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WEIGHTED INTERPOLATION INEQUALITIES AND EMBEDDINGS IN R^n

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1. Introduction. This paper is a continuation of [**3**] which initiated a systematic study of sufficient conditions for the weighted interpolation inequality of sum form,

$$(1.1) \qquad \int_\Omega N|D^j u|^p \leqq K\left\{\epsilon^{-\phi}\left(\int_\Omega W|u|^q\right)^{p/q} + \epsilon^{\theta}\left(\int_\Omega P|D^m u|^r\right)^{p/r}\right\},$$

to hold. Here ϕ, θ are non-negative functions of m, j, p, q, r, Ω is a bounded or unbounded domain in R^n, ϵ belongs to an interval $\Gamma = (0, \epsilon_0), u$ is in a certain Banach space $E(\Omega)$, and N, W, P are measurable real functions satisfying $N \geqq 0, W, P > 0$, as well as additional conditions stated below. Finally the constant K does not depend on u although it may depend on the other parameters.

Important examples of the weights N, W, P which will be investigated in some detail in the examples of section 3 include

$$(1.2) \qquad N(t) = |t|^{\beta},\ W(t) = |t|^{\gamma},\ P(t) = |t|^{\alpha},$$

$$(1.3) \qquad N(t) = d(t)^{\beta},\ W(t) = d(t)^{\gamma},\ P(t) = d(t)^{\alpha}$$

where $d(t) := \text{dist}\,(t, R^n \setminus \Omega)$,

or

$$(1.4) \qquad N(t) = h(t)^{\beta},\ W(t) = h(t)^{\gamma},\ P(t) = h(t)^{\alpha}$$

where h is positive measurable function approaching zero or infinity as t approaches some subset A of $\partial\Omega$.

We note that inequalities similar to (1.1) are of continuing importance to the spectral and perturbation theory of ordinary and partial differential operators. For example if $p = q = r = 2, j = 0, \epsilon = 1$, and the embedding operator determined by (1.1) is compact, then the spectrum of the corresponding operator is discrete. Applications for $n = 1$ to the determination of the lower bound of the spectrum are given in [**5**]. Additional applications of a similar kind may be found in the

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recent book of W. D. Evans and D. E. Edmunds [**4**]. For nonlinear problems also these kinds of embedding can be combined with degree theory or other devices to prove the existence of solutions.

The structure and methodology of the present paper is quite similar to [**3**]. However there are important differences and improvements in the details. In the earlier paper for example the restriction $m - j > n/r$ was imposed for $n > 1$. It is clear that such conditions rule out some desirable cases, e.g., $m = 1$, $j = 0, p = q = r = 2$, and $n \geqq 2$, which arise in the study of the Laplace operator with Dirichlet boundary conditions. This annoying condition is now essentially eliminated in Lemmas 2.2–2.3 below; indeed for $p \leqq r$ our theory requires only that $m > j$. The examples of [**3**] emphasized weights of type (1.2). We now can consider weights of type (1.2)–(1.4) on bounded and unbounded domains in a completely unified way. Also some of the conditions which in [**3**] were known to be sufficient for (1.1) are now seen to be "almost" necessary.

Some of the results of this paper generalize parts of excellent recent work of B. Opic [**17**] and P. Gurka and B. Opic [**7**], [**8**], [**9**]. These authors have obtained necessary and sufficient conditions for (1.1) as well as for compactness for the underlying embedding when $m = 1$, $j = 0$, and $p \geqq q = r$. We consider here arbitrary m, j, a wider class of weights and $\mathcal{L}^p$ norms with arbitrary p, q, r; however we wish to acknowledge the influence of their work in our development of parts (i) and (ii) of Lemma 2.2, Definition 3.1, Theorem 3.2 (a necessary condition) and in Theorems 4.1–4.2.

We turn now to a brief outline of the paper. The necessary and somewhat complicated notation peculiar to this subject is presented at the end of the present section. Section 2 contains a fundamental inequality on a cube first with unity weights (Lemma 2.2). Then N, W, P are introduced by means of Hölder and reverse Hölder arguments (Lemma 2.3). This section also contains the definitions of the fundamental integral averages $S_1(t)$ and $S_2(t)$. The boundedness of these averages amount to generalized A_p conditions and will imply (1.1). In section 3 our major result (Theorem 3.1) uses the Besicovitch covering lemma together with Lemma 2.3 to prove (1.1) for (i) $p \geqq \max\,(q, r)$, (ii) $p < \min\,(q, r)$, (iii) $q \leqq p < r$, and (iv) $r \leqq p < q$. This result with the modifications appropriate to the new setting follows the same pattern as the corresponding Theorem 3.1 of [**3**]. A large numbers of examples is then given to illustrate the theorem. Theorem 3.2 gives a necessary condition for (1.1). When applied to certain concrete examples it shows that the relations between the weights which were found to be sufficient for (1.1) are actually also necessary. Section 4 consists of a study of necessary and sufficient conditions that the mapping J determined by (1.1) is compact.

When $\Gamma = (0, \infty)$, it is straightforward to show by a calculus argument that (1.1) is equivalent to the multiplicative inequality

$$(1.5) \qquad \left(\int_\Omega N |D^j u|^p \right)^{1/p} \leqq K_1 \left(\int_\Omega W |u|^q \right)^{a/q} \left(\int_\Omega P |D^m u|^r \right)^{(1-a)/r}$$

where $a = \theta/(\phi + \theta)$. If $N = W = P = \omega$, it has been recently shown [**10**] that (1.5) holds if ω is of class A_p, $m = 2$, $j = 1$, $a = 1/2$, and $p = q = r$. In section 5 we show how the theory of section 3 leads to product inequalities when the weights are of class A_z for certain z.

We employ standard notation for spaces of smooth functions such as $C^m(\Omega)$, $C_0^\infty(\Omega)$, $W^{m,p}(\Omega)$, e.g., see [**1**], [**4**], or [**13**]. For a weight Z and a set $S \subset \Omega$, $\mathcal{L}^p(Z; S)$ denotes the space of equivalence classes of measurable functions with norm $\|u\|_{Z,p} = (\int_S Z|u|^p)^{1/p}$ with Z deleted when $Z \equiv 1$; $\|u\|_{\infty,S}$ is the essential supremum norm of u over S. If the context is clear norms even for different subspaces will be denoted by "$\|\cdot\|$". The subscript "loc" and subscript "0" mean respectively a local property or interior. The symbol $\hookrightarrow$ is used to denote an embedding. For $1 \leqq \eta \leqq \infty$, η' is the extended real number such that $1/\eta + 1/\eta' = 1$.

We define

$$|D^m u(x)| = \sum_{|\alpha|=m} |D^\alpha u(x)|$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index. Throughout we assume without further reference that

$$(1.6) \qquad 1 \leqq p,\ q,\ r < \infty.$$

(1.7) $\quad f(t)$ is a positive, measurable function on Ω.

The purpose of f is to define certain cubes. Q denotes a closed cube in R^n with sides of length L parallel to the coordinate axes. Frequently, $Q = Q_{t,\epsilon}$ where $Q_{t,\epsilon}$ is the cube with center t and $L = \epsilon f(t)$. Application of Theorem 3.1 requires proper choice of f. We also define

$$\mathcal{S}_\epsilon = \{Q_{t,\epsilon} : t \in \Omega\}$$

and

$$\Omega_\epsilon = \cup Q_{t,\epsilon} \quad \text{over} \quad Q_{t,\epsilon} \in \mathcal{S}_\epsilon.$$

$B(t, r)$ is the open ball of radius r and center t and C_x for $x \in Q$ denotes a spherical cone with vertex x, radius $L/2$, and central angle independent of x and such that $C_x \subset Q$.

Besides the basic assumptions of sign and measurability on the weights N, W, P, we also assume

$$(1.8) \qquad N,\ W^{-q'/q},\ P^{-r'/r} \in \mathcal{L}_{\mathrm{loc}}(\Omega_\epsilon).$$

These will not in general be the only restrictions on N, W, P. However, instead of giving a list of all possibilities which would be cumbersome to state, we remark that the weights will be supposed to satisfy the minimal regularity conditions in order that the statements involving them make sense, e.g., in $\int_Q N^{-\eta}$ assume that $N^{-\eta} \in \mathcal{L}_{\mathrm{loc}}(\Omega_\epsilon)$.

