}$. For $M \subset \mathcal{N}^+$, let $M^+$ denote the intersection of all elements in $\mathcal{N}$ that contain $M$ and $M^-$ denote the closure of the union of all elements in $\mathcal{N}$ that do not contain $M$. For $x \in \mathcal{N}^+$, let $x^+$, $x^-$ denote $\{x\}^+$, $\{x\}^-$, respectively. Write $x^+ = B(c_{x^+}, r_{x^+})$, and $x^- = B(c_{x^-}, r_{x^-})$ if $x^-$ is nonempty. For $x \in \mathcal{N}^+$ with $x^-$ being empty, let $x^\sharp = B(c_{x^\sharp}, r_{x^\sharp}) = x^+$. For $x$ with $x^-$ being nonempty, there exists a smallest

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$t_x$ with $0 \leq t_x \leq 1$ such that $x \in \partial B((1-t_x)c_{x^-} + t_x c_{x^+}, (1-t_x)r_{x^-} + t_x r_{x^+})$. To see this, note that the continuous function $\|x - ((1-t_x)c_{x^-} + t_x c_{x^+})\| - |(1-t_x)r_{x^-} + t_x r_{x^+}|$ is nonnegative when $t_x = 0$, nonpositive when $t_x = 1$ and monotonically decreasing on $[0,1]$. The intermediate value theorem leads to the result. Let $t'_x = \min\{2t_x, 1\}$ and let $x^\sharp = B(c_{x^\sharp}, r_{x^\sharp}) = B((1-t'_x)c_{x^-} + t'_x c_{x^+}, (1-t'_x)r_{x^-} + t'_x r_{x^+})$. It follows that for any bounded nest of balls $\mathcal{N}$, we have a unique lattice extension $\mathcal{N} \cup \{x^\sharp : x \in \mathcal{N}^+\}$ of $\mathcal{N}$.

Definition 2.5. We say that a bounded nest of balls $\mathcal{N}$ is *well behaved* if the function defined on $\mathcal{N}^+$: $x \mapsto r_{x^\sharp}$ is continuous.

There are examples of bounded nests of balls that are well behaved, such as a finite nest $\{[-r, r] \subset \mathbb{R} : r \in \{1, 2, \ldots, n\}\}$ or a continuous nest $\{[-r, r] \subset \mathbb{R} : r \in [1, 2]\}$.

The following lemma contains our technique for showing that the product of certain classes of lattices and topological spaces has low reflexivity index.

Lemma 2.6. *Let $\mathcal{H}_0$ be a nonzero linear subspace of $\mathcal{H}$ and $\mathcal{N}$ be a bounded nest of balls in $\mathcal{H}_0$ that is well behaved. Let $X \subset \mathcal{H}$ be a topological space with the A-property and E an A-part of X. Suppose that* $\dim(E) \geq \dim(\mathcal{H}_0)$ *in the sense of cardinality. Then the lattice $\mathcal{N} \times X$ with elements contained in $\mathcal{H}_0 \times \mathcal{H}$ is reflexive and has reflexivity index no more than* 2.

Proof. We use the notions introduced before Definition 2.5. From [13], we see that a sufficient condition for $\mathcal{N} \times X$ being reflexive is that for any $(a, b) \in \mathcal{N}^+ \times X$, the set $\{F((a, b)) : F \in \mathcal{F}\}$ is dense in $a^+ \times X$, where $\mathcal{F}$ is a set of maps generated by two elements in $\mathrm{Alg}\,(\mathcal{N} \times X)$.

Let $D$ denote the unit ball of $\mathcal{H}_0$. Note that for any $a \in \mathcal{N}^+$ and nonzero $p \in \mathcal{H}_0$, there is $s_{a,p} \geq 0$ such that $a + s_{a,p}p \in \partial(a^\sharp)$. Define a map $\phi : D \to C(\mathcal{N}^+)$ as follows. For $p \in D$ and $p = 0$, let $\phi(p)$ be the constant map on $\mathcal{N}^+$. For $0 < \|p\| \leq 1/2$, let $\phi(p)(a) = a + \min\{s_{a,p}, 5r^+\}p$. For $1/2 < \|p\| \leq 1$, let $\phi(p) = \phi((1/\|p\| - 1)p)$. It can be shown that $\phi$ is continuous with $C(\mathcal{N}^+)$ equipped with the sup-norm and that for any $a_1 \in a^\sharp$, there exists $p_1 \in D$ with $\|p_1\| \leq 2/5 < 1/2$ such that $\phi(p_1)(a) = a_1$.

Let $\{f_0, f_1\}$ be an $A$-set of $X$ relative to $E$ with $f_0$ the $A$-map. Fix some $b_0 \in E$. There exists $r_0$ with $0 < r_0 < \infty$ such that the set $D_0 = \{b \in X : \|b - b_0\| \leq r_0\}$ is contained in $E$. There is a homeomorphism $I_0$ from $D_0$ to the unit ball $D_1$ of a subspace of $\mathcal{H}$, and since $\dim(E) \geq \dim(\mathcal{H}_0)$, there is an operator $P : D_1 \to D$ that is a linear projection mapping $D_1$ onto $D$. Let $\psi = \phi P I_0$, so that $\psi : D_0 \to C(\mathcal{N}^+)$ is continuous.

