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CORRECTIONS TO "SEMINORMAL RINGS AND WEAKLY NORMAL VARIETIES"

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§ 1. Introduction

In "Seminormal rings and weakly normal varieties" we introduced the notion of a c-regular function on an algebraic variety defined over an algebraically closed field of characteristic zero. Our intention was to describe those k-valued functions on a variety X that become regular functions when lifted to the normalization of X, but without any reference to the normalization in the definition. That is, we aspired to identify the c-regular functions on X with the regular functions on the weak normalization of X.

Originally we defined a c-regular function to be a continuous k-valued function that is regular off the singular locus of X. This does not provide the desired characterization as is evidenced by the following observation (cf. Example 3.3). For any weakly normal singular curve X we can delete finitely many points from the normalization of X to obtain a curve X' that is homeomorphic to X via a birational morphism but is not isomorphic to X as varieties. A regular function on X' that is not the restriction of a regular function on the normalization of X provides us with an example of a continuous function on X that is regular off the singular locus but does not lift to a regular function on the normalization. This is due to the very special nature of the Zariski topology in dimension one. If we avoid one-dimensional components we can satisfactorily characterize those k-valued functions on X that lift to regular functions on the normalization of X but without referring to the normalization. This characterization hinges on the fact that if we avoid one-dimensional components a morphism onto a weakly normal variety that is a homeomorphism is necessarily an isomorphism (Theorem 3.7). In order to

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provide a unified treatment we will now define a c-regular function on a variety in terms of its normalization and subsequently offer alternative characterizations.

§ 2. Preliminaries and Notation

Let k be a fixed algebraically closed field of characteristic zero. When we use the term variety we assume that the underlying topological space is the set of closed points of a reduced, separated scheme of finite type over k. If A is a reduced, finitely generated k-algebra we let $\mathrm{Var}(A)$ denote the affine variety whose affine coordinate ring is A. For a variety X we let $S(X)$ denote the singular locus of X. By component of X we mean an irreducible component of X.

All rings in this paper are commutative with identity. For a ring A we let $R(A)$ denote the Jacobson radical of A. Let m be an element of an A-module M and x be a point of $\mathrm{Spec}(A)$ corresponding to the prime ideal P of A. We let m_x and $m(x)$ denote the images of m under the canonical homomorphisms $M \to M_x$ and $M \to M_x/PM_x$, respectively. We let $k(P)$ denote the residue class field A_x/PA_x.

We now recall Traverso's notion of seminormality. Let $A \subseteq B$ be an integral extension of rings. We define the seminormalization of A in B, denoted ${}^+_BA$ by

$$ {}^+_BA = \{b \in B \mid b_x \in A_x + R(B_x) \text{ for all } x \in \mathrm{Spec}(A)\}. $$

We say A is seminormal in B if $A = {}^+_BA$. If B is the normalization of A (i.e., the integral closure of A in its total ring of quotients) we write ${}^+A$ in lieu of ${}^+_BA$ and call ${}^+A$ the seminormalization of A. The ring A is said to be seminormal if $A = {}^+A$. A variety X is said to be seminormal at a point $x \in X$ if $\mathcal{O}_{X,x}$ is seminormal. X is said to be seminormal if X is seminormal at each point $x \in X$.

An integral extension $A \subseteq B$ of rings is said to be subintegral if for each prime ideal P of A there is a unique prime ideal Q of B lying over P and the canonical homomorphism $k(P) \to k(Q)$ is an isomorphism. Traverso characterized ${}^+_BA$ as the largest subintegral extension of A in B ([9], (1.1)). Traverso also showed that for a reduced noetherian ring A with finite normalization and a finite set of indeterminates X, A is seminormal if and only if the canonical homomorphism of Picard groups $\mathrm{Pic}(A) \to \mathrm{Pic}(A[X])$ is an isomorphism ([9], Theorem 3.6).