The underlying space $E(\Omega)$ for which (1.1) will be valid is constructed in the following way. Define

$$\mathcal{E} = \left\{ u \in W^{m,r}_{\mathrm{loc}}(\Omega) : \int_\Omega W|u|^q < \infty, \int_\Omega P|D^m u|^r < \infty \right\}$$

with a norm given by

$$\|u\|_{\mathcal{E}} = \left(\int_\Omega W|u|^q \right)^{1/q} + \left(\int_\Omega P|D^m u|^r \right)^{1/r}.$$

Additionally, let

$$E_0(\Omega) = \{ u \in C^m(\Omega) \cap \mathcal{E} : u \text{ has support in } \Omega \text{ if } \Omega_\epsilon \setminus \Omega \text{ is nonempty} \}.$$

Note that even if $\Omega_\epsilon \setminus \Omega$ is nonempty, the support of u need not be compact unless Ω is bounded. Under the assumptions given above, it can be shown that the topological closure of $E_0(\Omega)$ in $\mathcal{E}$ with respect to the norm $\|\cdot\|_{\mathcal{E}}$ is a Banach space which we denote by $E(\Omega)$.

Throughout K denotes a constant where meaning may change for different theorems. If it is necessary to distinguish the various K, subscripts are used.

2. Basic Embedding Lemmas. Fundamental to the theory of [**3**] was the following lemma.

LEMMA 2.1. *There exists a constant K depending only on m, n, and j such that if $u \in C^m(Q), 0 \leqq j \leqq m-1$, and $x \in Q$, then*

$$(2.1) \qquad |D^j u(x)| \leqq K \left\{ L^{-n-j} \int_q |u| + L^{m-n-j} \int_Q |D^m u| + \int_{C_x} |D^m u(y)||x-y|^{m-n-j} dy \right\}.$$

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A difficulty with this lemma which distinguishes the case $n > 1$ from the single variable setting is the term involving the cone C_x. This term necessitated the restriction $m - j > n/r$ in [**3**] and will still complicate to some extent the improvements given below. Our first result shows how the condition $m-j > n/r$ can be relaxed for unity weights.

LEMMA 2.2. *Let $v = \min\{q, r\}$ and $j < m$. There exists a constant K, depending only on m, n, and j, such that if $u \in C^m(Q)$ and any one of the following conditions hold:*

(i) $m - j > n/v$,
(ii) $m - j \leqq n/v$ *and* $m - j - n/v + n/p \geqq 0$,
(iii) $p \leqq r$,

then the inequality,

$$\int_Q |D^j u|^p \leqq K\left\{L^{-p(j+n/q-n/p)}\left(\int_Q |u|^q\right)^{p/q} + L^{p(m-j-n/r+n/p)}\left(\int_Q |D^m u|^r\right)^{p/r}\right\}, \tag{2.2}$$

holds.

Proof. First note that it is sufficient that L, the edge length of Q, equal one for the change of variables $s = (1/L)(t - \bar{t})$, where $\bar{t}$ is the center of the cube Q, transforms Q into a unit cube. Computation of the Jacobian and partial derivatives leads to the exponents on L in (2.2). For (i)–(ii) a further simplification is obtained by noting that is sufficient to prove (2.2) only for the case $q = r = v = \min\{q, r\}$. To see this in case $r < q$ suppose (2.2) holds on unit cubes with q replaced by r. Then by Hölder's inequality $(\int_Q |u|^r)^{p/r} \leqq (\int_Q |u|^q)^{p/q}$. The remainder of the proof for (i)–(ii) is basically an appeal to the Sobolev embedding theorem [**1**, p. 97]. First note that the Sobolev norm for $W^{m,v}(Q)$ is equivalent to the norm

$$\left(\int_Q |u|^v\right)^{1/v} + \left(\int_Q |D^m u|^v\right)^{1/v}$$

cf. [**1**, p. 79]. For $m-j > n/v$, the Sobolev embedding theorem gives $W^{m,v}(Q) \hookrightarrow C_B^j(Q)$. Since for some constant k, independent of u,

$$\|u\|_{C_B^j(Q)} = \sup_{x\in Q, |\alpha|\leqq j} |D^\alpha u(x)| \geqq k\left(\int_Q |D^j u|^p\right)^{1/p},$$

then (2.2) follows. For $m - j \leqq n/v$, the embedding theorem gives $W^{m,v}(Q) \hookrightarrow W^{j,p}(Q)$ if $v \leqq p \leqq nv/[n - (m - j)v]$ (for $m - j < n/v$) or $v \leqq p < \infty$

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(for $m-j=n/v$). These conditions are implied by (ii) when $p \geqq v$. Since Q has finite volume, $W^{j,v}(Q) \hookrightarrow W^{j,p}(Q)$ for $1 \leqq p < v$, so the proof of (ii) is complete.

For (iii) we start with Lemma 2.1 on a unit cube Q. Integration and application of Hölder's inequality to (2.1) gives for some constant K_1,

$$\text{(2.3)} \quad \int_Q |D^j u|^p \leqq K_1 \left\{ \left(\int_Q |u|^q \right)^{p/q} + \left(\int_Q |D^m u|^r \right)^{p/r} + \int_Q \left(\int_{C_x} |D^m u(y)||x-y|^{m-n-j} dy \right)^p dx \right\}.$$

If $0<\delta<1$ and $1<r$, two (for $p<r$) or one (for $p=r$) more applications of Hölder's inequality (the second time with index r/p) followed by Fubini's theorem yields that

$$\text{(2.4)} \quad \begin{aligned} &\int_Q \left(\int_{C_x} |D^m u(y)||x-y|^{m-n-j} dy \right)^p dx \\ &\leqq \int_Q \left(\int_{C_x} |x-y|^{-n+\delta r'} dy \right)^{p/r'} \\ &\quad \times \left(\int_{C_x} |D^m u(y)|^r |x-y|^{(m-j-\delta)r-n} dy \right)^{p/r} dx \\ &\leqq K_2 \int_Q \left(\int_Q |D^m u(y)|^r |x-y|^{(m-j-\delta)r-n} dy \right)^{p/r} dx \\ &\leqq K_2 \left(\int_Q \int_Q |D^m u(y)|^r |x-y|^{(m-j-\delta)r-n} dy dx \right)^{p/r} \\ &\leqq K_2 K_3 \left(\int_Q |D^m u(y)|^r dy \right)^{p/r} \end{aligned}$$

where

$$K_2 = \sup_{x \in Q} \int_{C_x} |x-y|^{-n+r'\delta} dy < \infty$$

$$K_3 = \sup_{y \in Q} \int_Q |x-y|^{(m-j-\delta)r-n} dx < \infty.$$

For $p=r=1$, only Fubini's theorem is needed to establish (2.4). The proof is completed by substitution of (2.4) into (2.3).

In order to incorporate nontrivial weights into Lemma 2.2 and use it as a replacement for Lemma 2.4 of [**3**], it is necessary to change the definition of G, S_1, S_2 of [**3**].

Definition 2.1. Let $\psi > 1, q^{-1} \leqq \theta_1 \leqq 1$, $r^{-1} \leqq \theta_2 \leqq 1$, let g and h be positive measurable functions on $Q = Q_{t,\epsilon}$ (so that $L = \epsilon f(t)$) and $1 < \eta, \xi \leqq \infty$. Set

$$\tilde{\eta} = \eta(\psi - 1)/\psi, \quad \tilde{\xi} = \xi(\psi - 1)/\psi,$$

$$G(g, \eta) = \left(\int_Q g^{-\tilde{\eta}}\right)^{1/\eta}, \quad G(h, \xi) = \left(\int_Q h^{-\tilde{\xi}}\right)^{1/\xi},$$

$$S_1(t) = f(t)^{-p(j+n/q-n/p\eta')}\left(L^{-n}\int_Q f^{\tilde{\eta}'} N^{\psi\tilde{\eta}'/(\psi-1)}\right)^{(\psi-1)/\psi\tilde{\eta}'} \cdot$$

$$\times \left(L^{-n}\int_Q W^{-\theta_1/(1-\theta_1)}\right)^{p(1-\theta_1)/\theta_1 q} \quad \text{if } \theta_1 < 1,$$

$$S_2(t) = f(t)^{p(m-j-n/r+n/p\xi')}\left(L^{-n}\int_Q h^{\tilde{\xi}'} N^{\psi\tilde{\xi}'/(\psi-1)}\right)^{(\psi-1)/\psi\tilde{\xi}'} \cdot$$

$$\times \left(L^{-n}\int_Q P^{-\theta_2/(1-\theta_2)}\right)^{p(1-\theta_2)/\theta_2 r} \quad \text{if } \theta_2 < 1.$$

If $\theta_1 = 1$ or $\theta_2 = 1$ we replace the integrals involving W or P by L^∞ norms, e.g. $(L^{-n}\int_Q W^{-\theta_1/(1-\theta_1)})^{p(1-\theta_1)/\theta_1 q}$ becomes $\|W^{-p/q}\|_{\infty,Q}$.

