Problem :—To find a point M external to triangle ABC such that lines from it to the vertices A, B and C make equal angles with the sides, that is $MCA_1 = MAB = MBC$.—It is evident that if $MCA_1 = MBC$ the point M lies on the circle O, which touches AC at C and passes through B. Again, if we take two equal and opposite pencils $A(BIMC)$ and $C(A_1EMb)$ we know that their intersections lie on an hyperbola, and observe that when AI is the bisector of A its corresponding ray CE of the second pencil is parallel to it. This gives the direction of the asymptotes; the second asymptote will be parallel to the external bisector of A. The conic is therefore a rectangular hyperbola. When the ray of the first pencil coincides with AC, the ray of the second corresponding to it will make with CA an angle equal to A, it is therefore parallel to AB, and the point of intersection is C; hence the second ray ab is a tangent at C, similarly BA is a tangent at A; therefore AC passes through the centre of the conic, and B′ is the centre. The point M where this conic cuts O is the required point M. Denote this hyperbola by γ, then on repeating this construction at B we get a second hyperbola β cutting the circle O_1 at M_1, which makes angle $M_1BA_2 = M_1AC = M_1CB$. (See Fig. on p. 157.)

The six angles thus obtained are equal. For the circles O and O_1 may be regarded as similarly and oppositely placed figures; A is their centre, and $AC:AB$ their ratio of similitude. Also γ and β are similarly and oppositely placed figures; A is also their centre, and $AC : AB$ their ratio of similitude. And as O is placed with respect to γ, so similarly is O_1 placed with respect to β; for the tangents to O and γ cut at C at an angle equal to A. The tangents to O_1 and β at B at an angle equal to A. Therefore the figure made up of O and γ is similar to the figure made up of O_1 and β, and therefore the point of intersection M in the first figure is a similar point to M_1 in the second, so that angle CAM = angle BAM_1, etc., and $MAB = M_1AC$. Therefore the six angles are equal.

The conic at C meets O in two points only, but while M may be considered a single point C itself is at once a single and double point, since $A(c)$ and $C(a)$ intersect at C, and $C(a)$ being a tangent C is again a double point.

If OC be produced to meet the perpendicular to BA at A, they meet at a point K on the conic and on the circle touching at A and passing through C, but this is not an M point.

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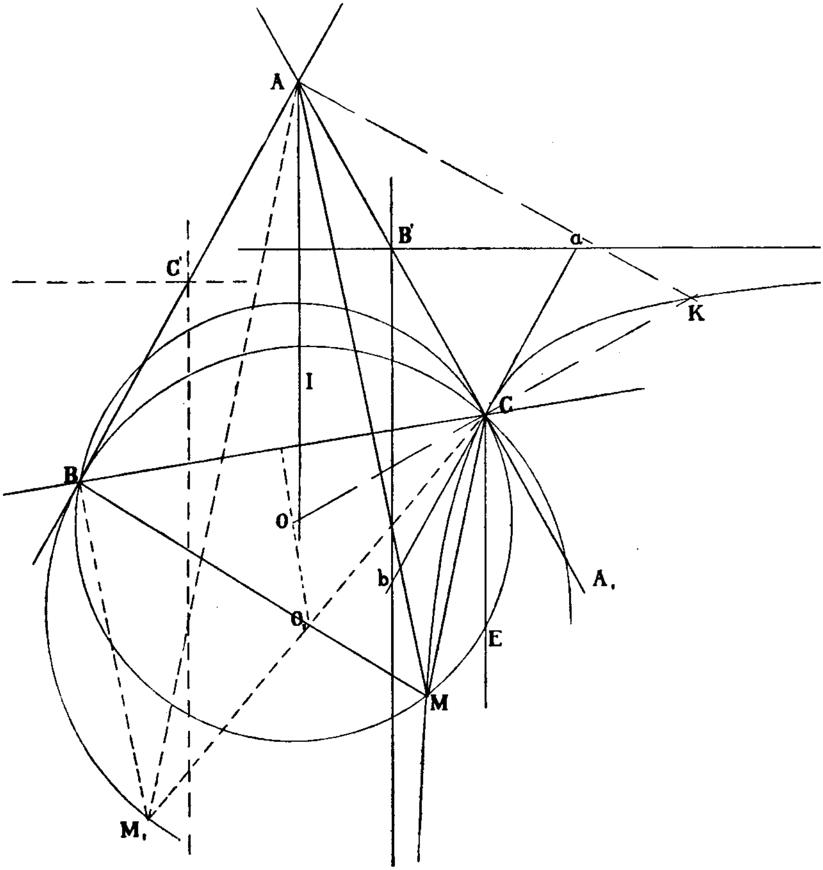

There are thus three sets of M points, one in each angle external to its base (but the angles of each set will in general be different), or, including the Brocard points, four sets of angles connected with the triangle, which have eight points with this property.

It is obvious from the construction that M being found, M_1 can be got by a parallel to AM through B cutting O_1, or a line from C through the intersection of OO_1, and BM, and cutting O_1 in M_1.

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