Proceedings of the Edinburgh Mathematical Society (2003) **46**, 501–509 ©
DOI:10.1017/S0013091502000652 Printed in the United Kingdom

POINCARÉ SERIES FOR SEVERAL PLANE DIVISORIAL VALUATIONS

F. DELGADO[1] AND S. M. GUSEIN-ZADE[2]

[1] *Department of Algebra, Geometry and Topology, University of Valladolid, 47005 Valladolid, Spain* (fdelgado@agt.uva.es)

[2] *Faculty of Mathematics and Mechanics, Moscow State University, Moscow 119899, Russia* (sabir@mccme.ru)

(Received 28 June 2002)

Abstract We compute the (generalized) Poincaré series of the multi-index filtration defined by a finite collection of divisorial valuations on the ring $\mathcal{O}_{\mathbb{C}^2,0}$ of germs of functions of two variables. We use the method initially elaborated by the authors and Campillo for computing the similar Poincaré series for the valuations defined by the irreducible components of a plane curve singularity. The method is essentially based on the notions of the so-called extended semigroup and of the integral with respect to the Euler characteristic over the projectivization of the space of germs of functions of two variables. The last notion is similar to (and inspired by) the notion of the motivic integration.

Keywords: multi-index filtrations; Poincaré series; divisorial valuations; germs of functions of two variables

2000 *Mathematics subject classification:* Primary 14B05; 16W70

We compute the (generalized) Poincaré series of the multi-index filtration defined by a finite collection of divisorial valuations on the ring $\mathcal{O}_{\mathbb{C}^2,0}$ of germs of functions of two variables. For one divisorial valuation, this series (which is the usual Poincaré series of the corresponding filtration) has been computed in [**8**]. For a collection of divisorial valuations the study of the Poincaré series was started by Delgado, Galindo and Núñez in [**6**]. They have conjectured the formula of Theorem 2 and recently have proved it using a completely different method. Here we use the method initially elaborated by the authors and by Campillo for computing the similar Poincaré series for the valuations defined by the irreducible components of a plane curve singularity. The corresponding formula (announced in [**2**]) has been proved in [**5**] using another method. The method used here is essentially based on the notions of the extended semigroup (similar to that from [**1**]) and of the integral with respect to the Euler characteristic over the projectivization $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ of the space of germs of functions of two variables. The last notion is similar to (and inspired by) the notion of the motivic integration (see, for example, [**7**]). It was introduced in [**3**] (see also [**4**]). The value of this method consists of the fact that it can be applied to some other multi-index filtrations in rings of functions (e.g. to the filtration

https://doi.org/10.1017/S0013091502000652 Published online by Cambridge University Press

on the ring of germs of functions on a rational surface singularity defined by components of the exceptional divisor of the minimal resolution).

Let $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ be a modification of the complex plane $\mathbb{C}^2$, i.e. a proper analytic map which is an isomorphism outside of the origin in $\mathbb{C}^2$ such that $\mathcal{D} = \pi^{-1}(0)$ is a normal crossing divisor. The modification π is obtained by a sequence of point blow-ups. The exceptional divisor $\mathcal{D}$ is the union of irreducible components E_σ ($\sigma \in G$), each of them is isomorphic to the complex projective line $\mathbb{CP}^1$.

Let E_σ be a component of the exceptional divisor $\mathcal{D}$. For a function g from the ring $\mathcal{O}_{\mathbb{C}^2,0}$ of germs of functions of two variables, let $v_\sigma(g)$ be the multiplicity of the lifting $g \circ \pi$ of the function g to the space $\mathcal{X}$ of the modification along the component E_σ ($v_\sigma(0) = \infty$). The map $v_\sigma : \mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\} \to \mathbb{Z}_{\geqslant 0}$ defines a valuation on the field of quotients of the ring $\mathcal{O}_{\mathbb{C}^2,0}$ (the divisorial valuation defined by the component E_σ). Let us fix r different components $E_{\sigma_1}, \dots, E_{\sigma_r}$ of the exceptional divisor $\mathcal{D}$, let $v_i := v_{\sigma_i}$, $\underline{v} := (v_1, \dots, v_r)$ and $\underline{v}(g) := (v_1(g), \dots, v_r(g))$ ($g \in \mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\}$). The map $\underline{v} : \mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\} \to \mathbb{Z}^r$ defines a multi-index filtration on the ring $\mathcal{O}_{\mathbb{C}^2,0}$: for $\underline{v} \in \mathbb{Z}^r$, the corresponding ideal is $J(\underline{v}) = \{g \in \mathcal{O}_{\mathbb{C}^2,0} : \underline{v}(g) \geqslant \underline{v}\}$. Note that we suppose $J(\underline{v})$ is defined for all $\underline{v} \in \mathbb{Z}^r$ (not only for $\underline{v} \in \mathbb{Z}^r_{\geqslant 0}$).