Define maps $F_0, F_1$ on $\mathcal{N}^+ \times X$ as follows. For any $(a, b) \in \mathcal{N}^+ \times X$, let

$$F_0(a,b) = \begin{cases} (a, f_0(b)) & \text{if } b \in X \setminus D_0, \\ (\psi(b)(a), f_0(b)) & \text{if } b \in D_0, \end{cases} \quad \text{and} \quad F_1(a,b) = (a, f_1(b)).$$

It can be shown that both of $F_0$ and $F_1$ are continuous. For any $(a_1, b_1) \in \mathcal{N}^+ \times X$ and $(a_2, b_2) \in a_1^+ \times X$, if $a_1 \in \partial a_1^+$, then there exists $p_1 \in D_0$ such that $\psi(p_1)(a_1) = a_2$. From the fact that $X$ has the $A$-property, we have $f_{\{f_0, f_1; X; E\}}(b_1) \simeq p_1$ and after replacing $f_0$ with $F_0$ and $f_1$ with $F_1$ in $f$, we have $F_{\{F_0, F_1\}}(a_1, b_1) \sim (a_1, p_1)$ and $F_0 F_{\{F_0, F_1\}}(a_1, b_1) \sim (a_2, f_0(p_1))$. Note that $g_{\{f_0, f_1\}}(f_0(p_1)) \simeq b_2$, and after a similar replacement of $f_0$ and

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$f_1$ with $F_0$ and $F_1$, respectively, we have $g$ replaced with $G$ and $GF_0F_{\{F_0,F_1\}}(a_1,b_1) \sim (a_2,b_2)$, which is our desired result.

If $a_1 \notin \partial(a_1^+)$, then there exists $q_1 \in D_0$ such that $\psi(q_1)(a_1) = s_1 \in \partial(a_1^\sharp)$ and $q_2 \in D_0$ such that $\psi(q_2)(s_1) = s_2 \in \partial(s_2^\sharp)$, and there exists a smallest integer $n$ with $1 \le n < \infty$ such that $\psi(q_n)(s_{n-1}) = s_n \in \partial a_1^+$. To see this, suppose that $t_1 \in (0,1)$ is the number satisfying $a_1 \in \partial B((1-t_1)c_{a_1^-} + t_1 c_{a_1^+}, (1-t_1)r_{a_1^-} + t_1 r_{a_1^+})$. Let $N_1$ be the smallest integer satisfying $2^{-N_1} \le t_1$, so that $n = N_1$. Denote the set of points $u$ in $E$ with the property that $\psi(u)(a_1) \in \partial a_1^\sharp$ by $\psi^{-1}(a_1 \to \partial a_1^\sharp)$. Then $g_{1\{f_0,f_1;\psi^{-1}(a_1\to\partial a_1^\sharp);E\}}(b_1) \simeq q_1$. We also have $g_{2\{f_0,f_1;\psi^{-1}(s_1\to\partial s_1^\sharp);E\}}(f_0(q_1)) \simeq q_2$ and similarly $g_{i+1\{f_0,f_1;\psi^{-1}(s_i\to\partial s_i^\sharp);E\}}(f_0(q_i)) \simeq q_{i+1}$ for all $i < n$. Let $q_{n+1} \in D_0$ be the point such that $\psi(q_{n+1})(s_n) = a_2$, so that $g_{n+1\{f_0,f_1;X;E\}}(f_0(q_n)) \simeq q_{n+1}$. Finally, we have $g_{n+2\{f_0,f_1\}}f_0(q_{n+1}) \simeq b_2$. Let $g^* = g_{n+2}\prod_{i=1}^{n+1}(f_0 g_i)$, so that $g^*_{\{f_0,f_1\}}(b_1) \sim b_2$. Replacing $f_0$ with $F_0$ and $f_1$ with $F_1$ in $g^*$, we obtain a finite composition $G^*$ and $G^*_{\{F_0,F_1\}}(a_1,b_1) \sim (a_2,b_2)$. This completes the proof that the lattice $\mathcal{N} \times X$ is reflexive and has reflexivity index no more than 2. □

REMARK 2.7. It will be shown in Example 2.10 that Lemma 2.6 may fail if the well-behaved property of the nest of balls is omitted.

From Lemma 2.6, we see that the condition '$\mathcal{N}$ is well behaved and $X$ has the $A$-property' is a sufficient condition for the product to have low reflexivity index.

Now we give our main results of this section. First, note that for $2 \le n < \infty$, $D^n$ can be viewed as the product of $D^{n-1}$ and a nest consisting of a single element $D^1$.