In 1980 Swan defined a seminormal ring as a reduced commutative ring A such that whenever $b, c \in A$ are such then $b^3 = c^2$ that there exists an element $a \in A$ such that $a^2 = b$ and $a^3 = c$. Swan pointed out that if the total ring of quotients of A is a product of fields then his notion and that of Traverso are equivalent. Swan constructed the seminormalization of a ring in a manner analogous to that used in the construction of the algebraic closure of a field. He used his modified version of seminormality to show that for an arbitrary ring A, $A/\mathrm{nil}(A)$ is seminormal if and only if $\mathrm{Pic}(A) \to \mathrm{Pic}(A[X])$ is an isomorphism. Here $\mathrm{nil}(A)$ denotes the nilradical of A.

§ 3. Revised Definitions and Theorems

Let π: $\tilde{X} \to X$ be the normalization of an affine variety $X = \mathrm{Var}(A)$. Recall that the seminormalization ${}^+A$ of A may be identified with the set of regular functions on X that are constant on the fibers of π ([4], Theorem 2.2). Let ${}^+X = \mathrm{Var}({}^+A)$ and let ρ: ${}^+X \to X$ be the induced morphism; ρ: ${}^+X \to X$ is called the seminormalization of X. As remarked above, any regular function on $\tilde{X}$ that is constant on the fibers of π determines a regular function on ${}^+X$.

Originally we defined a c-regular function on X to be a k-valued function on X that is globally continuous (with respect to the Zariski topology on both X and k) and is regular off the singular locus $S(X)$ of X. We were attempting to characterize those functions on X that lift to regular functions on ${}^+X$. We will give an example that shows this definition fails. First we prove a lemma.

LEMMA 3.1. *Let φ be a k-valued function on the affine variety $X = \mathrm{Var}(A)$. The function $(1, \varphi)$: $X \to X \times k$ is continuous if and only if every polynomial function in φ with coefficients in A is continuous.*

Proof. Suppose that $(1, \varphi)$ is continuous. Let $\psi = \sum a_i \varphi^i$, $a_i \in A$. Since proper closed subsets of k are finite sets of points, ψ is continuous if and only if $\psi^{-1}(\alpha)$ is closed in X for each α in k. Consider the regular function $g = \sum a_i t^i - \alpha$ on $X \times k = \mathrm{Var}(A[t])$. Since $\psi^{-1}(\alpha) = (1, \varphi)^{-1}(Z(g))$ we see that $\psi^{-1}(\alpha)$ is closed in X.

Conversely, assume that every polynomial in φ with coefficients in A defines a continuous k-valued function on X. Consider an element $h = \sum a_i t^i$ of $A[t]$ and let $\psi = \sum a_i \varphi^i$. Since $(1, \varphi)^{-1}(Z(h)) = \psi^{-1}(0)$ we know that $(1, \varphi)^{-1}(Z(h))$ is closed in X. Since every closed subset of $X \times k$ is a finite intersection of sets of the form $Z(h)$ this implies the continuity of $(1, \varphi)$.

DEFINITION 3.2. We say that a k-valued function φ on a variety X is p-continuous if $(1, \varphi)\colon X \to X \times k$ is continuous.

EXAMPLE 3.3. Let $X \subseteq A^2$ denote the affine plane curve defined by the equation $y^2 = x^2 + x^3$. The normalization of X is given by $\pi\colon A^1 \to X$ where $\pi(t) = (t^2 - 1, t(t^2 - 1))$.

Consider $V = A^1 - \{1\}$ and let $\eta = \pi|_V$. Notice that η is a birational morphism of affine varieties and is a homeomorphism. Let $f\colon V \to k$ denote the regular function defined by $f(t) = 1/(t - 1)$ and let $\varphi = f \circ \eta^{-1}$. Observe that φ is globally continuous and is regular at each nonsingular point of X. Since the affine coordinate ring of X is seminormal this shows our original definition of a c-regular function was incorrect. We shall further see that $(1. \varphi)\colon X \to X \times k$ is continuous and the graph of φ is closed in $X \times k$.