Let $d(\underline{v}) := \dim J(\underline{v})/J(\underline{v} + \underline{1})$ ($\underline{v} \in \mathbb{Z}^r$, $\underline{1} = (1, \dots, 1)$); one has $d(\underline{v}) < \infty$ (this follows from the fact that, for $k = \max v_i$, the kth power of the maximal ideal of the ring $\mathcal{O}_{\mathbb{C}^2,0}$ is contained in $J(\underline{v})$). Let $\mathcal{L} = \mathbb{Z}[[t_1, \dots, t_r, t_1^{-1}, \dots, t_r^{-1}]]$ be the set of formal Laurent series in the variables $t_1, \dots, t_r$. Elements of $\mathcal{L}$ are expressions of the form $\sum_{\underline{v} \in \mathbb{Z}^r} k(\underline{v}) \cdot \underline{t}^{\underline{v}}$ with $k(\underline{v}) \in \mathbb{Z}$, generally speaking, infinite in all directions (here $\underline{t} = (t_1, \dots, t_r)$, $\underline{t}^{\underline{v}} = t_1^{v_1} \cdot \dots \cdot t_r^{v_r}$). $\mathcal{L}$ is not a ring, but a $\mathbb{Z}[t_1, \dots, t_r]$-module (or even a $\mathbb{Z}[t_1, \dots, t_r, t_1^{-1}, \dots, t_r^{-1}]$-module). The ring of formal power series $\mathbb{Z}[[t_1, \dots, t_r]]$ can, in a natural way, be considered to be embedded into $\mathcal{L}$.

Let $L(t_1, \dots, t_r) = \sum_{\underline{v} \in \mathbb{Z}^r} d(\underline{v}) \cdot \underline{t}^{\underline{v}} \in \mathcal{L}$. One can understand that, along each line in the lattice $\mathbb{Z}^r$ parallel to a coordinate one, the coefficient $d(\underline{v})$ is the same for $\underline{v}$ from the non-positive part of the line, i.e. for $v'_{i_0} < v''_{i_0} < 0$, one has $d(v_1, \dots, v'_{i_0}, \dots, v_r) = d(v_1, \dots, v''_{i_0}, \dots, v_r)$. This implies that

$$P'(t_1, \dots, t_r) = L(t_1, \dots, t_r) \cdot \prod_{i=1}^{r} (t_i - 1)$$

is a series in $t_1, \dots, t_r$, i.e. an element of the subset $\mathbb{Z}[[t_1, \dots, t_r]] \subset \mathcal{L}$.

Definition 1. We call the series

$$P_V(t_1, \dots, t_r) = \frac{P'(t_1, \dots, t_r)}{t_1 \cdots t_r - 1}$$

the Poincaré series of the collection V of the divisorial valuations $v_1, \dots, v_r$.

For $r = 1$, $P(t)$ is the usual Poincaré series of the filtration on the ring $\mathcal{O}_{\mathbb{C}^2,0}$ defined by the valuation $v = v_1$.

https://doi.org/10.1017/S0013091502000652 Published online by Cambridge University Press

For a component E_σ of the exceptional divisor $\mathcal{D}$, let L'_σ be a germ of a smooth curve transversal to E_σ at a smooth point (i.e. not at an intersection point with another component of the exceptional divisor), let $L_\sigma = \pi(L'_\sigma) \subset (\mathbb{C}^2, 0)$ and let $g_\sigma = 0$ ($g_\sigma \in \mathcal{O}_{\mathbb{C}^2,0}$) be an equation of the curve L_σ, let $\underline{m}^\sigma := \underline{v}(g_\sigma)$. Let $\mathring{E}_\sigma$ be the 'smooth part' of the component E_σ, i.e. E_σ without intersection points with all other components of the exceptional divisor $\mathcal{D}$. Let $\chi(X)$ be the Euler characteristic of a space X.

Theorem 2.

$$P_V(t_1, \dots, t_r) = \prod_{\sigma \in G} (1 - \underline{t}^{\underline{m}^\sigma})^{-\chi(\mathring{E}_\sigma)}.$$

The image S_V of the map $\underline{v} : \mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\} \to \mathbb{Z}^r_{\geqslant 0}$ is called the semigroup of values of the divisorial valuations $v_1, \dots, v_r$.

