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THE EXTENT OF THE SEQUENCE SPACE ASSOCIATED WITH A BASIS

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1. Introduction. The associated sequence space S of a sequence of vectors $\{x_n\}$ in a Banach space consists of all scalar sequences (s_n) for which $\sum_{n=1}^{\infty} s_n x_n$ converges. My primary motivation in writing this paper was to present a new proof to a recent theorem of N. I. and V. I. Gurarii concerning limits of extent on S when $\{x_n\}$ is a basis of a uniformly convex or a uniformly smooth Banach space [**5**]. This theorem is stated as Theorem 2.4. Several interesting consequences of this theorem were noted by N. I. Gurarii in [**3**] and [**4**]. For instance he showed that for each pair of numbers p, q with $1 < p < q < \infty$ there is a basis $\{x_n\}$ of l^2 with $0 < \inf_n ||x_n|| < \sup_n ||x_n|| < \infty$ such that if $l^r \subset S$ then $r \leq p$ while if $l^s \supset S$ then $s \geq q$. Our Theorem 3.2 adds to this result in determining minimum sizes of l^s and maximum sizes of l^r for X a subspace of l^t or L^t. Finally in Theorem 3.3 we derive a summability property of a basis in terms of S (formula (3.1)). From this and the Gurarii theorem it follows that no basis for a uniformly convex or uniformly smooth space can be "purely conditional" (Corollary 3.4).

2. Growth numbers and the theorem of N. I. and V. I. Gurarii.

2.1 *Definition.* Let $\{x_i : i = 1, 2, \ldots\}$ be a sequence of vectors in a Banach space X having norm $|| \ ||$. The *associated sequence space* of (x_i), written $S(x_i)$ or simply S, consists of all scalar sequences (t_i) for which $\sum_i t_i x_i$ converges. The nth *growth number*, written $g(n, \{x_i\})$ or simply $g(n)$, is given by the formula

$$g(n, \{x_i\}) = \sup\{||\textstyle\sum_{i \in F} a_i x_i|| : F \text{ is any set of } n \text{ indices, } |a_i| \leq 1, i \in F\}.$$

The following proposition states a few obvious properties of the growth numbers, and we omit its proof.

2.2 Proposition. (a) $g(1) = \sup_n ||x_n||$.

(b) $g(n) \leq g(n+1)$ *for each* n.

(c) *If* $\{x_i\}$ *is a normalized sequence* (*i.e.*, $||x_i|| = 1$ *for each* i), *then* $1 \leq g(n) \leq n$ *for each* n.

(d) *If* $\{x_{i_k}\}$ *is any subsequence of* $\{x_i\}$, *then*

$$g(n, \{x_{i_k}\}) \leq g(n, \{x_i\}) \quad (n = 1, 2, \ldots).$$

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(e) *If* $\{x_{\pi(i)}\}$ *is any permutation of* $\{x_i\}$, *then*

$$g(n, \{x_{\pi(i)}\}) = g(n, \{x_i\}) \quad (n = 1, 2, \ldots).$$

The following theorem shows how the growth numbers of a vector sequence provides a measure of the size of the associated sequence space.

2.3 THEOREM. *Let* $\{x_i\}$ *be a sequence of vectors in a Banach space* X, *with* S *denoting the associated sequence space and* $g(n)$ *the nth growth number.*

(a) *If* $l^p \subseteq S$ *for* $p \geq 1$, *then* $g(n) = O(n^{1/p})$.

(b) *If* $g(n) = O(n^{1/p})$ *for* $p > 1$, *then* $l^s \subseteq S$ *for* $1 \leq s < p$.

(c) $c_0 \subseteq S$ *if and only if* $g(n) = O(1)$.

Proof. (a) It is well known (e.g., see [**7**]) that S is a BK-space (Banach space of sequences with continuous coefficients) given the norm

$$|||(t_i)||| = \sup_n ||\textstyle\sum_{i=1}^{n} t_i x_i||((t_i) \in S).$$

Moreover, the sequence $\{e_i : i = 1, 2, \ldots\}$ of coordinate vectors

$$e_i = \{\delta_{ij} : j = 1, 2, \ldots\}$$

forms a Schauder basis for S.