Let A denote the affine coordiate ring of X so that $A[z]$ is the affine coordinate ring of $X \times k$. Consider $\psi = \sum a_i \varphi^i$ where the coefficients are elements of A. Since $\psi = (\sum (a_i \circ \eta) f^i) \circ \eta^{-1}$ we see that ψ is continuous on X. By the preceding lemma we may conclude that φ is p-continuous.

As $\pi \times 1\colon A^1 \times k \to X \times k$ is a finite morphism it is a closed mapping. Let Γ denote the graph of f. Notice that Γ is the zero set of the regular function $(t - 1)z - 1$ on $A^1 \times k$ and hence Γ is closed in $A^1 \times k$. Since the graph of φ is $(\pi \times 1)(\Gamma)$ it is closed in $X \times k$.

Let $Y = X \times k \subseteq A^3$ and let $p\colon Y \to X$ denote the projection onto the first factor. Let $\psi = \varphi \circ p$. Since ψ is the composition of continuous functions ψ is continuous. We will show that ψ is not p-continuous. To see this we show that $z\psi - 1$ is not continuous where z is the third coordinate function on A^3.

Let $T = \{(x, y, z) \in Y \mid z\psi(x, y, z) - 1 = 0\} = \{(x, y, z) \in Y \mid z\varphi(x, y) = 1\}$. Observe that $T = \{(t^2 - 1, t(t^2 - 1), t - 1) \mid t \in k, t \neq 1\}$ and hence $\overline{T} = \{(t^2 - 1, t(t^2 - 1), t - 1) \mid t \in k\}$. Thus $z\psi - 1$ is not continuous on Y. Notice that $S(Y) = \{0\} \times k$ and hence ψ is regular off the singular locus of Y.

DEFINITION 3.4. Let $\pi\colon \tilde{X} \to X$ denote the normalization of a variety X. We define the sheaf of c-regular functions on X, denoted $\mathcal{O}_X^c$, as follows. For an open subset U of X define $\Gamma(U, \mathcal{O}_X^c) = \{\varphi\colon U \to k \mid \varphi \circ \pi \in \Gamma(\pi^{-1}(U), \mathcal{O}_{\tilde{X}})\}$.

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We say X is weakly normal at a point $x \in X$ if $\mathcal{O}_{X,x} = \mathcal{O}^c_{X,x}$. We say that X is weakly normal if $\mathcal{O}_X = \mathcal{O}^c_X$.

Notice that if U is an affine open subset of X then the set of c-regular functions on U may be identified with the seminormalization of $\Gamma(U, \mathcal{O}_X)$. Since the operations of seminormalization and localization commute ([4], Corollary 1.6), $\mathcal{O}^c_X$ is a coherent $\mathcal{O}_X$-module. As $\mathcal{O}^c_{X,x} = {}^+\mathcal{O}_{X,x}$ X is weakly normal at a point x of X if and only if X is seminormal at x.

We would like to point out that our definition of a weakly normal variety is not standard. Given an integral extension of rings $A \subseteq B$ there is an A-subalgebra A' of B called the weak normalization of A in B. It is characterized by the fact that it is the largest A-subalgebra C of B such that for each prime P of A there exists a unique prime Q in C lying over P and the induced homomorphism of fields $k(P) \to k(Q)$ is purely inseparable. If B is the normalization of A the ring A' is called the weak normalization of A. A is said to be weakly normal if $A = A'$. The scheme-theoretic operation of weak normalization was developed by Andreotti and Bombieri in [1] and is based on this algebraic formulation of weak normality. Since we are working over a field of characteristic zero the algebraic notions of seminormality and weak normality coincide and hence so do the geometric notions. Thus we feel free to use the expression "weakly normal variety" in our own sense. Recall that we say a variety is weakly normal if every c-regular function is regular.

