1

Basic Notions in Category Theory

1.1 Definition of a Category and Examples

If you want to define a category, it is not enough to specify the objects that you want to consider; you always have to say what kind of morphisms you want to allow.

Definition 1.1.1 A *category* $\mathcal{C}$ consists of

(1) A class of objects, Ob$\mathcal{C}$.
(2) For each pair of objects C_1 and C_2 of $\mathcal{C}$, there is a set $\mathcal{C}(C_1, C_2)$. We call the elements of $\mathcal{C}(C_1, C_2)$ the *morphisms from* C_1 *to* C_2 *in* $\mathcal{C}$.
(3) For each triple C_1, C_2, and C_3 of objects of $\mathcal{C}$, there is a composition law

$$\mathcal{C}(C_1, C_2) \times \mathcal{C}(C_2, C_3) \to \mathcal{C}(C_1, C_3).$$

We denote the composition of a pair (f, g) of morphisms by $g \circ f$.
(4) For every object C of $\mathcal{C}$, there is a morphism 1_C, called the *identity morphism on* C.

The composition of morphisms is associative, that is, for morphisms $f \in \mathcal{C}(C_1, C_2)$, $g \in \mathcal{C}(C_2, C_3)$, and $h \in \mathcal{C}(C_3, C_4)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

and identity morphisms do not change morphisms under composition, that is, for all $f \in \mathcal{C}(C_1, C_2)$,

$$1_{C_2} \circ f = f = f \circ 1_{C_1}.$$

We will soon see plenty of examples of categories. Despite the fact that for some categories this notation is utterly misleading, it is common to denote morphisms as arrows. If $f \in \mathcal{C}(C_1, C_2)$, then we represent f as $f : C_1 \to C_2$.

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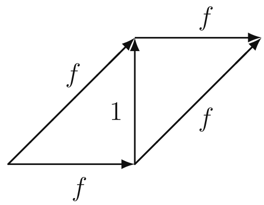
As for functions, we call $C_1 = s(f)$ the *source (or domain) of* f and $C_2 = t(f)$ the *target (or codomain) of* f for all $f \in \mathcal{C}(C_1, C_2)$.

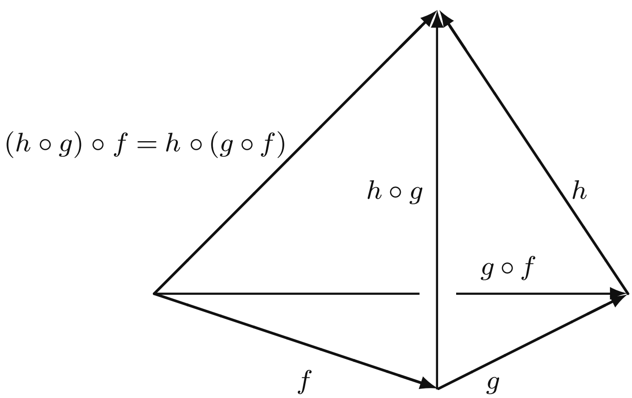
The identity morphism 1_C is uniquely determined by the object C: if both 1_C and $1'_C$ are identity morphisms on C, then

$$1_C = 1_C \circ 1'_C = 1'_C.$$

One can visualize the unit and associativity conditions geometrically. Omitting the objects from the notation, the rule $f \circ 1 = f = 1 \circ f$ can be expressed as two (glued) triangles,



and the associativity constraint corresponds to a tetrahedron.



These pictures are more than mere illustrations; this will become clear when we discuss nerves and classifying spaces of small categories in 11.1 and 11.2.

Remark 1.1.2 Sometimes one does not require that the morphisms constitute a set, but one would allow classes of morphisms as well. In such contexts, our definition would be called a *locally small category*, that is, one in which for every pair of objects C_1, C_2, $\mathcal{C}(C_1, C_2)$ is a set and not a class.