If $l^p \subseteq S$ the inclusion is continuous [**14**, § 11.3]. Hence there is $M > 0$ such that for $(t_i) \in l^p$

$$||\textstyle\sum_{i=1}^{\infty} t_i x_i|| \leq |||(t_i)||| \leq M||(t_i)||p$$

where $|| \quad ||_p$ is the usual norm on l^p. If F is a finite set of indices and $|a_i| \leq 1$ for $i \in F$ we thus have

$$\begin{aligned}||\textstyle\sum_{i\in F} a_i x_i|| &\leq |||\textstyle\sum_{i\in F} a_i e_i||| \\ &\leq M||\textstyle\sum_{i\in F} a_i e_i||_p \leq M n^{1/p}.\end{aligned}$$

(b) We define an extended real valued function N on the set of all sequences by the formula

$$N((t_i)) = \sup\{||\textstyle\sum_{i\in F} a_i t_{\pi(i)} x_i||: F \text{ is a finite set of indices, } |a_i| \leq 1, \pi \text{ is any permutation of indices}\}.$$

Then N is a balanced symmetric sequential norm in the sense of [**8**] and [**9**] so that S_N, the set of all (t_i) for which $N((t_i)) < \infty$, is a symmetric BK-space with the norm N. The sequence $\{e_i\}$ is a symmetric basis for its closed linear span S_N° in S_N. It is not hard to see (continuity of inclusion) that $(t_i) \in S_N^\circ$ if and only if $\sum_{n=1}^{\infty} t_i x_{\pi(i)}$ converges unconditionally in X for each permutation π on the indices. Consequently we have $S_N^\circ \subset S_N$.

It is obvious that

$$g(n) = N(e_1 + e_2 + \ldots + e_n).$$

Let $g_n' = g(n) - g(n-1)$ and let $(g_n')^\sigma$ consist of all sequences $(s_i)^n$ for which

$$(2.1) \qquad ||(s_i)||' = \sup\{\textstyle\sum_{i=1}^{\infty} |s_{\pi(i)}|g_i' : \pi \text{ is a permutation of indices}\} < \infty .$$

With the norm $|| \; ||'$, $(g_n')^\sigma$ is a symmetric BK-space, and $(g_n')^\sigma \subset S_N$ by [**9**, Proposition 3.2].

Let $h_n' = n^{1/p} - (n-1)^{1/p}$ and let $(h_n')^\sigma$ be defined by (2.1) with h_n' replacing g_n' in (2.1). If $g(n) = O(n^{1/p})$ then $(h_n')^\sigma \subseteq (g_n')^\sigma$ by [**8**, Proposition 3.6]. By [**9**, Proposition 3.6], $(h_n')^\sigma$ properly contains l^s for $1 < s < p$. Thus we have

$$\bigcup_{s<p} l^s \subseteq h'^\sigma = g'^\sigma \subseteq S_N.$$

For $s < p$ the inclusion of l^s in S_N is continuous so that

$$l^s \leq S_N{}^\circ \subset S,$$

since $\{e_i\}$ is a basis for l^s. The inclusion is obviously proper since

$$l^s \underset{\neq}{\subset} \bigcup_{n<p} l^r \subset S.$$

(c) If $c_0 \subseteq S$ then $\{g(n)\}$ is bounded by an argument like that used to prove (a).

If $\{g(n)\}$ is bounded then $\sum_n g_n' = \sum_n\{g(n) - g(n-1)\}$ converges so that $(g_n')^\sigma \supseteq m$ by [**8**, Proposition 3.2]. Thus we have $m \subseteq S_N$ which implies

$$c_0 \subseteq S_N{}^\circ \subseteq S.$$

We now present a new proof of the theorem of N. I. and V. I. Gurarii [**4**; **5**].

2.4 THEOREM. *Let $\{x_n\}$ be a Schauder basis in a Banach space X for which there is m and $M > 0$ such that $\inf_n||x_n|| \geq m$ and $\sup_n||x_n|| \leq M$; let S be the associated sequence space of $\{x_n\}$.*

(a) *If X is uniformly convex, then there is $r > 1$ such that $l^r \subset S$ and the inclusion is continuous.*

(b) *If X is uniformly smooth, then there is $s < \infty$ such that $S \subset l^s$ and the inclusion is continuous.*