The following results enable us to provide an alternative description of a c-regular function on a variety without one-dimensional components.

LEMMA 3.5. *Let Z be an affine variety, $U \subseteq Z$ an open dense subset, $u \in U$, $z \in Z - U$ and assume that there exists a component Z' of Z that contains z, meets U, and has dimension at least two. Then, there exists an irreducible closed subset H of Z such that $z \in H$, $u \notin H$ and $H \cap U \neq \emptyset$.*

Proof. We may and shall assume that Z is a closed subvariety of A^n for some positive integer n. For each component of $Z' - (U \cap Z')$, except possibly $\{z\}$, choose a point $z_i \neq z$ in that component. Let $L \subseteq A^n$ be a hypersurface such that $z \in L$, $u \notin L$ and $z_i \notin L$ for all i. Let H be an irreducible component of $Z' \cap L$ containing z. By Krull's Haupidealsatz H has the required porperties.

THEOREM 3.6. *Let f: $Y \to X$ be a morphism of varieties without one-*

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dimensional components defined over an algebraically closed field of arbitrary characteristic. If f is a homeomorphism (with respect to the Zariski topology on both X and Y), then f is a finite morphism.

Proof. Replacing X and Y be the respective unions of all components of dimension at least two we may assume that all components have dimension at least two.

By Grothendieck's form of Zariski's Main Theorem ([3], Théorème 8.12.6) there is an open immersion of varieties $g: Y \to Z$ and a finite morphism $h: Z \to X$ such that $h \circ g = f$. Identifying Y with its image in Z, and modifying f appropriately, we may assume that Y is an open subvariety of Z. Replacing Z by $\overline{Y}$ we may further assume that Y is dense in Z.

We claim that h is also a homeomorphism. Since h is a surjective closed mapping it suffices to show that h is injective. Suppose not. Then there is a point $x \in X$ such that the fiber $h^{-1}(x)$ contains at least 2 points. Let U be an affine open neighborhood of x in X and let $V = h^{-1}(U)$. Replacing X, Y and Z by U, $Y \cap V$ and V, respectively, we may assume that $h: Z \to X$ is a finite morphism of affine varieties and that all components of X and Z have dimension at least two.

Since $h^{-1}(x)$ contains more than one point and $h|_Y = f$ is a homeomorphism there exist points $y \in Y$ and $z \in Z - Y$ in $h^{-1}(x)$. By the preceding lemma there exists an irreducible closed subset H of Z such that $z \in H$, $y \notin H$ and $Y \cap H \neq \emptyset$.

Now $Y \cap H$ is a closed subset of Y implies $f(Y \cap H)$ is closed in X. But $x \notin f(Y \cap H) = h(Y \cap H)$, whereas $x \in h(H) = h(\overline{Y \cap H}) \subseteq \overline{h(Y \cap H)}$. This is the desired contradiction. Thus h is a homeomorphism. Hence $Y = Z$ and f is a finite morphism.

THEOREM 3.7. *Let X be a seminormal variety without one-dimensional components and let $f: Y \to X$ be a morphism of varieties. If f is a homeomorphism, then f is an isomorphism of varieties.*

Proof. By Theorem 3.6 f is a finite morphism. In particular we may reduce to the case where X and Y are affine with affine coordinate rings A and B, repectively. Then $A \subseteq B$ is an integral extension of rings and the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a homeomorphism. For suppose Q, Q' in $\mathrm{Spec}(B)$ both contract to P in $\mathrm{Spec}(A)$. Let N be a maximal ideal of B containing Q and set $M = N \cap A$. By Going Up there is a maximal ideal N' of B such that $Q' \subseteq N'$ and $N' \cap A = M$. Thus $N = N'$ so that every maximal ideal of B that contains Q also contains Q'. Hence $Q \subseteq Q'$. Similarly $Q' \subseteq Q$ and thus $Q = Q'$.