LEMMA 2.3. *Assume Definition* 2.1 *and set* $\tilde{p} = \psi p, \tilde{q} = \theta_1 q, \tilde{r} = \theta_2 r$, *and* $\tilde{v} = \min\{\tilde{r}, \tilde{q}\}$. *Assume* $\tilde{p}, \tilde{q}, \tilde{r}$, *and* $\tilde{v}$ *satisfy any one of the conditions: (i)* $m - j > n/\tilde{v}$, *(ii)* $m - j \leqq n/\tilde{v}$ *and* $m - j - n/\tilde{v} + n/\tilde{p} \geqq 0$, *(iii)* $\tilde{p} \leqq \tilde{r}$. *Then the inequality*

$$\int_Q N|D^j u|^p \leqq K\left\{\epsilon^{-p(j+n/q-n/p\eta')}G(g,\eta)S_1(t)\left(\int_Q W|u|^q\right)^{p/q}\right. \tag{2.5}$$
$$\left. + \epsilon^{p(m-j-n/r+n/p\xi')}G(h,\xi)S_2(t)\left(\int_Q P|D^m u|^r\right)^{p/r}\right\}$$

holds for all cubes $Q = Q_{t,\epsilon}$ *and for all* $u \in E(\Omega)$.

Proof. It is sufficient that $u \in C^m(Q)$. By Lemma 2.2,

$$\int_Q |D^j u|^{\tilde{p}} \leqq K\left\{L^{-\tilde{p}(j+n/\tilde{q}-n/\tilde{p}}\left(\int_Q |u|^{\tilde{q}}\right)^{\tilde{p}/\tilde{q}}\right. \tag{2.6}$$
$$\left. + L^{\tilde{p}(m-j-n/\tilde{r}+n/\tilde{p})}\left(\int_Q |D^m u|^{\tilde{r}}\right)^{\tilde{p}/\tilde{r}}\right\}$$

where $L = \epsilon f(t)$. The reverse Hölder inequality yields that

$$\int_Q |D^j u|^{\tilde{p}} \geqq \left(\int_Q N^{\psi/(\psi-1)}\right)^{1-\psi}\left(\int_Q N|D^j u|^p\right)^{\psi}. \tag{2.7}$$

On the other hand Hölder's inequality gives

$$(2.8)\qquad \left(\int_Q |u|^{\tilde q}\right)^{\tilde p/\tilde q} \leqq \left(\int_Q W^{-\theta_1/(1-\theta_1)}\right)^{\tilde p(1-\theta_1)/\theta_1 q}\left(\int_Q W|u|^q\right)^{\tilde p/q},$$

$$\left(\int_Q |D^m u|^{\tilde r}\right)^{\tilde p/\tilde r} \leqq \left(\int_Q P^{-\theta_2/(1-\theta_2)}\right)^{\tilde p(1-\theta_2)/\theta_2 r}\left(\int_Q P|D^m u|^r\right)^{\tilde p/r}.$$

Next we substitute (2.7), (2.8) into (2.6) and take the $1/\psi$th root to obtain, using $(a+b)^{1/\psi} \leqq a^{1/\psi} + b^{1/\psi}$,

$$\int_Q N|D^j u|^p \leqq K^{1/\psi}\left\{A\left(\int_Q W|u|^q\right)^{p/q} + B\left(\int_Q P|D^m u|^r\right)^{p/r}\right\}$$

where

$$A = L^{-p(j+n/\tilde q - n/\tilde p)}\left(\int_Q N^{\psi/(\psi-1)}\right)^{(\psi-1)/\psi}\left(\int_Q W^{-\theta_1/(1-\theta_1)}\right)^{p(1-\theta_1)/\theta_1 q}$$

$$B = L^{-p(m-j-n/\tilde r+n/\tilde p)}\left(\int_Q N^{\psi/(\psi-1)}\right)^{(\psi-1)/\psi}\left(\int_Q P^{-\theta_2/(1-\theta_2)}\right)^{p(1-\theta_2)/\theta_2 r}.$$

Again by Holder's inequality

$$\left(\int_Q N^{\psi/(\psi-1)}\right)^{(\psi-1)/\psi} = \left(\int_Q g^{-1} g N^{\psi/(\psi-1)}\right)^{(\psi-1)/\psi}$$

$$\leqq \left(\int_Q g^{-\tilde\eta}\right)^{(\psi-1)/\psi\tilde\eta}\left(\int_Q g^{\tilde\eta'} N^{\psi\tilde\eta'/(\psi-1)}\right)^{(\psi-1)/\psi\tilde\eta'}$$

so that from the definitions of G and S_1,

$$A \leqq L^{-p(j+n/\tilde q-n/\tilde p)}G(g,\ \eta)S_1(t)f(t)^{p(j+n/q-n/p\eta')}L^{np(1-\theta_1)/\theta_1 q}L^{n(\psi-1)/\psi\tilde\eta'},$$

which after simplification gives

$$A \leqq \epsilon^{-p(j+n/q-n/p\eta')}G(g,\ \eta)S_1(t).$$

This yields the first coefficient on the right hand side of (2.5); the proof for the second coefficient is similar.

Remark 2.1. Lemma 2.3 seems most useful in special cases.

(i) If N, W^{-1}, P^{-1} are locally bounded on $\Omega_\epsilon, g = h = 1$, and $\xi' = \eta' = 1$, then

$$S_1(t) \leqq f(t)^{-p(j+n/q-n/p)}\|N\|_{\infty,Q}\|W^{-p/q}\|_{\infty,Q}$$

$$S_2(t) \leqq f(t)^{p(m-j-n/r+n/p)}\|N\|_{\infty,Q}\|P^{-p/r}\|_{\infty,Q}.$$

(ii) For $\theta_1 = \theta_2 = 1$, then conditions (i), (ii), (iii) of Lemma 2.3 become, respectively, $m - j > n/v, m - j - n/v + n/p\psi \geqq 0, p \leqq r/\psi$. Note that even if $m - j > n/v (v = \min\{q, r\})$ or if $m - j < n/v$ and $m - j + n/v + n/p > 0$ or if $p < r$, if is always possible to choose θ_1, θ_2, and ψ so that (i) or (ii) of (iii) of Lemma 2.3 holds.

(iii) If $\theta_1 = 1/q$ and $\theta_2 = 1/r$, then the integrals involving W in Definition 2.1 become, respectively $(L^{-n} \int_Q W^{-q'/q})^{p/q'}$ and $(L^{-n} \int_Q P^{-p'/r})^{p/p'}$ while the conditions (i)–(iii) of Lemma 2.3 become respectively (i) $m - j > n$, (ii) $m - j \leqq n$ and $m - j - n + n/p\psi \geqq 0$, (iii) $p \leqq 1/\psi$ although the last condition is vacuous.

(iv) Other special cases are $\theta_1 = r/q, \theta_2 = 1$ when $q \geqq r$ or $\theta_1 = 1, \theta_2 = q/r$ when $q < r$.

These cases may be further refined or combined in various ways.

3. Sum Inequalities. We are now in a position to use Lemma 2.3 to obtain sum inequalities which relax the $m - j > n/r$ requirement of [**3**] although the derivation follows the same pattern as [**3**].

Lemma 2.3 gives the sum inequality on a generic cube $Q = Q_{t,\epsilon}$. Since $\mathcal{S}_\epsilon = \{Q_{t,\epsilon} : t \in \Omega\}$ covers Ω and every point t is the center of a cube in $\mathcal{S}_\epsilon$, the Besicovitch covering theorem (cf. [**11**, p. 2]) may be used to extract finitely many families $\Gamma_1, \ldots, \Gamma_s$ of mutually disjoint cubes in $\mathcal{S}_\epsilon$ to cover an arbitrary bounded subset A of Ω. Once this is done the remainder of the argument is routine. More specifically we consider the following conditions.

(H1) $$\sup_{t\in\Omega, \epsilon\in\Gamma} S_1(t) := s_1 < \infty.$$

(H2) $$\sup_{t\in\Omega, \epsilon\in\Gamma} S_2(t) := s_2 < \infty.$$

(H3) $$\int_\Omega g^{-\tilde{\eta}} < \infty.$$

(H4) $$\int_\Omega g^{-\tilde{\xi}} < \infty.$$

THEOREM 3.1. *Suppose the hypothesis of Lemma* 2.3 *holds. Let* $\eta = q/(q-p)$ *if* $q > p$ *and* $\xi = r/(r-p)$ *if* $r > p$. *Set* $\eta = \infty$ *and* $g = 1$ *if* $p \geqq q$ *and set* $\xi = \infty$ *and* $h = 1$ *if* $p \geqq r$. *Then the sum inequality* (1.1) *holds for all* $u \in E(\Omega)$ *in the following cases.*

(*i*) $p \geqq \max\{q, r\}$, (H1)–(H2) *hold,* $\theta = p(m - j - n/r + n/p)$, *and* $\phi = p(j + n/q - n/p)$.