THEOREM 2.8. *For $1 \le n < \infty$, $D^n$ has reflexivity index 2 and has the $A$-property.*

PROOF. Brouwer's fixed point theorem implies that for any continuous endomorphism $f$ on some $D^n$, $\mathrm{Lat}\{f\}$ contains a singleton. Thus, $\kappa(D^n) > 1$ for all $n \ge 1$ and $D^n$ has the $A$-property implies that $D^n$ has reflexivity index 2.

We show that all $D^n$ have the $A$-property by induction on $n$. First, $D^1$ has the $A$-property by Lemma 2.4. Suppose now that $D^m$ has the $A$-property. Then for $D^{m+1}$, which is homeomorphic to $D^1 \times D^m$, observe that $\{D^1\}$ is a well-behaved bounded nest of a single element. We use again the notions from Lemma 2.6. For any $(a_1,b_1)$ and $(a_2,b_2) \in D^1 \times D^m$, the finite composition of $F_0$ and $F_1$ constructed in Lemma 2.6 and denoted here by $F'$, which maps $(a_1,b_1)$ to some point close to $(a_2,b_2)$, satisfies $O_{F_0}(F',(a_1,b_1)) \cap D^1 \times (E \setminus D_0) = \varnothing$. Take $E_0 \subset E \setminus D_0$ which is homeomorphic to $E$ and take an open interval $I_0 \subset D^1$, so that $I_0 \times E_0$ is an $A$-part of $D^1 \times D^m$. It follows that $D^{m+1}$ has the $A$-property and the proof is complete. □

Theorem 2.8 will be helpful in proving some new results in following sections.

In [11], the authors determined the reflexivity index of some nests with a requirement that any two distinct elements in the nest have disjoint boundaries. Now we give some new results with this requirement removed. Note, for example, that if we call a closed interval of positive length a nontrivial closed interval, then the product of a

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bounded nest of nontrivial closed intervals and a topological space homeomorphic to $D^n$ can be viewed as a nest of spaces homeomorphic to $D^{n+1}$, and the intersection of boundaries of any two nonempty elements is nonempty.

THEOREM 2.9. *Let $\mathcal{N}$ be a well-behaved bounded nest of closed intervals. For $1 \le n < \infty$, the nest $\mathcal{N} \times D^n$ is reflexive and has reflexivity index* 2.

PROOF. Reflexivity follows from Lemma 2.6. Brouwer's fixed point theorem gives $\kappa(\mathcal{N} \times D^n) > 1$. Then $\kappa(\mathcal{N} \times D^n) = 2$ from Lemma 2.6 and Theorem 2.8. □

We give an example to show that the well-behaved property cannot be omitted in Lemma 2.6 and Theorem 2.9.

EXAMPLE 2.10. Let $\mathcal{N}_0 = \{[-r, r] : r \in [1, 2]\} \cup \{[-1, 0]\} \cup \varnothing$. Then $\mathcal{N}_0$ is a bounded nest of closed intervals which is not well behaved: the map $x \mapsto r_{x^\sharp}$ is not continuous at $x = -1$. We show that $\kappa(\mathcal{N}_0 \times D^1) = \infty$.

From [13], we see that $\mathcal{N}_0$ is reflexive. Suppose that $\mathrm{Lat}\,\mathcal{F}_0 = \mathcal{N}_0$. Let $C$ denote the set of all constant maps on $D^1$ and let $\mathcal{F}_1 = \{(f, \mathrm{id}) : f \in \mathcal{F}_0\} \cup \{(\mathrm{id}, g) : g \in C\}$ be a set of maps on $\mathcal{N}_0 \times D^1$. It can be shown that $\mathrm{Lat}\,\mathcal{F}_1 = \mathcal{N}_0 \times D^1$ and $\mathcal{N}_0 \times D^1$ is reflexive.

Suppose, in contrast, that there exists a finite set $\mathcal{F} = \{f_i : i = 1, 2, \ldots, n\}$ with $\mathrm{Lat}\,\mathcal{F} = \mathcal{N}_0 \times D^1$. For any $m \in \{1, \ldots, n\}$ and point $T \in \{-1\} \times D^1$, there exists $\epsilon_{m,T}$ such that for any point $Z$ with $Z \in O(T, \epsilon_{m,T}) = \{P \in [-2, 2] \times D^1 : \|P - T\| < \epsilon_{m,T}\}$, we have $f_m(Z) \in [-2, 1] \times D^1$. It follows that $\{O(T, \epsilon_{m,T}) : T \in \{-1\} \times D^1\}$ is an open covering of $\{-1\} \times D^1$ which has a finite subcovering, and there exists $\lambda_m < -1$ such that $[\lambda_m, -1] \times D^1$ is contained in the union of all sets in the subcovering. It follows that $f_m([\lambda_m, 1] \times D^1) \subset [\lambda_m, 1] \times D^1$. Note that each $f_i \in \mathcal{F}$ admits $\lambda_i < -1$ with $f_i([\lambda_i, 1] \times D^1) \subset [\lambda_i, 1] \times D^1$. Let $\lambda = \max\{\lambda_i : i = 1, \ldots, n\}$. It follows that $\lambda < -1$ and $[\lambda, 1] \times D^1 \in \mathrm{Lat}\,\mathcal{F}$, which is a contradiction. Thus, $\kappa(\mathcal{N}_0 \times D^1) = \infty$.