For $\sigma \in G$, the normal bundle ν_σ to the component E_σ in the space $\mathcal{X}$ of the modification is a power ν^{-k_σ} of the canonical line bundle ν over $E_\sigma \simeq \mathbb{CP}^1$. The initial part of the lifting $g \circ \pi$ of a function $g \in \mathcal{O}_{\mathbb{C}^2,0}$ to the space $\mathcal{X}$ of the modification is a section of the line bundle $\nu^{k_\sigma v_\sigma(g)}$. For $v_\sigma \in \mathbb{Z}$, let $\Gamma(E_\sigma, \nu^{k_\sigma v_\sigma})$ be the (finite-dimensional) space of sections of the line bundle $\nu^{k_\sigma v_\sigma}$, let $a_\sigma(g) \in \Gamma(E_\sigma, \nu^{k_\sigma v_\sigma(g)})$ be the initial part of the function $g \circ \pi$ (we call it the initial part of g with respect to the component E_σ). Let $k_i := k_{\sigma_i}$, $a_i(g) := a_{\sigma_i}(g)$, $\underline{a}(g) := (a_1(g), \dots, a_r(g)) \in \Gamma_{\underline{v}} := \bigoplus_{i=1}^r \Gamma(E_{\sigma_i}, \nu^{k_i v_i})$ and let $\Gamma = \bigcup_{\underline{v} \in \mathbb{Z}^r} \Gamma_{\underline{v}}$. Denote by $\Gamma^*_{\underline{v}} = \{(a_1, \dots, a_r) \in \Gamma_{\underline{v}} : a_i \neq 0,\ i = 1, \dots, r\}$, $\Gamma^* = \bigcup_{\underline{v} \in \mathbb{Z}^r} \Gamma^*_{\underline{v}}$ (in fact $\Gamma^*_{\underline{v}} = \emptyset$ for $\underline{v} \not\geqslant \underline{0}$).

One has a map $VA : \mathcal{O}_{\mathbb{C}^2,0} \setminus \{0\} \to \Gamma^*$ defined by $VA(g) = (\underline{v}(g), \underline{a}(g))$. The image $\hat{S}_V$ of this map is a semigroup with respect to the summation of the powers $\underline{v}$ and multiplication of the initial parts $\underline{a}$. Following [**1**], it is called the extended semigroup of the divisorial valuations $v_1, \dots, v_r$. There is a natural map (projection) of Γ^* to $\mathbb{Z}^r$, which sends the component $\Gamma^*_{\underline{v}}$ to $\underline{v} = (v_1, \dots, v_r)$, let $p : \hat{S}_V \to \mathbb{Z}^r_{\geqslant 0}$ be its restriction to $\hat{S}_V$ (its image is the semigroup S_V of the valuations $v_1, \dots, v_r$). The preimages $F_{\underline{v}} = p^{-1}(\underline{v}) \subset \hat{S}_V$ are called fibres of the extended semigroup $\hat{S}_V$.

One has the following description of the fibre $F_{\underline{v}}$. For $\underline{v} \in \mathbb{Z}^r$, there is a natural linear map $j_{\underline{v}} : J(\underline{v}) \to \Gamma_{\underline{v}}$ which sends a function $g \in J(\underline{v})$ to $(a_1, \dots, a_r) \in \Gamma_{\underline{v}}$, where a_i is the part of degree v_i of the lifting $g \circ \pi$ of the function g to the space $\mathcal{X}$ of the modification along the divisor E_{σ_i} (a_i may be equal to zero). Let $D(\underline{v}) \subset \Gamma_{\underline{v}}$ be the image of the map $j_{\underline{v}}$. It is not difficult to see that $D(\underline{v}) \simeq J(\underline{v})/J(\underline{v}+\underline{1})$ and $F_{\underline{v}} = D(\underline{v}) \cap \Gamma^*_{\underline{v}}$. Therefore, for $\underline{v} \in S_V$, the fibre $F_{\underline{v}}$ of the extended semigroup $\hat{S}_V$ is the complement to an arrangement of vector subspaces in the vector space $D(\underline{v})$.

The fibre $F_{\underline{v}}$ is invariant with respect to multiplication by non-zero constants. Let $\mathbb{P}F_{\underline{v}} = F_{\underline{v}}/\mathbb{C}^*$ be the projectivization of the fibre $F_{\underline{v}}$. It is the complement of an arrangement of projective subspaces in the projective space $\mathbb{P}D(\underline{v})$. Let $\mathbb{P}\hat{S}_V = \bigcup_{\underline{v} \in \mathbb{Z}^r_{\geqslant 0}} \mathbb{P}F_{\underline{v}}$ be the projectivization of the extended semigroup $\hat{S}_V$.

https://doi.org/10.1017/S0013091502000652 Published online by Cambridge University Press

Proposition 3.

$$P_V(t_1,\dots,t_r)=\sum_{\underline{v}\in\mathbb{Z}^r_{\geqslant 0}}\chi(\mathbb{P}F_{\underline{v}})\cdot\underline{t}^{\underline{v}}. \tag{1}$$

Proof. For $I\subset I_0=\{1,2,\dots,r\}$, let $\#I$ be the number of elements in I, and let $\underline{1}_I$ be the element of $\mathbb{Z}^r_{\geqslant 0}$, whose ith component is equal to 1 (respectively, to 0) if $i\in I$ (respectively, if $i\notin I$). Let $L_I\subset \varGamma_{\underline{v}}$ be the subspace $\{(a_1,\dots,a_r)\in\varGamma_{\underline{v}} : a_i=0 \text{ for } i\in I\}$.