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(ii) $p < \min\{q, r\}$, (H1)–(H4) *hold*, $\theta = p(m-j)$, *and* $\phi = pj$.
(iii) $q \leqq p < r$, (H1)–(H3) *hold*, $\theta = p(m-j-n/r+n/p)$, *and* $\phi = pj$.
(iv) $r \leqq p < q$, (H1), (H2), (H4) *hold*, $\theta = p(m-j)$, *and* $\phi = p(j+n/q-n/p)$.

Proof. In all cases application of Lemma 2.3 and addition over one of the families Γ_k discussed above yields that

$$(3.1) \qquad \int_{\gamma_k} N|D^j u|^p \leqq K_1 \sum_{Q\in\Gamma_k} \left\{ \epsilon^{-\phi} G(g,\ \eta) \left(\int_Q W|u|^q \right)^{p/q} + \epsilon^{\theta} G(h,\ \xi) \left(\int_Q P|D^m u|^r \right)^{p/r} \right\}$$

where $K_1 = K(s_1 + s_2)$, K as in Lemma 2.3, and $\gamma_k = \{\cup Q : Q \in \Gamma_k\}$. In (i), $\eta' = \xi' = G(g, \eta) = G(h, \xi) = 1$. We can then use the elementary inequality $\sum A_i^R \leqq (\sum A_i)^R, R \geqq 1$ to bound the two terms on the right of (3.1). For (ii) we use $\eta = (q/p)', \xi = (r/p)'$ and Hölder's inequality on each sum on the right of (3.1) to obtain

$$(3.2) \qquad \begin{aligned} \int_{\gamma_k} N|D^j u|^p &\leqq K_1 \left\{ \epsilon^{-\phi} \left(\int_{\gamma_k} g^{-\tilde{\eta}} \right)^{1/\eta} \left(\int_{\gamma_k} W|u|^q \right)^{p/q} \right. \\ &\qquad \left. + \epsilon^{\theta} \left(\int_{\gamma_k} \eta^{-\tilde{\xi}} \right)^{1/\xi} \left(\int_{\gamma_k} P|D^m u|^r \right)^{p/r} \right\} \\ &\leqq K_2 \left\{ \epsilon^{-\phi} \left(\int_\Omega W|u|^q \right)^{p/q} + \epsilon^{\theta} \left(\int_\Omega P|D^m u|^r \right)^{p/r} \right\} \end{aligned}$$

where $K_2 = K_1[(\int_\Omega g^{-\tilde{\eta}})^{1/\eta} + (\int_\Omega h^{-\tilde{\xi}})^{1/\xi}]$. Cases (iii) and (iv) are similar. Case (iii) requires that the first term on the right of (3.1) to be treated as in case (ii) and the second term to be treated as in case (i). In case (iv) we apply case (i) to the first term on the right of (3.1) and case (ii) to the second term. In all cases we end up with inequality (3.2). Thus

$$\begin{aligned} \int_A N|D^j u|^p &\leqq \sum_{k=1}^{s} \int_{\gamma_k} N|D^j u|^p \\ &\leqq sK_2 \left\{ \epsilon^{-\phi} \left(\int_\Omega W|u|^q \right)^{p/q} + \epsilon^{\theta} \left(\int_\Omega P|D^m u|^r \right)^{p/r} \right\}. \end{aligned}$$

Since A is an arbitrary bounded subset of Ω, the proof is complete.

Remark 3.1. (i) Below (Theorem 3.2) we will show that Theorem 3.1 is nearly necessary in that if $\inf[S_1(t)^{-1} + S_2(t)^{-1}] = 0$ over $t \in \Omega$, then (1.1)

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cannot hold. (Note that the contrapositive of Theorem 3.1 gives a necessary condition for a "nonequality", i.e., if (1.1) is false then not both s_1 and s_2 can be finite).

(ii) For $\Omega \neq R^n$ the definition of $E(\Omega)$ requires that $u = 0$ on $\partial\Omega$ in a generalized sense if $\Omega_\epsilon \backslash \Omega$ is nonempty. In many of the examples below $\Omega_\epsilon = \Omega$ so that this requirement on u may be dropped. Even if this is not the case standard results about extension operators can be applied to eliminate this restriction if $\partial\Omega$ is bounded, e.g. see Theorem 3.2 of [**3**].

We now give several examples of Theorem 3.1. The cases where the weights are of power type extend Examples 3.1 and 3.2 of [**3**] to cases where $m-j \leqq n/r$. The examples involving "distance to the boundary" are new and generalize some of Opic's results (principally in [**17**]). Unless stated otherwise, the weights are locally bounded above and below. Hence $S_1(t), S_2(t)$ can be computed according to Remark 2.1.

Example 3.1. Let $\Omega = R^n, 0 < \epsilon \leqq 1$, $P(t) = (1+|t|)^\alpha, W(t) = (1+|t|)^\gamma, f(t) = (1+|t|)^\Delta, N(t) = (1+|t|)^\beta y(t)$ where $\alpha, \beta, \gamma, \Delta$ are real numbers with $\Delta \leqq 1$ and $y(t)$ is a nonnegative measurable function on R^n with $\int_{R^n} y^\mu < \infty$ for some $\mu > 1$. We work out only case (i) of Theorem 3.1, i.e., $p \geqq \max\{q, r\}$, although the other cases are similar. In Definition 2.1 set $\theta_1 = \theta_2 = 1$ and $\psi = \mu/(\mu-1)$. Then since $g = h = \eta' = \xi' = 1$, a computation gives

$$S_1(t) = f(t)^{-p(j+n/q-n/p)}\left(L^{-n}\int_Q N^\mu\right)^{1/\mu} \|W^{-p/q}\|_{\infty,Q},$$

$$S_2(t) = f(t)^{p(m-j-n/r+n/p)}\left(L^{-n}\int_Q N^\mu\right)^{1/\mu} \|P^{-p/r}\|_{\infty,Q}.$$

Since $\epsilon, \Delta \leqq 1$, the ratio $(1+|t|)/(1+|s|)$ is bounded above and below by positive numbers for all $t \in R^n, s \in Q_{t,\epsilon}$. Thus for some constant k, independent of t and ϵ,

$$S_1(t) \leqq k[1+|t|]^a \epsilon^{-n/\mu}\left(\int_{R^n} y^\mu\right)^{1/\mu}, \tag{3.3}$$

$$a = -\Delta p(j+n/q-n/p) - n\Delta/\mu + \beta - p\gamma/q,$$

$$S_2(t) \leqq k[1+|t|]^b \epsilon^{-n/\mu}\left(\int_{R^n} y^\mu\right)^{1/\mu},$$

$$b = p\Delta(m-j-n/r+n/p) - n\Delta/\mu + \beta - p\alpha/r.$$

First apply Theorem 3.1 with ϵ fixed. By (3.3), s_1, s_2 are finite if a, b are nonpositive. This holds if we make $a = b$ and we require that

$$\Delta := (\alpha/r - \gamma/q)/(m+n/q-n/r) \leqq 1, \tag{3.4}$$

$$\beta \leqq p\gamma/q + n\Delta/\mu + \Delta p(j+n/q-n/p).$$

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Under (3.4), Theorem 3.1 yields an inequality (1.1) of the form

$$\int_{R^n}(1+|t|)^\beta y(t)|D^j u|^p dt \leqq K\left\{\epsilon^{-\phi-n/\mu}\left(\int_{R^n}(1+|t|)^\gamma|u|^q dt\right)^{p/q} + \epsilon^{\theta-n/\mu}\left(\int_{R^n}(1+|t|)^\alpha|D^m u|^r\right)^{p/r}\right\}, \tag{3.5}$$

$$\phi = p(j+n/q-n/p),$$

$$\theta = p(m-j-n/r+n/p),$$

for all $u \in E(R^n)$. Note that K is independent of ϵ for $\epsilon \in (0,1]$. The requirements of Lemma 2.3 for (3.5) to hold are that one of the following be true: (i) $m-j > n/v, v = \min\{q,r\}$, (ii) $m-j \leqq n/v$ and $m-j-n/v+n/p\psi \geqq 0$, (iii) $p\psi \leqq r$.