## 3. An application: determining the index of compact connected 2-manifolds

We wish to determine the reflexivity index of some more complicated topological spaces, such as the index of a Möbius strip or a Klein bottle. A direct evaluation appears to be difficult. Our method is to view these spaces as ones glued together by a finite number of homeomorphic images of closed balls, and then use Theorem 2.8 to show that all of them have low reflexivity index.

We say that a topological space $A$ is *glued together* by a set of spaces $\{A_i\}_{i\in\Lambda}$ if $A = \bigcup_{i\in\Lambda} A_i$ is path-connected and $A_i \cap A_j \subset \partial A_i \cap \partial A_j$ for any $i, j \in \Lambda$ and $i \ne j$.

LEMMA 3.1. *Let $X$ be a topological space that is glued together by $X_1, \ldots, X_m$ where $X_i$ is homeomorphic to $D^{n_i}$ for some $n_i$ with $1 \le n_i < \infty$, $i = 1, \ldots, m$. Then $X$ has the A-property.*

PROOF. Assume without loss of generality that $n_1 \ge n_i$ for $1 \le i \le m$. Write $D_i$ instead of $D^{n_i}$ for convenience and let $\phi_i : D_i \to X_i$ be the homeomorphisms. Note that there are projections $P_i$ from $D_1$ onto $D_i$ for all $i = 1, \ldots, m$. Let $D = \{x \in D_1 : \|x\| \le 1/2\}$.

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Since $\phi_1(D)$ has the $A$-property by Theorem 2.8, let $E$ be an $A$-part of $\phi_1(D)$, $\{f_0, f_1\}$ be an $A$-set of $\phi_1(D)$ relative to $E$ and $f_0$ the $A$-map. Take $m$ disjoint closed sets $B_1, B_2, \ldots, B_m$ such that each of them is contained in $E$ and is homeomorphic to $D_1$. Let $\psi_i : D_1 \to B_i$ denote the homeomorphisms from $D_1$ to $B_i$, for $i = 1, \ldots, m$.

We define a continuous map $F_0$ on $X$ as follows. For $x \in B_i$ for some $i$ and $\|\psi_i^{-1}(x)\| \le 1/4$, let $F_0(x) = \phi_i P_i(4\psi_i^{-1}(x))$. For $1/4 < \|\psi_i^{-1}(x)\| \le 1/2$, let $F_0(x) = F_0((1/2\|x\| - 1)x)$. For $1/2 < \|\psi_i^{-1}(x)\| \le 3/4$, let $F_0(x) = \alpha_i(\|x\|)$, where $\alpha_i : [1/2, 3/4] \to X$ is a path with initial point $\alpha_i(1/2) = \phi_i(0)$ and terminal point $\alpha_i(3/4) = \phi_1(0)$. This can be done since $X$ is path-connected. For $3/4 < \|\psi_i^{-1}(x)\| \le 1$, let $F_0(x) = \phi_1((4\|\psi_i^{-1}(x)\| - 3)\phi_1^{-1} f_0(x))$. For all $x \in \phi_1(D) \setminus \bigcup_{i=1}^m B_i$, let $F_0(x) = f_0(x)$. For $x \in \phi_1(D_1) \setminus \phi_1(D)$, let $F_0(x) = \phi_1((-2\|\phi_1^{-1}(x)\| + 2)\phi_1^{-1} f_0 \phi_1(\phi_1^{-1}(x)/2\|\phi_1^{-1}(x)\|))$. Finally, for $x \in X \setminus \phi_1(D_1)$, let $F_0(x) = \phi_1(0)$. It can be checked that $F_0$ is continuous.

Define $F_1$ on $X$ similarly. For $x \in \phi_1(D)$, let $F_1(x) = f_1(x)$. For $x \in \phi_1(D_1) \setminus \phi_1(D)$, let $F_1(x) = \phi_1((-2\|\phi_1^{-1}(x)\| + 2)\phi_1^{-1} f_1 \phi_1(\phi_1^{-1}(x)/2\|\phi_1^{-1}(x)\|))$, and for $x \in X \setminus \phi_1(D_1)$, let $F_1(x) = \phi_1(0)$. Then $F_1$ is also continuous.