Now let Q be a prime of B and set $P = Q \cap A$. Since the induced morphism $\mathrm{Var}(B/Q) \to \mathrm{Var}(A/P)$ is finite, separable and a homeomorphism we must have $k(P) = k(Q)$. In particular, A and B have the same total quotient ring. Since A is seminormal it follows that $A = B$ by Traverso's characterization of the seminormalization of A.

Remarks 3.8. Several people, including this author, have incorrectly asserted that a birational morphism $f\colon Y \to X$ which is a homeomorphism onto a seminormal variety X is an isomorphism of varieties (see [10], Proposition 2.4, [7], introductory remarks, [6], fact ii in introductory remarks and Fact 2.2 in the body of that paper, [5], main result asserts that a homeomorphic morphism of varieties is an isomorphism of varieties.). Andreotti and Bombieri correctly proved that if $f\colon Y \to X$ is a birational morphism onto a weakly normal variety and f is a universal homeomorphism, then f is an isomorphism of varieties ([1], Theorem 4). In their proof Zariski's Main Theorem is used to deduce that f is a finite morphism and hence an isomorphism. Another correct version of the assertion is that a finite birational morphism $f\colon Y \to X$ onto a weakly normal variety X which is a homeomorphism is an isomorphism of varieties ([4], Corollary 2.8).

Example 3.3 shows that the conclusion of Theorem 3.6 fails if some component of Y is one-dimensional. This enables us to exhibit a p-continuous function on Y with closed graph in $Y \times k$ that does not lift to a regular function on the normalization of Y. The next result says that this pathology can only occur in dimension one.

THEOREM 3.9. *Let X be a variety without one-dimensional components and consider a function $\varphi\colon X \to k$. Then, φ is c-regular if and only if φ is p-continuous with closed graph in $X \times k$.*

Proof. Since the question is local in nature we may and shall assume that $X = \mathrm{Var}(A)$ is affine. Let $\varphi\colon X \to k$ be a k-valued function. Let ${}^{+}X = \mathrm{Var}({}^{+}A)$ and let $\rho\colon {}^{+}X \to X$ denote the morphism induced by the inclusion $A \subseteq {}^{+}A$. Recall that $\rho\colon {}^{+}X \to X$ and $\rho \times 1\colon {}^{+}X \times k \to X \times k$ are homeomorphisms.

Suppose φ is c-regular. Then $\varphi \circ \rho$ is regular and consequently φ is

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p-continuous with closed graph in $X \times k$. Conversely, suppose φ is p-continuous with closed graph. Then $\psi = \varphi \circ \rho$ is p-continuous with closed graph. We wish to see that ψ is regular. For this it suffices to see that the restriction of ψ to each component of ${}^+X$ is regular. As the 0-dimensional components do not afford a problem, we may reduce to the following case: Y is an irreducible affine variety of dimension at least two such that the affine coordinate ring B of Y is seminormal and $\psi: Y \to k$ is p-continuous with closed graph.

Let $\eta = (1, \psi): Y \to Y \times k$. By assumption $\eta(Y)$ is a closed subvariety of $Y \times k$. Let $g: \eta(Y) \to Y$ denote the restriction to $\eta(Y)$ of the projection onto the first factor. The continuity of η implies g is a homeomorphism. By Theorem 3.7 g is an isomorphism.

Before we can discuss an alternative description of a c-regular function on a variety that has one-dimensional components, we need to establish several results.

LEMMA 3.10. *Let X be a normal variety of arbitrary dimension. If $\varphi: X \to k$ is p-continuous and its graph is closed in $X \times k$, then φ is regular.*

Proof. Since X is normal, φ is regular if and only if the restriction of φ to each component of X is regular. Hence we may and shall assume that X is irreducible.