One has

$$\begin{aligned}\chi(\mathbb{P}F_{\underline{v}})&=\chi(\mathbb{P}D(\underline{v}))-\chi\biggl(\bigcup_{i=1}^{r}\mathbb{P}(D(\underline{v})\cap L_{\{i\}})\biggr))\\&=\chi(\mathbb{P}D(\underline{v}))-\sum_{I\subset I_0,I\neq\emptyset}(-1)^{\#I-1}\chi(\mathbb{P}(D(\underline{v})\cap L_I))\\&=\sum_{I\subset I_0}(-1)^{\#I}\chi(\mathbb{P}(D(\underline{v})\cap L_I))=\sum_{I\subset I_0}(-1)^{\#I}\dim(D(\underline{v})\cap L_I)\end{aligned}$$

(we take into account that the Euler characteristic of the projectivization of a finite-dimensional complex vector space is equal to its dimension). Let us choose an element $\underline{w}$ of $\mathbb{Z}^r_{\geqslant 0}$, and let $b_{\underline{v}}=\dim J(\underline{v})/J(\underline{w})$. If $\underline{v}\leqslant\underline{w}-\underline{1}$, $\dim(D(\underline{v})\cap L_I)=b_{\underline{v}+\underline{1}_I}-b_{\underline{v}+\underline{1}}$ and therefore

$$\chi(\mathbb{P}F_{\underline{v}})=\sum_{I\subset I_0}(-1)^{\#I}(b_{\underline{v}+\underline{1}_I}-b_{\underline{v}+\underline{1}}).$$

This implies that the coefficient at the monomial $\underline{t}^{\underline{v}}$ in the series

$$\Bigl(\sum\chi(\mathbb{P}F_{\underline{v}})\cdot\underline{t}^{\underline{v}}\Bigr)\cdot(t_1\cdot\dots\cdot t_r-1)$$

is equal to

$$\sum_{I\subset I_0}(-1)^{\#I}(b_{\underline{v}+\underline{1}_I-\underline{1}}-b_{\underline{v}})-\sum_{I\subset I_0}(-1)^{\#I}(b_{\underline{v}+\underline{1}_I}-b_{\underline{v}+\underline{1}})$$

and, since $\sum_{I\subset I_0}(-1)^{\#I}=0$, also to

$$\sum_{I\subset I_0}(-1)^{\#I}(b_{\underline{v}-\underline{1}+\underline{1}_I}-b_{\underline{v}+\underline{1}_I})=\sum_{I\subset I_0}(-1)^{\#I}d(\underline{v}-\underline{1}+\underline{1}_I).$$

The coefficient at the monomial $\underline{t}^{\underline{v}}$ in the series

$$P'(\underline{t})=\Bigl(\sum d(\underline{v})\underline{t}^{\underline{v}}\Bigr)\cdot\biggl(\prod_{i=1}^{r}(t_i-1)\biggr)$$

is also equal to $\sum_{I\subset I_0}(-1)^{\#I}d(\underline{v}-\underline{1}+\underline{1}_I)$. □

In what follows we shall use the notion of the integral with respect to the Euler characteristic over the projectivization $\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}$ of the space $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of functions of n variables. This notion was introduced in [**3**] and was inspired by the notion of motivic

https://doi.org/10.1017/S0013091502000652 Published online by Cambridge University Press

integration (see, for example, [**7**]). The 'usual' notion of the integral with respect to the Euler characteristic (over a finite-dimensional space) can be found in [**9**].

Let $J^k_{\mathbb{C}^n,0}$ be the space of k-jets of functions at the origin in $(\mathbb{C}^n,0)$:

$$(J^k_{\mathbb{C}^n,0} = \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{m}^{k+1} \cong \mathbb{C}^{\binom{n+k}{k}}, \text{ where } \mathfrak{m} \text{ is the maximal ideal in the ring } \mathcal{O}_{\mathbb{C}^n,0}).$$

For a complex vector space L (finite or infinite dimensional) let $\mathbb{P}L = (L \setminus \{0\})/\mathbb{C}^*$ be its projectivization, let $\mathbb{P}^*L$ be the disjoint union of $\mathbb{P}L$ with a point (in some sense, $\mathbb{P}^*L = L/\mathbb{C}^*$). One has natural maps

$$\pi_k : \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \to \mathbb{P}^*J^k_{\mathbb{C}^n,0} \quad \text{and} \quad \pi_{k,\ell} : \mathbb{P}^*J^k_{\mathbb{C}^n,0} \to \mathbb{P}^*J^\ell_{\mathbb{C}^n,0}$$

for $k \geqslant \ell$. Over $\mathbb{P}J^\ell_{\mathbb{C}^n,0} \subset \mathbb{P}^*J^\ell_{\mathbb{C}^n,0}$ the map $\pi_{k,\ell}$ is a locally trivial fibration, the fibre of which is a complex affine space of dimension

$$\binom{n+k}{k} - \binom{n+\ell}{\ell}.$$

Definition 4. A subset $X \subset \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}$ is said to be cylindrical if $X = \pi_k^{-1}(Y)$ for a constructible subset $Y \subset \mathbb{P}J^k_{\mathbb{C}^n,0} \subset \mathbb{P}^*J^k_{\mathbb{C}^n,0}$.