For any $a, b \in X$, notice first that $f_1(a) \in \phi_1(D)$ for all $a \in X$. If $b \in \phi_1(D)$, then $f_{\{f_0, f_1; \phi_1(D); E\}}(f_1(a)) \simeq b$. Let $S \subset E$ be a set that is homeomorphic to $E$ and disjoint from $B_1, B_2, \ldots, B_m$, and $S$ is an $A_1$-part of $X$. Replacing $f_0$ with $F_0$ and $f_1$ with $F_1$ respectively in $f$, we get $F$, and $FF_{1\{F_0, F_1; X; S\}}(a) \simeq b$. If $b \notin \phi_1(D)$, note that $b \in X_{i_0}$ for some $1 \le i_0 \le m$ and it can be checked that there exists $c \in B_{i_0}$ with $F_0(c) = b$. It follows that $g_{\{f_0, f_1; \phi_1(D); E\}}(f_1(a)) \simeq c$. By the same replacement of $f_0$ with $F_0$ and $f_1$ with $F_1$ respectively in $g$, we get $G$, and we have $F_0 G F_{1\{F_0, F_1; X; S\}}(a) \simeq b$, completing the proof of the lemma. □

The main result of this section is as follows.

THEOREM 3.2. *The* 2*-torus and the Klein bottle have reflexivity index* 1*, and all other compact connected* 2*-manifolds have reflexivity index* 2*.*

PROOF. From [1], we see that among all compact connected 2-manifolds, only the 2-torus and the Klein bottle admit minimal maps. It follows that they have reflexivity index 1 and all the others have index no less than 2. From [12], we see that all closed surfaces admit finite triangulations and from Lemma 3.1, they all have reflexivity index 2. For a compact connected 2-manifold $Y$ with boundary, from [8], we see that $Y = X \setminus \bigcup_{i=1}^n O_i$, where $X$ is some compact connected 2-manifold without boundary and $O_i$ are disjoint open disks. Using the barycentric subdivision to the triangulation of $X$ finitely many times gives a new triangulation $X = \bigcup_{j=1}^m T_m$ with the property that each $O_i$ contains some $T_{j_i}$, for $j_i \in 1, \ldots, m$, $i = 1, \ldots, n$. It follows that $X \setminus \bigcup_{i=1}^n (T_{j_i} \setminus \partial(T_{j_i}))$, which is homeomorphic to $Y$, is glued together by $\{T_j : j = 1, \ldots, m\} \setminus \{T_{j_i} : i = 1, \ldots, n\}$ and thus has reflexivity index 2. □

## 4. Generalisations to noncompact cases

We wish to generalise our results to noncompact spaces and related lattices. This seems difficult in general since a noncompact space is never a continuous image

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of a closed set. Examples of spaces with unknown reflexivity index are given in Remark 4.7. However, there are some spaces whose reflexivity index can be evaluated, since a noncompact space can sometimes be written as a countable union of closed sets.

Let us start with a proposition that will be used later.

PROPOSITION 4.1. *Let* $E = \bigcup_{n\in\mathbb{Z}}(n - 1/4, n) \subset \mathbb{R}$. *There exists* $f_0, f_1 \in C(\mathbb{R})$ *such that for any* $a, b \in \mathbb{R}$, *we have* $f_{\{f_0,f_1;\mathbb{R};E\}}(a) \simeq b$.

PROOF. Let $\alpha \in (1/4, 1/2)$ be an irrational number and let $f_0(x) = x + \alpha$ and $f_1(x) = x + \alpha - 1, x \in \mathbb{R}$, so that $f_0$ and $f_1$ are continuous maps on $\mathbb{R}$.

If $b \in [[a], [a] + 1)$, using a method similar to that in Lemma 2.4, we have $f_{\{f_0,f_1;\mathbb{R};E\}}(a) \simeq b$. If $b \in [[a] - m, [a] - m + 1)$ for some positive integer $m$, note that there exists $N$ such that $f_1^N(a) \in [[a] - m, [a] - m + 1)$ and we have $g_{\{f_0,f_1;\mathbb{R};E\}}(f_1^N(a)) \simeq b$. Let $f = gf_1^N$ and the result follows. If $b \in [[a] + 1, [a] + 2)$, note that we have $g_{1\{f_0,f_1;([a]+1-\alpha,[a]+3/4);E\}}(a) \simeq [a] + (7 - 4\alpha)/8$. It follows that $f_0g_1(a) \in [[a] + 1, [a] + 2)$ and $g_{2\{f_0,f_1;\mathbb{R};E\}}(f_0g_1(a)) \simeq b$. Let $f = g_2f_0g_1$ and the result follows. It can be shown after repeating the construction for the case when $b \in [[a] + 1, [a] + 2)$ that $f_{\{f_0,f_1;\mathbb{R};E\}}(a) \simeq b$ for any $b \geq [a] + 2$. □

REMARK 4.2. Let $\mathbb{R}^+$ denote $\{x \in \mathbb{R} : x \geq 0\}$. If we replace $\mathbb{R}$ with $\mathbb{R}^+$ and replace $E$ with $\bigcup_{n\geq1,n\in\mathbb{Z}}(n - 1/4, n)$ in the above lemma, then the result still holds after replacing $f_1(x) = x + \alpha - 1$ with $f_1(x) = \max\{x + \alpha - 1, 0\}$.