Remark 1.1.3 Some authors require that the sets of morphisms in a category are pairwise disjoint. If (C_1, C_2) is different from (C'_1, C'_2), then

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$\mathcal{C}(C_1, C_2) \cap \mathcal{C}(C_1', C_2') = \varnothing$. This is related to the question of how you define a function: If X and Y are sets, then a function can be viewed as a relation $f \subset X \times Y$ with the property that, for all x in X, there is a unique $y \in Y$ with $(x, y) \in f$, or you could say that a function is a triple (X, Y, f) with $f \subset X \times Y$ with the same uniqueness assumption. In the latter definition, the domain and target are part of the data. In the first definition, f could also be a function for some other X' and Y'.

Definition 1.1.4

- A category $\mathcal{C}$ is *small* if the objects of $\mathcal{C}$ are a set (and not a proper class).
- A small category is *finite* if its set of objects is a finite set and every set of morphism is finite.
- A category is *discrete* if the only morphisms that occur in it are identity morphisms.

In particular, you can take any class X and form the *discrete category associated with* X, by declaring the elements of X to be the objects and by allowing only the identity morphisms as morphisms. If X is a set, then this category is small.

You are probably already familiar with several examples of categories.

Examples 1.1.5

- Sets: The category of sets and functions of sets. Here, the objects form a proper class.
- Gr: The category of groups and group homomorphisms.
- Ab: The category of abelian groups and group homomorphisms.
- K-vect: Here, K is a field and K-vect is the category of K-vector spaces and K-linear maps.
- R-mod: Here, R is an associative ring with unit and R-mod is the category of (left) R-modules and R-linear maps.
- Top: The category of topological spaces and continuous maps.
- Top_*: The category of topological spaces with a chosen basepoint and continuous maps preserving the basepoint.
- CW: The category of CW complexes and cellular maps.
- Ch: The category of (unbounded) chain complexes of abelian groups together with chain maps. Here, objects are families of abelian groups $(X_i)_{i \in \mathbb{Z}}$ with boundary operators $d_i : X_i \to X_{i-1}$

$$\cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots.$$

The d_i are linear maps and satisfy $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. We denote such a chain complex by (X_*, d). A chain map from (X_*, d) to (Y_*, d') is a

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family of linear maps $(f_i : X_i \to Y_i)_{i\in\mathbb{Z}}$ such that $f_i \circ d_{i+1} = d'_i \circ f_{i+1}$ for all $i \in \mathbb{Z}$, so the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_{n+2}} & X_{n+1} & \xrightarrow{d_{n+1}} & X_n & \xrightarrow{d_n} & X_{n-1} & \xrightarrow{d_{n-1}} & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \xrightarrow{d'_{n+2}} & Y_{n+1} & \xrightarrow{d'_{n+1}} & Y_n & \xrightarrow{d'_n} & Y_{n-1} & \xrightarrow{d'_{n-1}} & \cdots. \end{array}$$

We also consider the variant of the category of nonnegatively graded chain complexes, $\mathsf{Ch}_{\geq 0}$, where $X_i = 0$ for all negative indices i. An important variant is to allow different ground rings than the integers, so we might consider chain complexes of R-modules for some associative and unital ring R, and then, the boundary operators and chain maps are required to be R-linear.

There are other examples of categories where you might find the morphisms slightly nonstandard.

Examples 1.1.6

(1) Let Corr be the category of correspondences. Objects of this category are sets, and the morphisms $\mathrm{Corr}(S, T)$ between two sets S and T are the subsets of the product $S \times T$. If you have $U \subset R \times S$ and $V \subset S \times T$, then $U \times V$ is a subset of $R \times S \times S \times T$. You can take the preimage of $U \times V$ under the map $j : R \times S \times T \to R \times S \times S \times T$ that takes the identity on R and T and the diagonal map on S and then project with $p : R \times S \times T \to R \times T$. This gives the composition. The identity morphism on the set S is the diagonal subset

$$\Delta_S = \{(s, s) | s \in S\} \subset S \times S.$$

(2) Let X be a partially ordered set (poset, for short), that is, a nonempty set X together with a binary relation $\leq$ on X that satisfies that $x \leq x$ for all $x \in X$ (reflexivity), that $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity), and if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).