This definition means that the condition for a function $g \in \mathcal{O}_{\mathbb{C}^n,0}$ to belong to the set X is a constructible (and $\mathbb{C}^*$-invariant) condition on the k-jet j^kg of the germ g.

Definition 5. For a cylindrical subset $X \subset \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}$ ($X = \pi_k^{-1}(Y)$, $Y \subset \mathbb{P}J^k_{\mathbb{C}^n,0}$), its Euler characteristic $\chi(X)$ is defined as the Euler characteristic $\chi(Y)$ of the set Y.

Let $\psi : \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0} \to A$ be a function with values in an abelian group A.

Definition 6. We say that the function ψ is cylindrical if, for each $a \neq 0$, the set $\psi^{-1}(a) \subset \mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}$ is cylindrical.

Definition 7. The integral of a cylindrical function ψ over the space $\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}$ with respect to the Euler characteristic is

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^n,0}} \psi \, \mathrm{d}\chi := \sum_{a \in A, a \neq 0} \chi(\psi^{-1}(a)) \cdot a$$

if this sum makes sense in A. If the integral exists (makes sense), the function ψ is said to be integrable.

Let $\mathbb{Z}[[\underline{t}]] = \mathbb{Z}[[t_1,\dots,t_r]]$ be the group (with respect to the addition) of formal power series in the variables $t_1,\dots,t_r$. We have the map $\underline{v} : \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0} \to \mathbb{Z}^r_{\geqslant 0}$ and also the map $\underline{v} : \mathbb{P}\hat{S}_V \to \mathbb{Z}^r_{\geqslant 0}$. Let $\underline{t}^{\underline{v}}$ be the corresponding function(s) with values in $\mathbb{Z}[[\underline{t}]]$. One has that

$$\int_{\mathbb{P}\hat{S}_V} \underline{t}^{\underline{v}} \, \mathrm{d}\chi = \sum_{\underline{v} \in \mathbb{Z}^r_{\geqslant 0}} \chi(\mathbb{P}F_{\underline{v}})\underline{t}^{\underline{v}}. \tag{2}$$

For a topological space X, let $S^nX = X^n/S_n$ ($n \geqslant 0$) be the nth symmetric power of the space X. Here S_n is the symmetric group of permutations on n elements which acts

https://doi.org/10.1017/S0013091502000652 Published online by Cambridge University Press

on the Cartesian power X^n of the space X by permuting the factors. The symmetric power S^nX is the space of unordered n-tuples of points of the space X; S^0X is a point. One has the following formula (see, for example, [**5**]):

$$\sum_{n=0}^{\infty}\chi(S^nX)t^n=(1-t)^{-\chi(X)}. \tag{3}$$

Let

$$Y=\prod_{\sigma\in G}\Big(\bigcup_{n=0}^{\infty}S^n\mathring{E}_\sigma\Big)=\bigcup_{\{n_\sigma\}}\Big(\prod_{\sigma\in G}S^{n_\sigma}\mathring{E}_\sigma\Big),$$

where the union is over all sets $\{n_\sigma\}$ of non-negative integers n_σ, $\sigma\in G$.

Consider the map $\underline{v}:Y\to\mathbb{Z}^r_{\geqslant 0}$ which has the value $\sum_{\sigma\in G}n_\sigma\underline{m}^\sigma$ on the connected component $\prod_{\sigma\in G}S^{n_\sigma}\mathring{E}_\sigma$ of the space Y. One has

$$\int_Y\underline{t}^{\underline{v}}\,\mathrm{d}\chi=\prod_{\sigma\in G}\Big(\sum_{n=0}^{\infty}\chi(S^n\mathring{E}_\sigma)\underline{t}^{n\underline{m}^\sigma}\Big)=\prod_{\sigma\in G}(1-\underline{t}^{\underline{m}^\sigma})^{-\chi(\mathring{E}_\sigma)}. \tag{4}$$

Lemma 8. *For each $\underline{v}\in\mathbb{Z}^r_{\geqslant 0}$, the subset $\{g\in\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}:\underline{v}(g)=\underline{v}\}$ of $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ is cylindrical. Therefore, the function $\underline{t}^{\underline{v}(g)}$ on $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ with values in $\mathbb{Z}[[t_1,\dots,t_r]]$ is cylindrical.*