Note that if $H$ is a topological space homeomorphic to $\mathbb{R}^+$, then $\partial H$ is the homeomorphic image of $0 \in \mathbb{R}^+$. In the following proposition, we focus on the reflexivity index of topological spaces that are glued together by a set of homeomorphic images of $\mathbb{R}^+$ and $D^1$. These spaces appear often. For example, a space glued together by finitely many spaces homeomorphic to $D^1$ is called a *finite graph*.

PROPOSITION 4.3. (i) *A finite graph has reflexivity index no more than* 2.

(ii) *Let* $\{A_i\}_{i\in\Lambda}$ *be a nonempty countable set of topological spaces homeomorphic to* $\mathbb{R}^+$ *and* $\{B_j\}_{j\in\Gamma}$ *a countable set of spaces homeomorphic to* $D^1$. *Then a topological space* $X$, *which is glued together by all* $A_i$ *and* $B_j$, *where* $i \in \Lambda$ *and* $j \in \Gamma$, *has reflexivity index no more than* 2.

PROOF. (i) This follows immediately from Lemma 3.1.

(ii) Suppose first that $\Lambda = \Gamma = \{1, 2, \ldots\}$ are both countably infinite sets. For each positive integer $i$, let $\phi_i$ denote a homeomorphism from $\mathbb{R}^+$ to $A_i$ and write $\phi = \phi_1$ for convenience. Again, we let $\alpha \in (1/2, 1/4)$ denote an irrational number and $E$ denote $\phi(\bigcup_{n>0,n\in\mathbb{Z}}(n - 1/4, n)) \subset A_1$. Define continuous maps $f_0$, $f_1$ on $X$ as follows. For $x \in A_1 \setminus E$, let $f_0(x) = \phi(\phi^{-1}(x) + \alpha)$. For each positive integer $N$, let $r_N : [N - 1/4, N] \to X$ be a path with initial point $\phi(N - 1/4 + \alpha)$, terminal point $\phi(N + \alpha)$, and with the property that $B_N$ and all $\phi_i([0, N])(1 \leq i \leq N)$ are contained in $r_N([N - 1/4, N])$. If $x \in E$ and moreover $N_0 - 1/4 < \phi^{-1}(x) < N_0$, let $f_0(x) = r_{N_0}(\phi^{-1}(x))$. Finally, for $x \in X \setminus A_1$, let $f_0(x) = f_0(\phi(0))$. Then $f_0$ is continuous.

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For $x \in A_1$, let $f_1(x) = \phi(\max\{0, \phi^{-1}(x) + \alpha - 3/4\})$ and for $x \in X \setminus A_1$, let $f_1(x) = \phi(0)$, then $f_1$ is also continuous. Now we show that $\mathrm{Lat}\{f_0, f_1\} = \{X, \emptyset\}$. Note that $f_0(A_1) = X$ from our construction of $f_0$. It follows that for any $b \in X$, there exists $c \in A_1$ such that $f_0(c) = b$. For any $a \in X$, since $f_1(a) \in A_1$, from Remark 4.2, we have $f_{\{f_0,f_1;X;E\}}(f_1(a)) \simeq c$. Thus, $f_0 f f_1(a) \sim b$ and the desired result follows.

If either $\Lambda$ or $\Gamma$ is finite, remove the extra $\phi_i([0, N])$'s and $B_j$'s that are assumed to be contained in those $r_N([N - 1/4, N])$'s, and the same result holds. □

In the following proposition, we show that $\mathbb{R}^n$ has reflexivity index 2 for all $n < \infty$. Equivalently, $\mathbb{C}^n$ and open balls in finite dimensional spaces also have index 2.

Proposition 4.4. *$\mathbb{R}^n$ has the A-property for any $1 \le n \le \infty$. Moreover, $\mathbb{R}^n$ has reflexivity index 2 for $1 \le n < \infty$.*

Proof. We show first that $\mathcal{H} = \mathbb{R}^\infty$ has the $A$-property. Suppose that $\{\xi_m\}_{m=1}^\infty$ is an orthonormal basis of $\mathcal{H}$. Define continuous maps $f_0, f_1$ on $\mathcal{H}$ as follows. Let $1/4 < \alpha < 1/2$ be an irrational number and let $E = \bigcup_{N \ge 1, N \in \mathbb{Z}} (N - 1/4, N) \subset \mathbb{R}$. For $x = (x_1, x_2, \ldots) \in \mathcal{H}$ with $x_1 \notin E$, let $f_0(x) = x + \alpha\xi_1$.

For a positive integer $N$, let $r_N : [N - 1/4, N] \to \mathcal{H}$ be a path with initial point 0, terminal point 0 and such that $\{sN\xi_i : -1 \le s \le 1\} \subset r_N([N - 1/4, N])$ for $1 \le i \le N$. For $x_1 \in E$ and moreover $N_0 - 1/4 < x_1 < N_0$, for some positive integer $N_0$, let $f_0(x) = x + \alpha\xi_1 + r_{N_0}(x_1)$, so $f_0$ is continuous. Define $f_1$ by $f_1(x) = x + (\alpha - 3/4)\xi_1$ for all $x \in \mathcal{H}$. Let $S = \{x \in \mathcal{H} : \|x + \xi_1/8\| < 1/8\}$ and $S$ is an $A_1$-part of $\mathcal{H}$.