We consider such a poset as a category, and by abuse of notation, we call this category X. Its objects are the elements of X, and the set of morphisms $X(x, y)$ consists of exactly one element if $x \leq y$. Otherwise, this set is empty.

(3) Quite often, we will view categories as diagrams. For instance, let [0] be the category with one object and one morphism, the identity on that object.

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Similarly, let $[1] = \{0, 1\}$ be the category with two objects 0 and 1, coming with their identity morphisms and one other morphism from 0 to 1. This corresponds to the poset $0 < 1$ viewed as a category: $0 \to 1$.

When we draw diagrams like that, we usually omit the identity morphisms and we don't draw composites in posets. For every poset $[n] = 0 < 1 < \cdots < n$, we get the corresponding category $0 \to 1 \to \cdots \to n$.

(4) Let X be a topological space and let $\mathfrak{U}(X)$ denote its family of open subsets of X. We can define a partial order on $\mathfrak{U}(X)$ by declaring that $U \leq V$ if and only if $U \subset V$.

(5) If $\mathcal{C}$ is an arbitrary category and if C is an object of $\mathcal{C}$, then the endomorphisms of C, $\mathcal{C}(C, C)$ form a monoid, that is, a set with a composition that is associative and possesses a unit. Thus, every category can be thought of as a *monoid with many objects*.

Conversely, if $(M, \cdot, 1)$ is a monoid with composition $\cdot$ and unit 1, then we can form the category that has one object $*$ and has M as its set of endomorphisms. We denote this category by $\mathcal{C}_M$.

There are several constructions that build new categories from old ones.

Definition 1.1.7

- We will need the *empty category*. It has no object and hence no morphism.
- If we have two categories $\mathcal{C}$ and $\mathcal{D}$, then we can build a third one by forming their *product* $\mathcal{C} \times \mathcal{D}$. As the notation suggests, the objects of $\mathcal{C} \times \mathcal{D}$ are pairs of objects (C, D) with C an object of $\mathcal{C}$ and D an object of $\mathcal{D}$. Morphisms are pairs of morphisms:

$$\mathcal{C} \times \mathcal{D}((C_1, D_1), (C_2, D_2)) = \mathcal{C}(C_1, C_2) \times \mathcal{D}(D_1, D_2),$$

and composition and identity morphisms are formed componentwise:

$$(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1), \quad 1_{(C,D)} = (1_C, 1_D).$$

This is indeed a category.

- Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can also form their disjoint union, $\mathcal{C} \sqcup \mathcal{D}$. Its objects consist of the disjoint union of the objects of $\mathcal{C}$ and $\mathcal{D}$. One defines

$$(\mathcal{C} \sqcup \mathcal{D})(X, Y) := \begin{cases} \mathcal{C}(X, Y), & \text{if } X, Y \text{ are objects of } \mathcal{C}, \\ \mathcal{D}(X, Y), & \text{if } X, Y \text{ are objects of } \mathcal{D}, \\ \varnothing, & \text{otherwise.} \end{cases}$$

- If we want a limited amount of interaction between $\mathcal{C}$ and $\mathcal{D}$, we can form the *join of* $\mathcal{C}$ *and* $\mathcal{D}$, denoted by $\mathcal{C} * \mathcal{D}$. The objects of $\mathcal{C} * \mathcal{D}$ are the disjoint union of the objects of $\mathcal{C}$ and the objects of $\mathcal{D}$, and as morphism, we have

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$$(\mathcal{C}*\mathcal{D})(X, Y) = \begin{cases} \mathcal{C}(X, Y), & \text{if } X \text{ and } Y \text{are objects of } \mathcal{C}, \\ \mathcal{D}(X, Y), & \text{if } X \text{ and } Y \text{ are objects of } \mathcal{D}, \\ \{*\}, & \text{if } X \text{ is an object of } \mathcal{C} \text{ and } Y \text{ is an object of } \mathcal{D}, \\ \varnothing, & \text{otherwise.} \end{cases}$$

So the join is not symmetric: There are morphisms from $\mathcal{C}$ to $\mathcal{D}$ but not from $\mathcal{D}$ to $\mathcal{C}$.