Proof. (a) Since $\{x_n\}$ is a Schauder basis for X there is $\delta > 0$ such that for each n

$$(2.2) \qquad ||\textstyle\sum_{i=1}^{n} a_i x_i - \sum_{i=n+1}^{\infty} a_i x_i|| \geq \delta$$

whenever $||\sum_{i=1}^{n} a_i x_i|| = 1$ or $||\sum_{i=n+1}^{\infty} a_i x_i|| = 1$. (Let

$$\delta = \inf_n\{||P_n||_n{}^{-1}, ||I - P_n||^{-1}\} \quad \text{where} \quad P_n(\textstyle\sum_{i=1}^{\infty} a_i x_i) = \sum_{i=1}^{n} a_i x_i;$$

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see [**1**, p. 67]). Since X is uniformly convex there is $\epsilon > 0$ such that

$$(2.3) \qquad ||x + y|| \leq 2(1 - \epsilon)$$

whenever $||x||$ and $||y||$ are ≤ 1 and $||x - y|| \geq \delta$ [**1**, p. 112]. Using these two facts we shall prove that

$$(2.4) \qquad g(2n) \leq 2(1 - \epsilon)g(n)$$

for each n. Let F be any set of $2n$ indices and let $|a_i| \leq 1$ for $i \in F$. Let F_1 be the set of the n smallest indices in F and F_2 the complement of F_1 in F. If

$$C = \max\{||\textstyle\sum_{i \in F_1} a_i x_i||, ||\sum_{i \in F_2} a_i x_i||\} \neq 0$$

we then have

$$||\textstyle\sum_{i \in F} (a_i/C)x_i - \sum_{i \in F_2} (a_i/C)x_i|| \geq \delta$$

by (2.2). By (2.3)

$$(2.5) \qquad ||\textstyle\sum_{i \in F} a_i x_i|| \leq 2(1 - \epsilon)C \leq 2(1 - \epsilon)g(n).$$

If $C = 0$ then (2.5) is obviously true. The inequality (2.4) now quickly follows from (2.5).

We now use (2.4) to show that there is $r > 0$ for which $g(n) = O(n^{1/r})$. From (2.4) it follows that $g(2^n) \leq M2^n(1 - \delta)^n$ for $n = 0, 1, 2, \ldots$. Let d_n be defined by

$$d_n = [2(1 - \delta)]^{\log_2 n}.$$

Since $\{g(k)\}$ and $\{d_k\}$ are increasing sequences and

$$2Md_{2^n} \geq g(2^{n+1})$$

for each n, it follows that $g(n) = O(d_n)$. If p is any number with $1 > 1/p \geq 1 + \log_2(1 - \delta)$, we have $d_n = O(n^{1/p})$ since

$$\begin{aligned} \log_2(d_n/n^{1/p}) &= \log_2 n\{1 + \log_2(1 - \delta) - 1/p\} \\ &\leq 0. \end{aligned}$$

Thus if $1/r > 1 + \log_2(1 - \delta)$, $l^r \subset S$ by Theorem 2.2 (b).

(b) If X is uniformly smooth then X^* is uniformly convex [**1**, (8), p. 114] and X is reflexive. The coefficient functionals $\{f_i\}$ of the basis $\{x_i\}$ form a basis of X^*. Since $\inf_n||x_n|| > 0$, $\sup_n||f_n|| < \infty$. Thus by the argument of (a) there is $s' > 1$ such that $l^{s'} \subset T$ where T is the associated sequence space of $\{f_n\}$. The inclusion $l^{s'} \subset T$ implies $T^\gamma \subset l^s$ where

$$T^\gamma = \{(s_i) : \sup_n\{|\textstyle\sum_{i=1}^n s_i t_i| < \infty \text{ for each } (t_i) \in T\}$$

and $1/s + 1/s' = 1$. However, $T^\gamma = S$ the associated sequence space of $\{x_n\}$ by [**7**, Corollary 3.3].

3. Further results on the associated sequence space.

3.1 PROPOSITION. *Suppose* $\{x_n\}$ *is a sequence of vectors in a Banach space* X *such that* $l^p \subset S(\{x_n\})$.

(a) *If* $X = l^1$, l^r *or* L^r *with* $1 < r \leq 2$ *and* $p > 2$, *then* $\sum_{n=1}^{\infty} ||x_n||^s < \infty$ *where* $s = 2p/(p-2)$.

(b) *If* $X = l^r$ *or* L^r *with* $2 \leq r < \infty$ *and* $p > r$, *then* $\sum_{n=1}^{\infty} ||x_n||^s < \infty$ *where* $s = pr/(p-r)$.