The above example may also be worked out with a different hypothesis on $y(t)$. Assume instead that $\sup[(\text{vol } Q)^{-1}\int_Q y^\mu] < \infty$ for some $\mu > 1$ where the sup is taken over all cubes Q whose edge length does not exceed $(1+|t|)^\Delta$ where t is the center of Q. This bound for y is then used in the above expressions for S_1, S_2. The conditions replacing (3.4) and (3.5) are those obtained by setting $\mu = \infty$ in (3.5) and leaving (3.4) unchanged; these are the same conditions as given in part (i) of Example 3.1 of [**3**]. Thus the conclusions of that example apply to many situations with $m-j \leqq n/r$. For fixed ϵ (in which case (3.5) may regarded as an embedding), it is not necessary to require that the above sup be over as large a set of cubes. For example, with $\alpha = \gamma = \beta = 0$ and $\epsilon = 1$, we require only that $\sup \int_Q y^\mu < \infty$ over all cubes of edge length one. In other words, if for some $\mu > 1$, $\sup \int_Q y^\mu < \infty$ over all cubes of edge length one, then

$$\int_{R^n} y|D^j u|^p \leqq K\left\{\left(\int_{R^n}|u|^q\right)^{p/q} + \left(\int_{R^n}|D^m u|^r\right)^{p/r}\right\}$$

for all $u \in W^{m,r}_{\text{loc}}(R^n)$ for which the above right hand side is finite provided one of the following hold: (i) $m-j > n/v$, (ii) $m-j \leqq n/v$ and $m-j-n/v+n/p\psi \geqq 0$, (iii) $p\psi \leqq r$ where $\psi = \mu/(\mu-1)$, $v = \min\{q,r\}$.

Example 3.2. Suppose Ω is bounded with respect to $d(t)$, i.e., d is bounded on Ω. Set $N(t) = d(t)^\beta, P(t) = d(t)^\alpha, W(t) = d(t)^\gamma,\ f(t) = d(t)^{\Delta_1}, g(t) = d(t)^{\Delta_2}$, and $h(t) = d(t)^{\Delta_3}$ where all exponents are real and $\Delta_1 \geqq 1$. Further set $\Gamma = (0, \epsilon_0)$ where $\epsilon_0 = \delta^{1-\Delta_1}/\sqrt{n}$ and $\delta = \sup\{d(t) : t \in \Omega\}$.

First we show that if $x \in Q_{t,\epsilon}, t \in \Omega$, then

$$\frac{1}{2} \leqq \frac{d(x)}{d(t)} \leqq \frac{3}{2}. \tag{3.6}$$

Note that (3.6) implies that on each cube $Q_{t,\epsilon}$, W, P, N are bounded above and below by positive constants. To establish (3.6), the triangle inequality gives for $v \in R^n \setminus \Omega$ that $d(t) \leqq |t-v| \leqq |t-x| + |x-v|$; it follows from this inequality that $d(t) \leqq |t-x| + d(x)$. Further, $|x-t| \leqq (\sqrt{n}/2)f(t)\epsilon$ so that a division by $d(t)$ gives

$$1 \leqq \frac{d(x)}{d(t)} + (\sqrt{n}/2)\epsilon d(t)^{\Delta_1 - 1} \leqq \frac{d(x)}{d(t)} + \frac{1}{2}$$

by the definition of ϵ_0. This establishes the left of (3.6); the other half follows by a similar argument. Note also that by choice of ϵ_0, $|x-t| \leqq d(t)/2$ so that $Q_{t,\epsilon} \subset \Omega$. This implies that $\Omega_\epsilon = \Omega$ so that there is no requirement that u vanish on $\partial\Omega$.

First consider case (i) of Theorem 3.1 with $p \geqq \max\{q, r\}$. By the above remarks there are constants k_1, k_2, independent of t and ϵ such that

$$S_1(t) \leqq k_1 f(t)^{-p(j+n/q-n/p)} N(t) W(t)^{-p/q},$$
$$S_2(t) \leqq k_2 f(t)^{p(m-j-n/r+n/p)} N(t) P(t)^{-p/r}.$$

Hence (H1)–(H2) are equivalent to respectively,

$$\begin{aligned} &\beta \geqq p\gamma/q + \Delta_1 p(j + n/q - n/p) \\ &\beta \geqq p\alpha/r + \Delta_1 p(m - j - n/r + n/p). \end{aligned} \tag{3.7}$$

If we set the two right hand sides of (3.7) equal this defines Δ_1 and leads to the requirement

$$\Delta_1 := (\alpha/r - \gamma/q)/(m + n/q - n/r) \geqq 1$$

and the requirement for β is

$$\begin{aligned} (\beta/p)(m - n/r + n/q) \geqq\ & (\gamma/q)(m - j - n/r + n/p) \\ & + (\alpha/r)(j + n/q - n/p). \end{aligned}$$

This result agrees with the sufficiency condition given by B. Opic for embedding in [**17**] when $m = 1, j = 0, q = r, p \geqq q$, and $\gamma = \alpha - p$. Note here in the Definition 2.1 we take $\theta_1 = \theta_2 = 1$ and we are free to choose $\psi > 1$ arbitrarily. Thus the three conditions of Lemma 2.3 are: (i) $m - j > n/v, v = \min(q, r)$, (ii) $m - j \leqq n/v$ and $m - j - n/v + n/p > 0$, (ii) $p < r$. These same choices apply to the other cases listed below.

Turning now to case (ii) of Theorem 3.1, we have $p < \min\{q, r\}, \eta = q/(q-p), \xi = r/(r-p)$. The condition (H3) is that $\int_\Omega d(t)^{-\Delta_2\eta(\psi-1)/\psi} dt < \infty$. This condition imposes additional restrictions on Ω. Since $d(t) \longrightarrow 0$ as $t \longrightarrow \partial\Omega$, it

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imposes at the very least that $\Delta_2\eta(\psi-1)/\psi<1$; further restraints are imposed by the finiteness of the integral away from the boundary. We simply content ourselves here to determine what are the further restrictions on the parameters given that (H3) and (H4) hold. We see that for some constants k_1, k_2 independent of t and ϵ,

$$S_1(t) \leqq k_1 d(t)^a, a = -pj\Delta_1 + \Delta_2(\psi-1)/\psi + \beta - \gamma p/q,$$
$$S_2(t) \leqq k_2 d(t)^b, b = p(m-j)\Delta_1 + \Delta_3(\psi-1)/\psi + \beta - \alpha p/r.$$

These will be finite provided we have

$$\Delta_1 := (\alpha/r - \gamma/q)/m \geqq 1, \tag{3.8}$$

and

$$\beta > pj\Delta_1 + \gamma p/q = \alpha p/r - p(m-j)\Delta_1 + (m-j)p\gamma/mq + p\alpha j/rm, \tag{3.9}$$

and require that ψ is sufficiently near one so that $a, b \geqq 0$.

Similar calculations may be made in the cases $r \leqq p < q$ and $q \leqq p < r$. We omit the details.

Example 3.3. The previous example can be generalized in several ways. The following seems to be a useful compromise between breadth and adaptability. Again let Ω be bounded with respect to $h(t)$ and set $N(t) = h(t)^\beta, W(t) = h(t)^\gamma, P(t) = h(t)^\alpha$ where h is a positive measurable function on R^n which is bounded on compact subsets of Ω. In many cases h will be of the form $h(t) = g(d(t))$; however a different example would be $h(t) = \prod_{i=1}^n |t_i|^{\lambda_i}$.

We consider two cases: (i) $h(t) \longrightarrow 0$, and (ii) $h(t) \longrightarrow \infty$ as t approaches some subset A of $\partial\Omega$ in some predefined way. In case (i) assume that h is Lipschitz on Ω with uniform constant K. Take $f(t) = h(t)^\Delta$ for some $\Delta \geqq 1$ and let $\epsilon_0 = \delta^{1-\Delta}/K\sqrt{n}$ where $\delta = \sup h(t), t \in \Omega$. The by an argument similar to that given in Example 3.2, we have that for $t \in \Omega$ and $x \in Q_{t,\epsilon}, \epsilon \leqq \epsilon_0, \frac{1}{2} \leqq h(x)/h(t) \leqq \frac{3}{2}$. These inequalities imply that N, P, W are locally bounded above and below by positive constants. The remaining steps of the analysis parallels Example 3.2 and yields the same relationships between β, α, and γ as (3.8)–(3.9).

In case (ii) we assume that h is locally Lipschitz and suppose there is a $\Delta \leqq 1$ such that for each $t \in \Omega$ the Lipschitz constant K_t for h on $B(t, d(t)/2)$ satisfies

$$\sup_{t\in\Omega} K_t h(t)^{\Delta-1} := K_0 < \infty. \tag{3.10}$$

Setting $f(t) = h(t)^\Delta, \epsilon_0 = 1/K_0\sqrt{n}$, we again have by an argument like that of Example 3.2 that for $t \in \Omega$ and $x \in Q_{t,\epsilon}, \epsilon \leqq \epsilon_0, \frac{1}{2} \leqq h(x)/h(t) \leqq \frac{3}{2}$. It will be found that Theorem 3.1 applies if (3.8)–(3.9) hold with the inequality signs reversed.