- Let $\mathcal{C}$ be an arbitrary category. Let $\mathcal{C}^o$ be the category whose objects are the same as the ones of $\mathcal{C}$ but where

$$\mathcal{C}^o(C, C') = \mathcal{C}(C', C).$$

We denote by f^o the morphism in $\mathcal{C}^o(C, C')$ corresponding to $f \in \mathcal{C}(C', C)$.

The composition of $f^o \in \mathcal{C}^o(C, C')$ and $g^o \in \mathcal{C}^o(C', C'')$ is defined as $g^o \circ f^o := (f \circ g)^o$. The category $\mathcal{C}^o$ is called the *dual category of* $\mathcal{C}$ or the *opposite category of* $\mathcal{C}$.

If you consider the preceding example of the category $\mathcal{C}_M$ from above, then the dual $(\mathcal{C}_M)^o$ is the category associated with the opposite of the monoid M, M^o. Here, M^o has the same underlying set as M, but the multiplication is reversed:

$$m \cdot^o m' := m' \cdot m.$$

1.2 EI Categories and Groupoids

Definition 1.2.1 We call a morphism $f \in \mathcal{C}(C, C')$ in a category $\mathcal{C}$ an *isomorphism* if there is a $g \in \mathcal{C}(C', C)$, such that $g \circ f = 1_C$ and $f \circ g = 1_{C'}$.

We denote g by f^{-1}, because g is uniquely determined by f.

Definition 1.2.2

- A category $\mathcal{C}$ is an *EI category* if every endomorphism of $\mathcal{C}$ is an isomorphism.
- A category $\mathcal{C}$ is a *groupoid* if every morphism in $\mathcal{C}$ is an isomorphism.

Of course, every groupoid is an EI category. In any EI category, the endomorphisms of an object form a group.

Examples 1.2.3

- Consider the category $\mathcal{I}$ whose objects are the finite sets $\mathbf{n} = \{1, \ldots, n\}$ with $n \geq 0$ and $\mathbf{0} = \varnothing$. The morphisms in $\mathcal{I}$ are injective functions. Hence, the

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endomorphisms of an object **n** constitute the symmetric group on n letters, Σ_n, and $\mathcal{I}$ is an EI category.

- Dually, let Ω be the category of finite sets and surjections, that is, Ω has the same objects as $\mathcal{I}$, but $\Omega(\mathbf{n}, \mathbf{m})$ is the set of surjective functions from the set **n** to the set **m**. Again, the endomorphisms of **n** consist of the permutations in Σ_n, and Ω is an EI category.
- Let $\mathcal{C}$ be any category. Then one can build the associated *category of isomorphisms of* $\mathcal{C}$, $\mathrm{Iso}(\mathcal{C})$. This has the same objects as $\mathcal{C}$, but we take

$$\mathrm{Iso}(\mathcal{C})(C_1, C_2) = \{f \in \mathcal{C}(C_1, C_2) | f \text{ is an isomorphism}\}.$$

Hence, for all categories $\mathcal{C}$, the category $\mathrm{Iso}(\mathcal{C})$ is a groupoid. We call the category $\mathrm{Iso}(\mathcal{I}) = \mathrm{Iso}(\Omega)$ the *category of finite sets and bijections*, Σ, that is, Σ has the same objects as Ω but

$$\Sigma(\mathbf{n}, \mathbf{m}) = \begin{cases} \Sigma_n, & \text{if } n = m, \\ \varnothing, & \text{otherwise.} \end{cases}$$