Proof. This follows from the fact that, for $g\in\mathfrak{m}^s$, $v_i(g)\geqslant s$, i.e. the multiplicity of the lifting $g\circ\pi$ of the function g along the component E_i is at least s. □

Therefore, the function $\underline{t}^{\underline{v}}=\underline{t}^{\underline{v}(g)}$ on $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}$ is integrable (since $\sum_{v\in\mathbb{Z}^r_{\geqslant 0}}\ell(\underline{v})\cdot\underline{t}^{\underline{v}}\in\mathbb{Z}[[t_1,\dots,t_r]]$ for any integers $\ell(\underline{v})$).

Proposition 9.

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}}\underline{t}^{\underline{v}}\,\mathrm{d}\chi=\int_{\mathbb{P}\hat{S}_V}\underline{t}^{\underline{v}}\,\mathrm{d}\chi. \tag{5}$$

Proof. The natural map $VA=(\underline{v},\underline{a}):\mathcal{O}_{\mathbb{C}^2,0}\setminus\{0\}\to\hat{S}_V$ is $\mathbb{C}^*$-invariant and thus can be considered as a map $VA:\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}\to\mathbb{P}\hat{S}_V$. For $\underline{v}=(v_1,\dots,v_r)\in\mathbb{Z}^r_{\geqslant 0}$, let $N=1+\max_{1\leqslant i\leqslant r}v_i$ and let $Z_{\underline{v}}=\{j^Ng\in\mathbb{P}J^N_{\mathbb{C}^2,0}:\underline{v}(g)=\underline{v}\}\subset\mathbb{P}J^N_{\mathbb{C}^2,0}$ (recall that the condition $\underline{v}(g)=\underline{v}$ is determined by the N-jet j^Ng of the germ g). Therefore, $\{g\in\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}:\underline{v}(g)=\underline{v}\}=\pi_N^{-1}(Z_{\underline{v}})$. The natural map $Z_{\underline{v}}\to\mathbb{P}F_{\underline{v}}$ (the restriction of the map $VA:\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}\to\mathbb{P}\hat{S}_V$) is a locally trivial fibration whose fibre is a complex affine space. Therefore, $\chi(Z_{\underline{v}})=\chi(\mathbb{P}F_{\underline{v}})$ and thus

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}}\underline{t}^{\underline{v}}\,\mathrm{d}\chi=\sum_{\underline{v}\in\mathbb{Z}^r_{\geqslant 0}}\chi(Z_{\underline{v}})\cdot\underline{t}^{\underline{v}}=\sum_{\underline{v}\in\mathbb{Z}^r_{\geqslant 0}}\chi(\mathbb{P}F_{\underline{v}})\cdot\underline{t}^{\underline{v}}=\int_{\mathbb{P}\hat{S}_V}\underline{t}^{\underline{v}}\,\mathrm{d}\chi.$$

□

Proposition 10.

$$\int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}}\underline{t}^{\underline{v}}\,\mathrm{d}\chi=\int_Y\underline{t}^{\underline{v}}\,\mathrm{d}\chi. \tag{6}$$

https://doi.org/10.1017/S0013091502000652 Published online by Cambridge University Press

Proof. Let us fix $\underline{V} \in \mathbb{Z}^r_{\geqslant 0}$ and suppose that the modification $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ is such that, for any $g \in \mathcal{O}_{\mathbb{C}^2,0}$ with $\underline{v}(g) \leqslant \underline{V}$, the strict transform of the curve $\{g = 0\}$ intersects the exceptional divisor $\mathcal{D}$ of the modification π only at smooth points of $\mathcal{D}$. Such a modification can be obtained, say, from the initial one by a sequence of additional blow-ups at intersection points of components of the exceptional divisor. Though the space Y depends on the modification, it is obvious that, for the space Y constructed for the new modification, the integral $\int_Y \underline{t}^{\underline{v}} \,\mathrm{d}\chi$ is the same (since the Euler characteristics of the smooth parts of the additional divisors are equal to zero).

Let $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}(\underline{V})$ be the set $\{g \in \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0} : \underline{v}(g) \leqslant \underline{V}\}$. From Lemma 8 it follows that the set $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}(\underline{V})$ is cylindrical, i.e. for N big enough, $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}(\underline{V}) = \pi_N^{-1}(\mathbb{P}J^N(\underline{V}))$ for a constructible subset $\mathbb{P}J^N(\underline{V}) \subset \mathbb{P}J^N_{\mathbb{C}^2,0}$.