An example of case (i) would be $h(t) = \exp(-1/d(t))$, and for case (ii) $h(t) = \exp(1/d(t))$. For the first choice of $h(t)$ K_t is uniformly bounded on Ω, while for the second we have on the ball $B(t, d(t)/2)$ that $K_t \leqq 4d(t)^{-2}\exp(2/d(t))$ and (3.10) will be satisfied if $\Delta < -1$.

Another example of case (i) can be obtained by taking $h(t) = \prod_{i=1}^{n} |t_i|$ where A contains points having some coordinates zero. If we also assume Ω is bounded, then a straightforward argument shows that h is Lipschitz on Ω.

Example 3.4. Let the weights be as in Example 3.2. Assume $p \geqq \max\{q, r\}$ and Ω is unbounded with respect to $d(t)$. Since $d(t)$ takes on both large and small values, we take $f(t) = d(t)$. With $\epsilon_0 = \frac{1}{\sqrt{n}}$, it follows as in Example 3.2 that $\Omega_\epsilon = \Omega$ and $1/2 \leqq d(x)/d(t) \leqq 3/2$ for $x \in Q_{t,\epsilon}, \epsilon \leqq \epsilon_0$. Further s_1 and s_2 will be finite when equality holds in (3.8)–(3.9). In particular for $\Omega = R^n/\setminus\{0\}, d(t) = |t|$.

Example 3.5. If Ω is bounded and $0 \in \partial\Omega$, the choices $N(t) = |t|^\beta, W(t) = |t|^\gamma, P(t) = |t|^\alpha$ give the same results as Example 3.2 provided $f(t) = |t|^{\Delta_1}$. However, Ω is a proper subset of Ω_ϵ generally. If Ω is unbounded and $0 \notin \partial\Omega$, then we get the same type of results as in Example 3.1 with $f(t) = |t|^{\Delta_1}, \Delta_1 \leqq 1$. This is because the ratio $|t|/(1 + |t|)$ is bounded above and below on Ω by positive constants.

To state the next theorem concerning necessary conditions it is convenient to give a formal definition of a property we have been using implicitly in many previous examples.

Definition 3.1. A weight Z is said to be bounded above and below with respect to the family of cubes $\mathcal{S}_\epsilon$ if for each $Q_{t,\epsilon} \in \mathcal{S}$ there exist positive constants C_Z and D_Z (depending on t and ϵ) such that for all $x \epsilon Q_{t,\epsilon}, C_Z \leqq Z(x)/Z(t) \leqq D_Z$. If the ratio D_Z/C_Z is bounded independent of t, ϵ, we say Z is strongly bounded with respect to $\mathcal{S}_\epsilon$.

Remark 3.2. This property is similar to B. Opic's condition C [**17**]; in Examples 3.1–3.5 it has allowed us to compute $S_1(t)$ and $S_2(t)$ by moving weights outside of integrals.

THEOREM 3.2. *Assume the hypothesis of Lemma* 2.3 *holds,* $p \geqq \max\{q, r\}$, *and* $g = h = \eta' = \xi' = 1$ *in Definition* 2.1. *Suppose* N, W, P *are strongly bounded with respect to* $\mathcal{S}_\epsilon$. *Then if for some* $\epsilon \in \Gamma$,

$$(3.11)\qquad \inf_{t\in\Omega}[S_1(t)^{-1} + S_2(t)^{-1}] = 0,$$

the sum inequality (1.1) does not hold.

Proof. Let ρ be a C_0^∞ function with support in the interior of the unit cube Q_0 centered at the origin. By (3.11) there is a sequence of cubes $Q_k = Q_{t_k,\epsilon}$ in S_ϵ such that $S_1(t_k)^{-1} + S_2(t_k)^{-1} \longrightarrow 0$ as $k \longrightarrow \infty$. Define ρ_k by $\rho_k(t) = \rho([t-t_k]/2\epsilon f(t_k))$. Then ρ_k is a C_0^∞ function with support in Q_k. Moreover for each multi-index $\alpha, D^\alpha \rho_k = (D^\alpha \rho)(2\epsilon f(t_k))^{|\alpha|}$. Set $K_{i,s} = \int_{Q_0} |D^i \rho|^s$. Since (1.1) holds we have that

$$
\begin{aligned}
(3.12) \quad C_N N(t_k)[2\epsilon f(t_k)]^{-jp+n} K_{j,p} &\leqq \int_{Q_k} N|D^j \rho_k|^p \\
&\leqq K\left\{\epsilon^{-\phi}\left(\int_{Q_k} W|\rho_k|^q\right)^{p/q} \right.\\
&\quad \left. + \epsilon^\theta\left(\int_{Q_k} P|D^m \rho_k|^r\right)^{p/r}\right\} \\
&\leqq K\{\epsilon^{-\phi} D_W^{p/q} W(t_k)^{p/q}[2\epsilon f(t_k)]^{np/q} K_{0,q}^{p/q} \\
&\quad + \epsilon^\theta D_P^{p/r} P(t_k)^{p/r}[2\epsilon f(t_k)]^{-pm+np/r} K_{m,r}^{p/r}\}.
\end{aligned}
$$

From Definition 2.1,

$$
\begin{aligned}
S_1(t_k) &\leqq f(t_k)^{-p(j+n/q-n/p)} D_N N(t_k) C_W^{-p/q} W(t_k)^{-p/q}, \\
S_2(t_k) &\leqq f(t_k)^{p(m-j-n/r+n/p)} D_N N(t_k) C_P^{-p/r} P(t_k)^{-p/r}.
\end{aligned}
$$

Substitution of these inequalities into (3.12) leads to

$$
\begin{aligned}
(2\epsilon)^{-jp+n} K_{j,p} \leqq K\{&\epsilon^{-\phi}(D_W/C_W)^{p/q}(D_N/C_N) S_1(t_k)^{-1}(2\epsilon)^{np/q} K_{0,q}^{p/q} \\
&+ \epsilon^\theta (D_P/C_P)^{p/r}(D_N/C_N) S_2(t_k)^{-1}(2\epsilon)^{-pm+np/r} K_{m,r}^{p/r}\}.
\end{aligned}
$$

This inequality yields a contradiction since $S_1(t_k)^{-1} + S_2(t_k)^{-1}$ tends to zero as k becomes unbounded.

In the examples above where $S_1(t) = S_2(t)$, the condition $\sup S_1(t) < \infty$, $t \in \Omega$, becomes a necessary and sufficient condition for (1.1) to hold for fixed ϵ and under the other assumptions of Theorem 3.2. Thus for example, on R^n, with $N(t) = (1+|t|)^\beta, W(t) = (1+|t|)^\gamma, P(t) = (1+|t|)^\alpha, f(t) = (1+|t|)^\Delta$ with

$$
\Delta := (\alpha/r - \gamma/q)/(m + n/q - n/r) \leqq 1,
$$

$p \geqq \max\{q, r\}$, and assuming one of the following hold: (i) $m - j > n/v, v = \min\{q, r\}$, (ii) $m - j \leqq n/v$ and $m - j - n/v + n/p > 0$, then we have that a necessary and sufficient condition for (1.1) is

$$
\beta \leqq p\gamma/q + \Delta p(j + n/q - n/p).
$$

4. Compact Mappings and Embeddings. For a fixed ϵ inequality (1.1) determines a continuous map J from the Banach space $E(\Omega)$ into $\mathcal{L}^p(N;\Omega)$ defined by $J(u) = D^\alpha u$ where α is a fixed multi-index with $|\alpha| = j$. When $j = 0$ this map is called an embedding. In the present section we develop conditions which guarantee that J is compact. These conditions will generally be necessary or "almost" necessary as well as sufficient. In addition to the work of Opic and Gurka and Opic, extensive results on embedding in weighted Sobolev space may be found in the texts by A. Kufner [**12**] and H. Triebel [**18**]. We use the following notation throughout this section.