- If G is a group, then we denote by $\mathcal{C}_G$ the category with one object $*$ and $\mathcal{C}_G(*, *) = G$ with group multiplication as composition of maps. Then, $\mathcal{C}_G$ is a groupoid. Hence, every group gives rise to a groupoid. Vice versa, a groupoid can be thought of as a *group with many objects*.
- Let X be a topological space. The *fundamental groupoid of* X, $\Pi(X)$, is the category whose objects are the points of X, and $\Pi(X)(x, y)$ is the set of homotopy classes of paths from x to y:

$$\Pi(X)(x, y) = [[0, 1], 0, 1; X, x, y].$$

The endomorphisms $\Pi(x, x)$ of $x \in X$ constitute the fundamental group of X with respect to the basepoint x, $\pi_1(X, x)$.
- Another important example of a groupoid is the *translation category of a group*. If G is a discrete group, then we denote by $\mathcal{E}_G$ the category whose objects are the elements of the group and

$$\mathcal{E}_G(g, h) = \{hg^{-1}\}, \ g \xrightarrow{hg^{-1}} h.$$

This category has the important feature that there is precisely one morphism from one object to any other object, so every object has equal rights.

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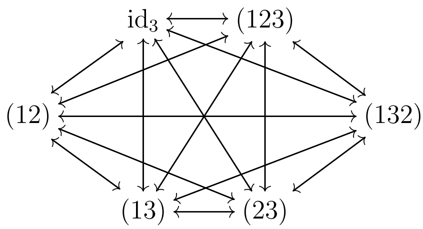
For the symmetric group on three letters, Σ_3, the diagram of objects and (nonidentity) morphisms looks as follows:



Here, we use cycle notation for permutations. Note that the upper-right triangle depicts the translation category of the cyclic group of order three.

1.3 Epi- and Monomorphisms

Often, we will need morphisms with special properties. In the category of sets, one can use elements in order to test whether a function is surjective or injective. In a general category, we do not have a notion of elements, but we always have sets of morphisms. Epimorphisms and monomorphisms are defined using morphisms as test objects. For the category of sets, this is straightforward: if a function $f\colon S \to T$ is injective, then $f(s_1) = f(s_2)$ implies that $s_1 = s_2$. So, this is also true for morphisms $h_1, h_2\colon U \to S$. If $f \circ h_1 = f \circ h_2$, then $h_1 = h_2$. A similar consideration applies to surjective functions.

Definition 1.3.1 Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(C_1, C_2)$. Then, f is an *epimorphism* if for all objects D in $\mathcal{C}$ and all pairs of morphisms $h_1, h_2\colon C_2 \to D$, the equation $h_1 \circ f = h_2 \circ f$ implies that $h_1 = h_2$.

Epimorphisms are therefore right-cancellable.

Exercise 1.3.2 Beware, epimorphisms might not be what you think they are. Consider the category of commutative rings with unit and show that the unique morphism from the integers into the rational numbers is an epimorphism.

Remark 1.3.3 Of course, every identity morphism is an epimorphism and the composition of two epimorphisms is an epimorphism. If $g \circ f$ is an epimorphism, then so is g, because $h_1 \circ g = h_2 \circ g$ implies that $h_1 \circ g \circ f = h_2 \circ g \circ f$, and by assumption, this yields $h_1 = h_2$. Note that every isomorphism is an epimorphism.

Proposition 1.3.4 *Epimorphisms in the category of sets and functions are precisely the surjective functions.*

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Proof Let $f: X \to Y$ be a surjective function of sets and let $h_1, h_2: Y \to Z$ be functions with $h_1 \circ f = h_2 \circ f$. Let $y \in Y$ be an arbitrary element. Then, there is an $x \in X$ with $f(x) = y$. Hence

$$h_1(y) = h_1(f(x)) = h_2(f(x)) = h_2(y),$$

and thus, $h_1 = h_2$. Conversely, let $f: X \to Y$ be an epimorphism. Let $Z = \{z_1, z_2\}$ with $z_1 \neq z_2$ and define $h_1: Y \to Z$ as $h_1(y) = z_1$ for all $y \in Y$ and let h_2 be the test function of the image of f, that is, $h_2(y) = z_1$ if y is in the image of f and $h_2(y) = z_2$ if y lies outside of the image of f. Then, $h_1 \circ f = h_2 \circ f$ is the constant function with value z_1, and hence, $h_1 = h_2$, which implies that f is surjective. □

Similarly, one can prove that epimorphisms of topological spaces are surjective continuous maps. It is more involved to show that epimorphisms of groups are surjective group homomorphisms. See, for example, [**Bo94-1**, 1.8.5.d].