Define a map $I : \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}(\underline{V}) \to Y$ in the following way. For a function $g \in \mathcal{O}_{\mathbb{C}^2,0}(\underline{V})$ with $\underline{v}(g) \leqslant \underline{V}$, the strict transform of the curve $\{g = 0\}$ intersects the exceptional divisor $\mathcal{D}$ at 'the smooth part' $\dot{\mathcal{D}} = \bigcup_{\sigma \in G} \mathring{E}_\sigma$ of it. The intersection points with the component E_σ counted with the corresponding multiplicities (the intersection numbers) represent an element of the appropriate symmetric power of the space $\mathring{E}_\sigma$. The collection of these sets for all $\sigma \in G$ is an element, $I(g)$, of Y.

One can see that $\underline{v}(I(g)) = \underline{v}(g)$ (here $\underline{v}$ are the corresponding maps from Y and from $\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}(\underline{V})$ to $\mathbb{Z}^r_{\geqslant 0}$, respectively). Moreover, the image of the map I contains the union $Y^{\underline{V}}$ of all the components of Y with $\underline{v} \leqslant \underline{V}$.

Lemma 11. *For g and g' from $\mathcal{O}_{\mathbb{C}^2,0}(\underline{V})$, $I(g) = I(g')$ if and only if $g' = \alpha g + h$, where $\alpha \in \mathbb{C}^*$ and $v_\sigma(h) > v_\sigma(g)$ $(= v_\sigma(g'))$ for any $\sigma \in G$.*

Proof. Consider the liftings $\tilde{g} = g \circ \pi$ and $\tilde{g}' = g' \circ \pi$ of the functions g and g' to the space $\mathcal{X}$ of the modification and let $\psi = \tilde{g}'/\tilde{g}$ be a meromorphic function on $\mathcal{X}$. If $g' = \alpha g + h$ with $\alpha \in \mathbb{C}^*$ and $v_\sigma(h) > v_\sigma(g)$ for any $\sigma \in \Gamma$, then $\psi|_{\mathcal{D}} \equiv \alpha$. Therefore, zeros and poles of ψ cancel each other on the exceptional divisor $\mathcal{D}$ and thus $I(g) = I(g')$. In the other direction, if $I(g) = I(g')$, then zeros and poles of ψ cancel each other on the exceptional divisor $\mathcal{D}$ and thus ψ is constant (say, equal to α) on $\mathcal{D}$. In this case $v_\sigma(g' - \alpha g) > v_\sigma(g) = v_\sigma(g')$. □

Lemma 11 implies that the preimage of a point of the space Y under the map I is a complex affine space. Since the Euler characteristic of a complex affine space is equal to 1, the Fubini formula (applied to the map $I : \mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}(\underline{V}) \to Y^{\underline{V}}$) implies that

$$\int_{Y^{\underline{V}}} \underline{t}^{\underline{v}} \,\mathrm{d}\chi = \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}(\underline{V})} \underline{t}^{\underline{v}} \,\mathrm{d}\chi.$$

Since this equation holds for any $\underline{V} \in \mathbb{Z}^r_{\geqslant 0}$, one has

$$\int_{Y} \underline{t}^{\underline{v}} \,\mathrm{d}\chi = \int_{\mathbb{P}\mathcal{O}_{\mathbb{C}^2,0}} \underline{t}^{\underline{v}} \,\mathrm{d}\chi.$$

□

Theorem 2 is a direct consequence of formulae (1), (2), (4), (5) and (6).

https://doi.org/10.1017/S0013091502000652 Published online by Cambridge University Press

Remark 12. Let $C \subset (\mathbb{C}^2, 0)$ be a curve singularity with r irreducible components such that its strict transform under the modification π consists of germs of smooth curves transversal to the divisors $E_{\sigma_1}, \dots, E_{\sigma_r}$. For this curve, there is defined the Poincaré series $P_C(t_1, \dots, t_r)$ which appears to be a polynomial (for $r > 1$) and which coincides with the Alexander polynomial of the link $C \cap S^3_\varepsilon \subset S^3_\varepsilon$ (see [**5**]). The formula for $P_C(\underline{t})$ from [**5**] (which coincides with the A'Campo type formula of Eisenbud and Neumann for the Alexander polynomial) and Theorem 2 imply the equation

$$P_V(t_1, \dots, t_r) = \frac{P_C(t_1, \dots, t_r)}{\prod_{i=1}^r (1 - \underline{t}^{\underline{m}^{\sigma_i}})}.$$

Remark 13. It is more usual to consider the Hilbert function $h(\underline{v}) = \dim \mathcal{O}_{\mathbb{C}^2,0}/J(\underline{v})$ of the filtration instead of $d(\underline{v})$. For convenience suppose that it is defined for all $\underline{v} \in \mathbb{Z}^r$ (evidently $h(\underline{v}) = 0$ if $\underline{v} \leqslant \underline{0}$, i.e. if $v_i \leqslant 0$ for all $i = 1, \dots, r$). Let

$$H(\underline{t}) := \sum_{\underline{v} \in \mathbb{Z}^r} h(\underline{v}) \cdot \underline{t}^{\underline{v}} \in \mathcal{L}.$$