Suppose that $a = (a_1, a_2, \ldots) \in \mathcal{H}$. We show first that given any real number $t$ and positive integer $L$, we have $f_{\{f_0,f_1;\mathcal{H};S\}}(a) \simeq a + t\xi_L$. If $L = 1$, then the result follows from Proposition 4.1. If $L > 1$, there exists $c > 0$ such that $r_{[c]+1}(c) = t\xi_L$. It follows that $h_{1\{f_0,f_1\}}(a) \simeq a + (c - a_1)\xi_1$, $f_0(a + (c - a_1)\xi_1) = a + (c - a_1 + \alpha)\xi_1 + t\xi_L$ and $h_{2\{f_0,f_1\}}(a + (c - a_1 + \alpha)\xi_1 + t\xi_L) \simeq a + t\xi_L$. We have $h_2 f_0 h_{1\{f_0,f_1\}}(a) \simeq a + t\xi_L$. For any positive integer $m$, let $P_m$ denote the projection onto the linear subspace spanned by $\{\xi_1, \ldots, \xi_m\}$. Given any $b = (b_1, b_2, \ldots) \in \mathcal{H}$ and $\epsilon > 0$, there exists $M > 0$ such that $\|(1 - P_M)a\| < \epsilon$ and $\|(1 - P_M)b\| < \epsilon$. It follows that $F_{1\{f_0,f_1\}}(P_M a) \simeq P_M a + (b_1 - a_1)\xi_1$, $F_{2\{f_0,f_1\}}(P_M a + (b_1 - a_1)\xi_1) \simeq P_M a + (b_1 - a_1)\xi_1 + (b_2 - a_2)\xi_2, \ldots, F_{M\{f_0,f_1\}}(P_M a + \Sigma_{i=1}^{M-1}(b_i - a_i)\xi_i) \simeq P_M a + \Sigma_{i=1}^{M}(b_i - a_i)\xi_i) = P_M b$. That is, $(\Pi_{i=1}^M F_i)_{\{f_0,f_1\}}(P_M a) \simeq P_M b$. Since $\epsilon$ can be chosen arbitrarily small, this completes the proof that $\mathcal{H}$ has the $A$-property.

If $n$ is finite, replacing $\mathcal{H}$ with $\mathbb{R}^n$ and removing the '$\epsilon - M$' argument gives the same result.

Finally, from [3], we see that all noncompact manifolds do not admit minimal maps, and thus $\kappa(\mathbb{R}^n) > 1$ for all $1 \le n < \infty$. It follows that $\kappa(\mathbb{R}^n) = 2$ for all $1 \le n < \infty$. □

It was shown in [11] that unit spheres have reflexivity index no more than 2. In the next proposition, we show that they all have the $A$-property.

Proposition 4.5. *For $1 \le n \le \infty$, the unit sphere $S^n$ has the A-property.*

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PROOF. Since $S^n$ is glued together by two spaces homeomorphic to $D^n$ when $n$ is finite, the desired result follows from Lemma 3.1. Consider the case when $n = \infty$. Let $\{\xi_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathbb{R}^{\infty}$. By a rotation operator rotating the subspace span$\{\xi_n, \xi_{n+1}\}$, we mean a linear operator $R$ from $\mathbb{R}^{\infty}$ to itself such that $R\xi_i = \cos 2\pi\theta\xi_n + \sin 2\pi\theta\xi_{n+1}$ if $i = n$, $R\xi_i = -\sin 2\pi\theta\xi_n + \cos 2\pi\theta\xi_{n+1}$ if $i = n+1$ and $R\xi_i = \xi_i$ if $i \notin \{n, n+1\}$ for some $n \geq 1$ and $\theta \in \mathbb{R}$.

Let $\{\theta_i\}_{i=1}^{\infty}$ be a sequence of irrational numbers that are rationally independent. Define multi-rotation operators $f_0, f_1$ as follows: let

$$\begin{aligned} f_0\xi_1 &= \xi_1, \\ f_0\xi_{2n} &= \cos 2\pi\theta_n\xi_{2n} + \sin 2\pi\theta_n\xi_{2n+1}, \\ f_0\xi_{2n+1} &= -\sin 2\pi\theta_n\xi_{2n} + \cos 2\pi\theta_n\xi_{2n+1}, \\ f_1\xi_{2n-1} &= \cos 2\pi\theta_n\xi_{2n-1} + \sin 2\pi\theta_n\xi_{2n}, \\ f_1\xi_{2n} &= -\sin 2\pi\theta_n\xi_{2n-1} + \cos 2\pi\theta_n\xi_{2n}, \end{aligned}$$

for $n \geq 1$. From [11], $\mathrm{Lat}(\{f_0, f_1\}) = \{\emptyset, S^{\infty}\}$.