Definition 4.1. Let $\{\Omega_i\}$ be a chain of subdomains of Ω such that $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_i \subset \Omega$, and $\Omega = \cup_{(i)}\Omega_i$. Define $J_i, J^i : E(\Omega) \longrightarrow \mathcal{L}^p(N;\Omega)$ for a fixed multi-index $\alpha, |\alpha| = j$, by

$$J_i u(x) = \begin{cases} D^\alpha u(x) & \text{if } x \in \Omega_i, \\ 0 & \text{otherwise} \end{cases}$$

$$J_i u(x) = \begin{cases} D^\alpha u(x) & \text{if } x \in \Omega \setminus \Omega_i, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.1. (i) The chain $\{\Omega_i\}$ will be defined with reference to a subset A of $\partial\Omega$ where the weights have singular behavior. Thus if Ω is unbounded and the weights have singular behavior only at infinity, then we set $\Omega_i = Q(i) \cap \Omega$ where $Q(i)$ is a cube centered at the origin with edges of length $2i$. On the other hand if N, W, P are powers of $d(t)$, set $\Omega_i = \{t : d(t) > 1/i\}$.
(ii) We have the trivial facts:

$$J^i = J - J_i,$$

$$\|J^i\| = \sup_{u \in E(\Omega), \|u\|_\epsilon = 1} \left(\int_{\Omega \setminus \Omega_i} N|D^\alpha u|^p \right)^{1/p},$$

$$\|J^{i+1}\| \leqq \|J^i\|.$$

To replace (H1) we introduce the condition:

(H1$^\#$) For each $\epsilon \in (0, \epsilon_0)$,

$$\lim_{i \to \infty} s_{1,i} = 0 \quad \text{where} \quad s_{1,i} := \sup\{S_1(t) : t \in \Omega \setminus \Omega_i\}.$$

LEMMA 4.1. *If each J_i is compact, then J is compact if and only if $\|J^i\| \longrightarrow 0$ as $i \longrightarrow \infty$.*

Proof. Since $J^i = J - J_i$, the "if" part is a consequence of the fact that the uniform limit of compact linear operators is compact. Suppose we do not have $\|J^i\| \longrightarrow 0$ as $i \longrightarrow \infty$. Then there is a $\delta > 0$ and a sequence $\{u_i\}$ such that

$$\|J^i(u_i)\| \geqq \delta, \quad \|u_i\| = 1. \tag{4.1}$$

By definition of J^i, (4.1) implies that

$$\text{(4.2)} \qquad \begin{aligned} \|J(u_i)\|^p &= \|J_i(u_i)\|^p + \|J^i(u_i)\|^p \\ &\geqq \|J_i(u_i)\|^p + \delta^p. \end{aligned}$$

Since J is compact, $\{J(u_i)\}$ contains a convergent subsequence which we take to be $\{J(u_i)\}$ itself, say $J(u_i) \longrightarrow w$ as $i \longrightarrow \infty$. Since

$$\|J_i(u_i) - w\|^p \leqq \|w\|^p_{\Omega\setminus\Omega_i} + \|J(u_i) - w\|^p,$$

We also have $J_i(u_i) \longrightarrow w$ as $i \longrightarrow \infty$. Hence in (4.2) we conclude that

$$\|w\|^p \geqq \|w\|^p + \delta^p$$

which is a contradiction.

THEOREM 4.1. *Suppose the hypothesis of Theorem* 3.1 *holds and that each map J_i in Definition 4.1 is compact. Then the mapping J is compact in each of the following cases.*

(i) (H1)$^\#$, (H2) hold and $p \geqq \max\{q, r\}$.
(ii) (H1)$^\#$, (H2)–(H4) hold and $p < \min\{q, r\}$.
(iii) (H1)$^\#$, (H2), (H3) hold and $q \leqq p < r$.
(iv) (H1)$^\#$, (H2), (H4) hold and $r \leqq p < q$.

Proof. For $\epsilon \in (0, \epsilon_0)$ and $\delta > 0$, it is possible by (H1)$^\#$ to choose $I = I(\delta, \epsilon)$ such that $s_{1,i} < \delta$ for $i \geqq I$. For $i > I$ and $t \in \Omega_i \setminus \Omega_I$, we have by Lemma 2.3 that with $Q = Q_{t,\epsilon}$,

$$\int_Q N|D^\alpha u|^p \leqq K\left\{\epsilon^{-p(j+n/q-n/p\eta')}\delta G(g,\ \eta)\left(\int_Q W|u|^q\right)^{p/q} + \epsilon^{p(m-j-n/r+n/p\xi')}G(h,\ \xi)s_2\left(\int_Q P|D^m u|^r\right)^{p/r}\right\}.$$

By the Besicovitch covering argument of Theorem 3.1 we conclude that in each of the above cases there is a constant K_1 such that for $i > I$,

$$\text{(4.3)} \qquad \int_{\Omega_i\setminus\Omega_I} N|D^\alpha u|^p \leqq K_1[\delta\epsilon^{-p(j+n/q-n/p\eta')} + \epsilon^{p(m-j-n/r+n/p\xi')}]\|u\|^p_\epsilon.$$

Since i is arbitrary in (4.3), $\Omega \setminus \Omega_I$ may be substituted for $\Omega_i \setminus \Omega_I$ in (4.3). Now the exponent $m - j - n/r + n/p\xi' > 0$ in all four cases above; hence we have

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that (4.3) implies $\|J^i\| \longrightarrow 0$ as $i \longrightarrow \infty$. The conclusion now follows by Lemma 4.1.

Remark 4.2. The Rellich-Kondrasŏv theorem gives conditions under which the mappings J_i are compact. For example if each Ω_i is bounded with minimally smooth boundary, and N, W, P are locally bounded above and below, then J_i is compact, cf. [**4**, p. 263], [**1**, p. 144], if (i) $m - j - n/r > 0$ or if (ii) $m - j < n/r, m - j - n/r + n/p > 0$ or if (iii) $m - j = n/r$. Recall we require $1 \leqq p, q, r < \infty$ throughout.

Under certain conditions we can show the condition $(\mathrm{H1})^{\#}$ is almost necessary for J to be compact.

THEOREM 4.2. *Assume the hypothesis of Lemma* 2.3 *holds,* $p \geqq \max\{q, r\}$, *and* $g = h = \eta' = \xi' = 1$ *in Definition 2.1. Suppose* N, W, P *are strongly bounded with respect to* $\mathcal{S}_\epsilon$. *Then the map* J *is not compact if for some* $\epsilon > 0$,

$$\lim_{k\to\infty}\left\{\inf_{t\in\Omega\setminus\Omega_k} S_1(t)\right\} > 0 \quad \textit{and} \quad \lim_{k\to\infty}\left\{\inf_{t\in\Omega\setminus\Omega_k} S_2(t)\right\} > 0. \tag{4.4}$$

Proof. Condition (4.4) implies that there is a $\delta > 0$ and a sequence $\{t_k\}$ such that $t_k \epsilon \Omega \setminus \Omega$ and $S_1(t_k) > \delta, S_2(t_k) > \delta$. Let $\rho_k(t)$ be as in the proof of Theorem 3.2. Suppose J is compact; hence by Lemma 4.1, $\|J^k\| \longrightarrow 0$ as $k \longrightarrow \infty$. Thus $\|J^k(\rho_k)\|/\|\rho_k\|_\epsilon \longrightarrow 0$ as $k \longrightarrow \infty$. From the estimates in the proof of Theorem 3.2 we have that

$$C_N N(t_k)[2\epsilon f(t_k)]^{-jp+n} K_{j,p} \leqq \int_{Q_k} N|D^j \rho_k|^p = \|J^k(\rho_k)\|^p, \tag{4.5}$$

$$\begin{aligned} \|\rho_k\|_\epsilon &= \left(\int_{Q_k} W|\rho_k|^q\right)^{1/q} + \left(\int_{Q_k} P|D^m \rho_k|^r\right)^{1/r} \\ &\leqq D_W^{1/q} W(t_k)^{1/q}[2\epsilon f(t_k)]^{n/q} K_{0,q}^{1/q} \\ &\quad + D_P^{1/r} P(t_k)^{1/r}[2\epsilon f(t_k)]^{-m+n/r} K_{m,r}^{1/r}. \end{aligned}$$

The same estimates for $S_1(t)$ and $S_2(t)$ hold as in the proof of Theorem 3.2 and this leads to the estimate,

$$\begin{aligned} N(t_k)^{-1/p}\|\rho_k\|_\epsilon &\leqq T[S_1(t_k)^{-1/p} f(t_k)^{-j+n/p} + S_2(t_k)^{-1/p} f(t_k)^{-j+n/p}] \\ &\leqq 2T\delta^{-1/p} f(t_k)^{-j+n/p}, \end{aligned} \tag{4.6}$$

where T is a constant depending on ϵ, N, W, and P, but not on k. The estimates (4.5) and (4.6) contradict the fact that $\|J^k(\rho_k)\|/\|\rho_k\| \longrightarrow 0$ as $k \longrightarrow \infty$. Hence J is not compact.

Remark 4.3. The practical consequence of Theorem 4.1 is that the mapping J in the examples of Section 3 is compact if the inequalities associated with (H1) are made strict, e.g., in (3.4) require a strict inequality for β. When the parameters are such that $S_1(t) = S_2(t)$ the condition may become a necessary and sufficient condition for compactness, e.g., as in (3.4).

5. A Product Inequality. As pointed out in the introduction, (1.1) is equivalent to the multiplicative inequality (1.5) if $\Gamma = (0, \infty)$. Hence if $\Gamma = (0, \infty)$ a multiplicative inequality as well as a sum inequality will be true. Various special cases and illustrations of this simple observation were given in [**3**] assuming $m - j > n/r, P^{-1}$ locally bounded, etc. We state one result here which is an immediate corollary of case (i) of Theorem 3.1.