One has $H(\underline{t}) = L(\underline{t})(1 + \underline{t}^{\underline{1}} + \underline{t}^{\underline{2}} + \cdots)$, where the right-hand side makes sense since $d(\underline{v}) = 0$ for $\underline{v} \leqslant \underline{0}$. From the definition it is not clear that one can directly restore the Laurent series $L(\underline{t}) \in \mathcal{L}$ from the Poincaré series $P_V(\underline{t})$ since, in $\mathcal{L}$, there are elements which become equal to zero after multiplication by $\prod_{i=1}^r (t_i - 1)$ (e.g. $\sum_{k=-\infty}^{+\infty} t_1^k$) and $L(\underline{t})$ is not a power series but contains (infinitely many) terms with negative exponents. However, this can be done as follows. Let

$$\tilde{L}(\underline{t}) = \sum_{\underline{v} \in \mathbb{Z}^r_{\geqslant 0}} d(\underline{v}) \cdot \underline{t}^{\underline{v}} \in \mathbb{Z}[[t_1, \dots, t_r]],$$

$$\tilde{P}'(\underline{t}) = \tilde{L}(\underline{t}) \cdot \prod_{i=1}^r (t_i - 1), \qquad \tilde{P}(\underline{t}) = \tilde{P}'(\underline{t})/(t_1 \cdots t_r - 1).$$

Since $\tilde{L}(\underline{t})$, $\tilde{P}'(\underline{t})$ and $\tilde{P}(\underline{t})$ are power series, they can be restored one from another, e.g.

$$\tilde{L}(\underline{t}) = \frac{\tilde{P}(\underline{t}) \cdot (t_1 \cdots t_r - 1)}{\prod_{i=1}^r (t_i - 1)}.$$

Moreover, the power series $\tilde{L}(\underline{t})$ evidently determines the Laurent series $L(\underline{t})$ (since, for $\underline{v} \not\leqslant -\underline{1}$, $d(\underline{v}) = d(\max(v_1, 0), \dots, \max(v_r, 0))$; for $\underline{v} \leqslant -\underline{1}$, $d(\underline{v}) = 0$). On the other hand, the formula

$$\tilde{P}(\underline{t}) = \sum_{I \subset I_0} (-1)^{\#I} \frac{[P_V(\underline{t})(t_1 \cdots t_r - 1)]|_{\{t_i = 1 \text{ for } i \in I\}}}{t_1 \cdots t_r - 1}$$

($I_0 = \{1, \dots, r\}$) allows us to determine the series $\tilde{P}(\underline{t})$ (and thus the series $\tilde{L}(\underline{t})$) from the series $P_V(\underline{t})$.

Acknowledgements. F.D. was partly supported by grant Mcyt BFM2001-2251. S.M.G.-Z. was partly supported by the grants RFBR-01-01-00739, INTAS-00-259 and Mecd SAB2000–0168.

https://doi.org/10.1017/S0013091502000652 Published online by Cambridge University Press

References

1. A. Campillo, F. Delgado and S. M. Gusein-Zade, The extended semigroup of a plane curve singularity, *Proc. Steklov Institute Math.* **221** (1998), 139–156.
2. A. Campillo, F. Delgado and S. M. Gusein-Zade, The Alexander polynomial of a plane curve singularity, and the ring of functions on the curve, *Russ. Math. Surv.* **54** (1999), 634–635.
3. A. Campillo, F. Delgado and S. M. Gusein-Zade, Integration with respect to the Euler characteristic over the projectivization of the space of functions and the Alexander polynomial of a plane curve singularity, *Russ. Math. Surv.* **55** (2000), 1148–1149.
4. A. Campillo, F. Delgado and S. M. Gusein-Zade, Integrals with respect to the Euler characteristic over the space of functions and the Alexander polynomial, *Proc. Steklov Institute Math.* **238** (2002), 134–147.
5. A. Campillo, F. Delgado and S. M. Gusein-Zade, The Alexander polynomial of a plane curve singularity via the ring of functions on it, *Duke Math. J.*, in press.
6. F. Delgado, C. Galindo and A. Núñez, Generating sequences and Poincaré series for a finite set of plane divisorial valuations, preprint, Valladolid University.
7. J. Denef and F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration. *Invent. Math.* **135** (1999), 201–232.
8. C. Galindo, On the Poincaré series for a plane divisorial valuation, *Bull. Belg. Math. Soc.* **2** (1995), 65–74.
9. O. Y. Viro, *Some integral calculus based on Euler characteristic. Topology and geometry. Rohlin Seminar*, Lecture Notes in Mathematics, vol. 1346, pp. 127–138 (Springer, 1988).

https://doi.org/10.1017/S0013091502000652 Published online by Cambridge University Press