Let $Y = (1, \varphi)(X) \subseteq X \times k$. By our assumptions, Y is an irreducible closed subvariety of $X \times k$. Let q denote the restriction of projection onto the first factor to Y. Then the morphism $q: Y \to X$ is a homeomorphism with topological inverse $(1, \varphi)$. Hence q is an isomorphism by Zariski's Main Theorem ([2], Corollaire 2, p. 137).

The next result is a consequence of Swan's construction of the seminormalization of a reduced ring.

PROPOSITION 3.11 ([8], Theorem 4.1). *Let $\sigma: A \to B$ be a ring homomorphism of reduced rings and let $i_A: A \to {}^+A$ and $i_B: B \to {}^+B$ denote the inclusion maps into the respective seminormalizations. Then there exists a unique ring homomorphism ${}^+\sigma: {}^+A \to {}^+B$ such that ${}^+\sigma \circ i_A = i_B \circ \sigma$.*

Our original construction of the weak normalization of a variety remains valid. Let $\pi: \tilde{X} \to X$ denote the normalization of the variety X. Briefly, we set $X^w = \tilde{X}/\sim$ where $y \sim z \Leftrightarrow \pi(y) = \pi(z)$. The structure sheaf on X^w is defined by considering regular functions on $\tilde{X}$ that are constant on the fibers of π. Notice that the sheaf of c-regular functions on X may be identified with the structure sheaf of the weak normalization of X. Recall that if $X = \mathrm{Var}(A)$ is affine we are just describing the semi-normalization ${}^{+}X = \mathrm{Var}({}^{+}A)$ of X. We summarize this construction below.

PROPOSITION 3.12. *Let X be a variety. Then there exists an essentially unique pair (X^w, ρ_X) consisting of a weakly normal variety X^w together with a finite birational morphism $\rho_X\colon X^w \to X$ that is a homeomorphism. By essentially unique we mean that if (Y, μ) is any other such pair, then there is a unique morphism $\tau\colon Y \to X^w$ such that $\rho_X \circ \tau = \mu$ and τ is an isomorphism. The pair (X^w, ρ_X) is called the weak normalization of X.*

By combining Propositions (3.11) and (3.12) we can deduce the following.

COROLLARY 3.13. *Let $f\colon Y \to X$ be a morphism of varieties and let (Y^w, ρ_Y), (X^w, ρ_X) denote the respective weak normalizations. Then there exists a unique morphism $f^w\colon Y^w \to X^w$ such that $\rho_X \circ f^w = f \circ \rho_Y$.*

The following assertion is a direct consequence of Corollary 3.13.

COROLLARY 3.14. *Let $i\colon Y \subseteq X$ be a closed subvariety. If $\varphi\colon X \to k$ is c-regular then $\varphi|_Y$ is again c-regular.*

PROPOSITION 3.15. *Let X be a variety of arbitrary dimension and $\varphi\colon X \to k$ a k-valued function. Then, the following are equivalent:*

(a) *φ is c-regular,*

(b) *the graph of φ is closed in $X \times k$ and for every morphism of varieties $f\colon Y \to X$ the composition $\varphi \circ f$ is p-continuous, and*

(c) *the graph of φ is closed in $X \times k$ and $\varphi \circ \pi$ is p-continuous where $\pi\colon \tilde{X} \to X$ denotes the normalization of X.*

Proof. Let $\rho = \rho_X\colon X^w \to X$ denote the weak normalization of X. (a) $\Rightarrow$ (b): Suppose $\varphi\colon X \to k$ is c-regular. Then $\varphi \circ \rho$ is regular on X^w. Recall that ρ and $\rho \times 1\colon X^w \times k \to X \times k$ are homeomorphisms. Since the graph of the regular function $\varphi \circ \rho$ is closed in $X^w \times k$ the graph of φ is closed in $X \times k$. Now let $f\colon Y \to X$ be a morphism of varieties and let $\rho_Y\colon Y^w \to Y$ denote the weak normalization of Y. By (3.13) there exists a unique morphism $f^w\colon Y^w \to X^w$ such that $\rho_X \circ f^w = f \circ \rho_Y$. Since $\varphi \circ \rho_X \circ f^w$ is a regular function $(1, \varphi \circ \rho_X \circ f^w)\colon Y^w \to Y^w \times k$ is a morphism.