THEOREM 5.1. *Suppose there are numbers* $\psi > 1, q^{-1} \leqq \theta_1 \leqq 1$, *and* $r^{-1} \leqq \theta_2 \leqq 1$ *such that the following hold with* $f(t) = 1$ *in* (1.7) *and* $\Omega = R^n$.

(i) $\sup_{t\in R^n, \epsilon>0} [\epsilon^{-n} \int_{Q_{t,\epsilon}} N^{\psi/(\psi-1)}]^{(\psi-1)/\psi} [\epsilon^{-n} \int_{Q_{t,\epsilon}} W^{-\theta_1/(1-\theta_1)}]^{p(1-\theta_1)/\theta_1 q} := s_1 < \infty.$

(ii) $\sup_{t\in R^n, \epsilon>0} [\epsilon^{-n} \int_{Q_{t,\epsilon}} N^{\psi/(\psi-1)}]^{(\psi-1)/\psi} [\epsilon^{-n} \int_{Q_{t,\epsilon}} P^{-\theta_2/(2-\theta_1)}]^{p(1-\theta_2)/\theta_2 q} := s_2 < \infty.$

(iii) $p \geqq \max\{q, r\}$.

(iv) One of the conditions (i), (ii), or (iii) of Lemma 2.3 *holds.*

Then for all $u \in E(R^n)$,

$$\int_{R^n} N|D^j u|^p \leqq K \left(\int_{R^n} W|u|^q\right)^{pa/q} \left(\int_{R^n} P|D^m u|^r\right)^{p(1-a)/r} \tag{5.1}$$

where $a = (m - j - n/r + n/p)/(m - n/r + n/q)$ *and* K *is independent of* u.

Our motivation for stating this case comes from an interesting result which has recently been proved by C. Gutierrz and R. Wheeden [**10**]. A corollary of their Theorem 5 is

$$\int_{R^n} \omega|Du|^p \leqq K \left(\int_{R^n} \omega|u|^p\right)^{1/2} \left(\int_{R^n} \omega|D^2 u|^p\right)^{1/2}$$

for all $u \in C^{(2)}(R^n)$ with compact support if ω satisfies the condition

$$A_p : \sup_{Q\subset R^n} \left(\frac{1}{|Q|} \int_{R^n} \omega\right) \left(\frac{1}{|Q|} \int_{R^n} \omega^{-1/(p-1)}\right)^{p-1} := C_p < \infty$$

where Q is a cube in R^n and $|Q|$ is the measure of Q. We will show that the conditions (i) and (ii) of Theorem 5.1 are sometimes implied by A_p conditions

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although in the special case $p = q = r, m = 2$, and $j = 1$ the result falls short of that of Gutierrz and Wheeden.

Take $W = N^{q/p}$ and $P = N^{r/p}$ and suppose $N \in A_z$ for some z (to be specified later). Then for some $\delta > 0$, depending only on z and C_z,

$$(5.2) \qquad \left(\frac{1}{|Q|}\int_{R^n} N^{1+\delta}\right)^{1/(1+\delta)} \leqq \frac{C_z}{|Q|}\int_{R^n} N,$$

See [**6**, p. 397]. Choose ψ so that $\psi/(\psi-1) = 1+\delta$. Now $W^{-\theta_1/(1-\theta_1)} = N^{-q\theta_1/p(1-\theta_1)} = N^{-1/(z_1-1)}$ where $z_1 = 1 + p(1-\theta_1)/q\theta_1$. Hence (i) of Theorem 5.1 holds if $N \in A_{z_1}$. Similarly (ii) of Theorem 5.1 holds if $N \in A_{z_2}$ with $z_2 = 1 + p(1-\theta_2)/r\theta_2$. Since $A_p \subset A_s$ for $p < s$, [**6**, p. 394], the best result is obtained by choosing θ_1, θ_2 as small as possible to be consistent with the other conditions of Theorem 5.1.

The easiest case to discuss is when $q = r$, and $m - j > n/q$. In this case choose ψ so that $\psi/(\psi-1) = 1+\delta$ where δ is as in (5.2). We choose θ_1 as small as possible while preserving $q^{-1} \leqq \theta_1 \leqq 1$. For $m - j > n$, take $\theta_1 = q^{-1}$ (now (i) of Lemma 2.3 holds) so that $z_1 = 1 + p(q-1)/q$. For $n/q < m - j \leqq n$, take $\theta_1 = n/q(m-j)$ (now (ii) of Lemma 2.3 holds) so that $z_1 = 1+p[q(m-j)-n]/qn$. In both cases take $\theta_2 = \theta_1$. Thus we have for $p \geqq q = r, W = P = N^{q/p}$, that (5.1) holds if

$$(5.3) \qquad m - j > n, \quad N \in A_z, \quad z = 1 + p(q-1)/q,$$

or if

$$(5.4) \qquad n/q < m - j \leqq n, \quad N \in A_z, \quad z = 1 + p[q(m-j) - n]/qn.$$

Note that for $p = q$ (5.3) gives $N \in A_p$ while (5.4) gives $N \in A_z, z = q(m-j)/n \leqq q = p$, so that the conclusion is weaker that the Gutierrz and Wheeden result. On the other hand, the indices p, q, r are not required to be equal and the function u need not have compact support in Theorem 5.1. Clearly one may work out A_p conditions in Theorem 5.1 in other cases, e.g., $q \neq r$, as well as in the other three cases of Theorem 3.1. However the calculations are more involved in these situations.

Remark 5.1. (i) For $n = 1$ many product inequalities are known for weights N, W, P, e.g., see [**2**], [**14**], [**15**]. The case $n = 1, N = W = P = 1$ is due to Landau. Good bounds for the constant K in many cases have been derived by M. Kwong and A. Zettl. (ii) That ω satisfy an A_p condition is far from a necessary condition for a product inequality. It was proven in [**3**, Theorem 5.1] that another sufficient condition for a product inequality on R^n is that $p = q = r, m-j > n/r$, and $N = W = P$ be a nondecreasing function in each coordinate. (iii) Necessary and sufficient conditions for compact support functions and power type weights have been given by Lin [**16**].

References

1. R. A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.

2. R. C. Brown and D. B. Hinton, *Necessary and sufficient conditions for one variable weighted interpolation inequalities*, J. London Math. Soc. **(2)35** (1987), 439–453.

3. R. C. Brown and D. B. Hinton, *Weighted interpolation inequalities of sum and product form in* R^n, Proc. London Math. Soc. **(3)56** (1988), 261–280.

4. D. E. Edmunds and W. D. Evans, "Spectral Theory and Differential Operators," Oxford University Press, 1987.

5. W. D. Evans, N. K. Kwong, and A. Zettl, *Lower bounds for the spectrum of ordinary differential operators*, J. Differential Eqs. **48** (1983), 123–155.

6. J. Garcia-Cuerva and J. Rubio De Francia, "Weighted norm inequalities and related topics," North Holland, Amsterdam, 1985.

7. P. Gurka and B. Opic, *Continuous and compact imbeddings of weighted Sobolev spaces I*, Czech. Math. J. **38** (1988), 611–617.

8. P. Gurka and B. Opic, *Continuous and compact imbeddings of weighted Sobolev spaces II*, Czech. Math. J. **39** (1989), 78–94.

9. P. Gurka and B. Opic, *N-dimensional Hardy inequality and imbedding theorems for weighted Sobolev spaces on unbounded domains*. To appear in Proceedings, Summer School on Function Spaces, etc., Sodankyla, Finland.

10. C. E. Gutierrz and R. L. Wheeden, *Interpolation inequalities with weights*. to appear.

11. Miguel De Guzmann, *Differentiation of Integrals in* R^n, in "Lecture Notes in Mathematics 481," Springer, Berlin, 1975.

12. A. Kufner, "Weighted Sobolev Spaces," John Wiley and Sons, Chichester, 1985.

13. A. Kufner, O. John, and S. Fučik, "Function Spaces," Noordhoff International Publishing, Leyden, 1977.

14. M. K. Kwong and A. Zettl, *Ramifications of Landau's inequality*, Proc. Roy. Soc. Edinburgh **A86** (1980), 175–212.

15. M. K. Kwong and A. Zettl, *Weighted norm inequalities of sum form involving derivatives*, Proc. Roy. Soc. Edinburgh **A88** (1981), 121–134.

16. C. S. Lin, *Interpolation inequalities with weights*, Comm. Partial Differential Eqs. **11** (1986), 1515–1538.

17. B. Opic, *Necessary and sufficient conditions for imbedding in weighted Sobolev spaces*, Căsopis pro pěst Math. **114** (1989), 165–175.

18. H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North Holland, Amsterdam, 1978.

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