Dual to the notion of an epimorphism is the one of a monomorphism.

Definition 1.3.5 Let $\mathcal{C}$ be a category. A morphism $f \in \mathcal{C}(C_1, C_2)$ is a *monomorphism* if $f^o \in \mathcal{C}^o(C_2, C_1)$ is an epimorphism.

So, monomorphisms are left-cancellable. For all objects D in $\mathcal{C}$ and all $h_1, h_2: D \to C_1$, the equation $f \circ h_1 = f \circ h_2$ implies that $h_1 = h_2$.

One can use morphism sets as test objects because the very definition of mono- and epimorphisms gives the following criteria.

Proposition 1.3.6 *A morphism $f \in \mathcal{C}(C_1, C_2)$ is a monomorphism if and only if for all objects D in $\mathcal{C}$, the induced map $\mathcal{C}(D, f): \mathcal{C}(D, C_1) \to \mathcal{C}(D, C_2)$ is an injective function of sets.*

Dually, a morphism $f \in \mathcal{C}(C_1, C_2)$ is an epimorphism if and only if for all objects D in $\mathcal{C}$, the induced map $\mathcal{C}(f, D): \mathcal{C}(C_2, D) \to \mathcal{C}(C_1, D)$ is an injective function of sets.

Monomorphisms of sets are injective functions. Monomorphisms of commutative rings with unit are injective ring homomorphisms: Let $f: R_1 \to R_2$ be a monomorphism of commutative rings with unit. Consider the polynomial ring in one variable over the integers, $\mathbb{Z}[X]$. A morphism $h: \mathbb{Z}[X] \to R_1$ determines and is determined by the element $r = h(X)$ of R_1. For all $r, s \in R_1$, we define $h_1: \mathbb{Z}[X] \to R_1$ via $h_1(X) = r$ and $h_2: \mathbb{Z}[X] \to R_1$ via $h_2(X) = s$. If $f(r) = f(s)$, then $f \circ h_1 = f \circ h_2$, and thus, $h_1 = h_2$, which implies $r = s$, so f is injective. The converse is easy to see.

There are categories where monomorphisms should be handled with care, that is, where monomorphisms do not behave like injective maps. A standard example is the category of divisible abelian groups (see Exercise 1.3.13).

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A topologically minded example is the category of connected Hausdorff topological groups [**HM13**, A3.10].

Exercise 1.3.7 Show that, in the category of monoids, the inclusion of the additive monoid of natural numbers into the integers is a monomorphism and an epimorphism.

Definition 1.3.8 A morphism $r \in \mathcal{C}(C_1, C_2)$ is called a *retraction* if there is an $s \in \mathcal{C}(C_2, C_1)$, such that $r \circ s = 1_{C_2}$. In this situation, s is called a *section* and C_2 is a *retract of* C_1.

Proposition 1.3.9 *Retractions are epimorphisms, and sections are monomorphisms.*

Proof We only prove the first claim; the second is dual. Let r be a retraction with section s. If $h_1 \circ r = h_2 \circ r$, then $h_1 = h_1 \circ r \circ s = h_2 \circ r \circ s = h_2$. □

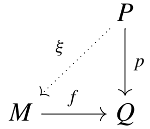
Remark 1.3.10 Be careful: the converse of the preceding statement is often wrong. For instance, let $\mathcal{C}$ be the category of groups. Then, a surjective group homomorphism f does not have a section in general. There is a section of the underlying function on sets, but this section does not have to be a group homomorphism in general.