Let $E = \{x \in S^{\infty} : \|x - \xi_1\| < 1\}$, then $x = (x_1, x_2, x_3, \ldots) \in E$ if and only if $x_1 > 1/2$. Since $(x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)/(2x_1 - 1)$ is a homeomorphism from $E$ to $\mathbb{R}^{\infty}$, we see that $E$ is an $A_1$-part of $S^{\infty}$.

Let $P_n$ denote the projection onto the subspace span$\{\xi_i : 1 \leq i \leq n\}$ for a positive integer $n$. Given any $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in S^{\infty}$ and $0 < \epsilon < 1$, there exists an even number $N \geq 4$ such that $\|P_N a\| \geq 1 - \epsilon$, $\|P_N b\| \geq 1 - \epsilon$. Inductively, for $1 \leq n \leq N-1$, there exists a rotation operator $S_n$ rotating the subspace span$\{\xi_n, \xi_{n+1}\}$ such that $P_n(\prod_{i=1}^{n} S_i)(a) = 0$. Now, $S_{N-1}$ can be chosen so that the $N$th coordinate of $(\prod_{i=1}^{N-1} S_i)(P_N a)$ is positive, and we have $(P_N \prod_{i=1}^{N-1} S_i)a = \|P_N a\|\xi_N$. From [9], we see that the SOT-closure $\overline{\{f_0^i\}_{i=1}^{\infty}}$ of $\{f_0^i\}_{i=1}^{\infty}$ contains all rotation operators rotating the subspace span$\{\xi_{2n}, \xi_{2n+1}\}$ and $\overline{\{f_1^i\}_{i=1}^{\infty}}$ contains all rotation operators rotating span$\{\xi_{2n-1}, \xi_{2n}\}$ for $n \geq 1$. It follows that $h_{1\{f_0,f_1;S^{\infty};E\}}(a) \simeq S_1 a$ and inductively, $h_{n\{f_0,f_1\}}((\prod_{i=1}^{n-1} h_i)a) \sim (\prod_{i=1}^{n} S_i)a$ for $1 < n \leq N-1$. Note that $f_0$ acts on the points whose first coordinates are close to 0; in these constructions, the symbol '$\sim$' can be replaced with '$\simeq$'. It follows that $(\prod_{i=1}^{N-1} h_i)_{\{f_0,f_1\}}(a) \simeq d$, where $d$ denotes $\|P_N a\|\xi_N + (1 - P_N)a$.

Denote $\|P_N(b)\|\xi_N + (1 - P_N)b$ by $c$. Inductively, for $1 \leq n \leq N-1$, there is a rotation operator $T_n$ rotating span$\{\xi_{N-n}, \xi_{N-n+1}\}$ so $(P_N - P_{N-n})(b - (\prod_{i=1}^{n} T_i)(c)) = 0$; $T_{N-1}$ can be chosen so that $P_1(b - (\prod_{i=1}^{n} T_i)(c)) = 0$. Similarly, $k_{1\{f_0,f_1\}}(c) \simeq T_1 c$ and inductively for $1 < n \leq N-1$, $k_{n\{f_0,f_1\}}((\prod_{i=1}^{n-1} k_i)c) \simeq (\prod_{i=1}^{n} T_i)c$. It follows that $(\prod_{i=1}^{N-1} k_i)_{\{f_0,f_1\}}(c) \simeq b$.

Denote $\prod_{i=1}^{N-1} h_i$ by $F_1$ and $\prod_{i=1}^{N-1} k_i$ by $F_2$. Since

$$\|F_2F_1a - b\| \leq \|F_2c - b\| + \|F_2\|\,\|F_1a - c\| \leq \|F_2c - b\| + \|F_1a - d\| + \|d - c\|$$

and $\|d - c\| < \epsilon\|\xi_N\| + 2\epsilon = 3\epsilon$, letting $\epsilon \to 0$, we have $(F_2F_1)_{\{f_0,f_1\}}(a) \simeq b$. This implies that $S^{\infty}$ has the $A$-property. □

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Combining Lemma 2.6 and Propositions 4.4 and 4.5 together gives the following proposition.

PROPOSITION 4.6. *Let $\mathcal{N}$ be a well-behaved bounded nest of closed intervals. For $1 \le n \le \infty$, the nests $\mathcal{N} \times \mathbb{R}^n$ and $\mathcal{N} \times S^n$ both have reflexivity index no more than* 2*.*

REMARK 4.7. It would be interesting to determine the reflexivity index of $D^\infty$, $\mathbb{R}^\infty$, or a space glued together by an infinite number of spaces homeomorphic to $D^1$.

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BINGZHANG MA, School of Mathematics,
East China University of Science and Technology,
Shanghai, PR China
e-mail: bingzhangmath@foxmail.com

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