Thus $\varphi \circ f \circ \rho_Y = \varphi \circ \rho_X \circ f^w$ is p-continuous.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a): Suppose the graph of φ is closed in $X \times k$ and $\varphi \circ \pi \colon \tilde{X} \to k$ is p-continuous. Then the graph of $\varphi \circ \pi$ is closed and hence $\varphi \circ \pi$ is regular by Lemma 3.10.

Remarks 3.16. Say a k-valued function $\varphi\colon X \to k$ is stably p-continuous if for every morphism of varieties $f\colon Y \to X$ the function $\varphi \circ f$ is p-continuous. Say φ is strongly p-continuous if $\varphi \circ \pi$ is p-continuous where $\pi\colon \tilde{X} \to X$ denotes the normalization of X. Example 3.3 demonstrates that a p-continuous function need not be stably p-continuous. We offer another example below.

We now see that one must demand stable p-continuity, or at least strong p-continuity, in characterizing the c-regular functions on a variety with one-dimensional components. Once again assume that X is a variety without one-dimensional components. We do not know whether or not a p-continuous function $\varphi\colon X \to k$ that is regular off the singular locus of X is c-regular. If we assume φ is strongly p-continuous and regular off $S(X)$ then φ is c-regular (see the proofs of Lemma 2.5 and Proposition 2.6 in [4]).

EXAMPLE 3.17. For each $x \in k$ choose a square root $\varphi(x)$ of x. Let $p\colon A^2 \to A^1$ denote projection onto the first factor and let $\psi = \varphi \circ p$. We will show that φ is a p-continuous function. Let $\eta = \sum a_i \varphi^i$ where the coefficients are regular functions on A^1 and consider the zero set of η. Now $\sum a_i(x)\varphi(x)^i = \sum_{i \text{ even}} a_i(x)x^{i/2} + \varphi(x)\sum_{i \text{ odd}} a_i(x)x^{[i-1]/2} = b(x) - \varphi(x)c(x)$ where b and c are regular functions on A^1. If b and c are identically zero, then the zero set of η is the entire affine line. Otherwise the zero set of η is a subset of the zero set of the regular function $b^2 - xc^2$. As the latter is a finite set of points so is the former. Hence η is continuous.

ψ is continuous (it is the composition of continuous functions) but not p-continuous. Let x and y denote the coordinate functions on A^2 and consider $\rho(x, y) = \psi(x, y) - y$. Let $T = \{(x, y) \in A^2 \mid \rho(x, y) = 0\}$ and let $Z = \{(x, y) \in A^2 \mid x - y^2 = 0\}$. Now Z is an irreducible closed variety of dimension one and T is a proper subset of Z. Since the only proper closed subsets of Z are finite sets and T is infinite, T is not closed in Z and hence T is not closed in A^2. Thus ψ is not p-continuous.

Notice that the original function φ is integral over $k[x]$, the affine coordinate ring of A^1. This shows that ([4], Proposition 2.14) which asserted "If $\mathcal{O}_X$ denotes the sheaf of c-regular functions on X and $\mathscr{C}_X$ denotes the sheaf of continuous k-valued functions on X then $\mathcal{O}_{X,x}$ can be viewed as the integral closure of $\mathcal{O}_{X,x}$ in $\mathscr{C}_{X,x}$" is incorrect and that no analogue is possible.

The subsequent results of "Seminormal rings and weakly normal varieties" are correct as stated. Those results rely primarily on the algebraic aspects of weak normality.

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