Proof. (a) If $l^p \subset S$ then for each (s_i) in l^p, $(s_i x_i)$ is unconditionally convergent. This is because $(s_i(e_i))$ converges unconditionally, the inclusion from l^p into S is continuous, and the operator $T(s_i) = \sum_{i=1}^{\infty} s_i x_i$ is continuous, from S into X. Consequently $(||s_i x_i||)$ is in l^2 for each $(s_i) \in l^p$ (see [**1**, p. 63]). This means that $(||x_i||)$ determines a diagonal operator from l^p into l^2, so that $(||x_i||)$ must be in the space l^s where $s = 2p/(p-2)$. (See [**10**]; note however that the values in the 3rd column of the table on p. 48 should be $l^{pq/(p-q)}$ and $l^{(pq-p+q)/pq}$.)

(b) The proof of this assertion is like that of (a).

3.2 THEOREM. *Let* $\{x_n\}$ *be a basis of a Banach space* X *such that* $\inf_n ||x_n|| = m > 0$ *and* $\sup_n ||x_n|| = M < \infty$. *Denote the associated sequence space of* $\{x_n\}$ *by* S.

(a) *If* X *is Hilbert space and* $S \supset l^p$, *then* $p \leq 2$; *if* $S \subset l^p$, *then* $p \geq 2$.

(b) *If* X *is a subspace of* l^r *or* L^r *with* $1 < r < 2$ *and* $l^p \supset S$, *then* $p \leq 2$; *if* $S \supset l^p$, *then* $p \geq r$.

(c) *If* X *is a subspace of* l^r *or* L^r *with* $2 < r < \infty$ *and* $l^p \subset S$, *then* $p \leq r$; *if* $S \supset l^p$ *then* $p \geq 2$.

Proof. We shall prove only (c); the proofs of (a) and (b) are similar.

(c) If $l^p \subset S$ then $p \leq r$ by (b) of 3.1 plus the hypothesis that $\inf_n ||x_n|| > 0$. The space S is isomorphic to a subspace of l^r or L^r so S^γ is isomorphic to a subspace of $l^{r'}$ or $L^{r'}$ where $1 < r < 2$ and $1/r + 1/r' = 1$. Since S is reflective $\{e_i\}$ forms a basis in S and since $\sup_n ||x_n|| < \infty$, $\inf_n ||e_n|| > 0$. The inclusion $S \supset l^p$ implies $S^\gamma \subset l^{p'}$ where $1/p + 1/p' = 1$. By (a) of 3.1 we conclude that $p' \leq 2$ or $p \geq 2$.

3.3 THEOREM. *Let* $\{x_n\}$ *be a basis of a Banach space* X *such that* $\inf_n ||x_n|| = m > 0$ *and* $\sup_n ||x_n|| = M < \infty$. *Denote the associated sequence space of* $\{x_n\}$ *by* S *and the biorthogonal sequence of coefficient functionals by* $\{f_n\}$. *If for* $1 \leq p < \infty$ *either* (a) $S \subset l^p$, *or* (b) $l^{p'} \subset S$ *where* $1/p + 1/p' = 1$, *then for each* $x \in X$ *and* $f \in X^*$ *we have*

$$\sum_{n=1}^{\infty} |f_n(x) f(x_n)|^p < \infty. \tag{3.1}$$

Proof. The conclusion that (a) implies (3.1) follows trivially from the fact that $(f_n(x)) \in l^p$ and $\sup_n ||f_n|| < \infty$.

If (b) is valid then $l^{p'} \subset M(S)$ the multiplier algebra of S [**6**]. For, if $(u_i) \in l^{p'}$ and $(s_i) \in S$, then (s_i) is a bounded sequence because $m > 0$ so that $(u_i s_i) \in l^p \subset S$. Since $l^{p'} \subset M(S)$, $M(S)^\gamma \subset l^p$. But $M(S)^\gamma$ contains all sequences of the form $(f_n(x)f(x_n))$ where $x \in X$ and $f \in X^*$ [**11**].

3.4 COROLLARY. *Let $\{x_n\}$ be a basis for a Banach space X with coefficient functionals $\{f_n\}$. If X is either uniformly convex or uniformly smooth there is $p < \infty$ such that for each x in X and f in X^**

$$\sum_{n=1}^{\infty} |f(x_n) f_n(x)|^p < \infty.$$

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