The example of the category of commutative rings with units shows that there are categories where morphisms f that are epimorphisms and monomorphisms do not have to be isomorphisms: take $f\colon \mathbb{Z} \to \mathbb{Q}$. It is a monomorphism because it is injective, and it is an epimorphism but certainly not an isomorphism.

You might know the notions of projective and injective modules from homological algebra. The following is the categorical analog of these properties.

Definition 1.3.11

- An object P in a category $\mathcal{C}$ is called *projective* if for every epimorphism $f\colon M \to Q$ in $\mathcal{C}$ and every $p\colon P \to Q$, there is a $\xi \in \mathcal{C}(P, M)$ with $f \circ \xi = p$:



- Dually, an object I in a category $\mathcal{C}$ is called *injective* if for every monomorphism $f\colon U \to M$ in $\mathcal{C}$ and every $j\colon U \to I$, there is a $\zeta \in \mathcal{C}(M, I)$ with $\zeta \circ f = j$:

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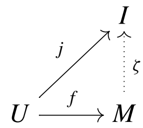

Remark 1.3.12 We think of the morphism ξ as a *lift* of p to M and of the morphism ζ as an *extension* of j to M. Note that uniqueness of the morphisms ξ and ζ is *not* required.

In the category of sets, every object is injective and projective, assuming the axiom of choice for projectivity. In the category of left R-modules for R an associative ring with unit, projectivity and injectivity are precisely defined as in homological algebra. Examples of projective modules are free modules or R-modules of the form Re, where e is an idempotent element of R, that is, $e^2 = e$. Injective $\mathbb{Z}$-modules, that is, injective abelian groups, are divisible abelian groups. These are abelian groups A, such that $nA = A$ for all natural numbers $n \neq 0$. Thus, $\mathbb{Q}$ and the discrete circle $\mathbb{Q}/\mathbb{Z}$ are injective abelian groups.

Exercise 1.3.13 Show that in the category of divisible abelian groups, the canonical projection map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism.

Projectivity and injectivity are preserved by passing to retracts.

Proposition 1.3.14 *If P is a projective object of a category $\mathcal{C}$ and if $i \colon U \to P$ is a monomorphism in $\mathcal{C}$ with a retraction $r \colon P \to U$, then U is projective. Similarly, if $i \colon J \to I$ is a monomorphism with retraction $r \colon I \to J$ and I is injective, then J is injective.*

Proof Let $f \colon M \to Q$ be an epimorphism. If U maps to Q via g, then P maps to Q via $g \circ r$. Thus, there is a morphism $\xi \colon P \to M$ with $f \circ \xi = g \circ r$, and therefore, $\xi \circ i$ satisfies $f \circ \xi \circ i = g \circ r \circ i = g$.

In the second case, if U maps to M via the monomorphism f and $j \colon U \to J$, then $i \circ j \colon U \to I$ has an extension $\zeta \colon M \to I$ with $\zeta \circ f = i \circ j$, and hence, $r \circ \zeta$ is the required extension of j to M. □

We also get certain splitting properties for injective and projective objects.

Proposition 1.3.15 *If $q \colon Q \to P$ is an epimorphism and if P is projective, then q has a section. Dually, if $j \colon I \to J$ is a monomorphism and I is injective, then j has a retraction.*

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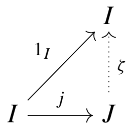
Proof We show the second claim and leave the first claim as an exercise. Consider the diagram



By the injectivity of I, we get an extension of 1_I to J, ζ, satisfying $\zeta \circ j = 1_I$. Thus, ζ is a retraction for j. □

1.4 Subcategories and Functors

Definition 1.4.1 Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ *of* $\mathcal{C}$ consists of a subcollection of objects and morphisms of $\mathcal{C}$, called the objects and morphisms of $\mathcal{D}$, such that

- for all objects D_1, D_2 of $\mathcal{D}$, there is a set of morphisms $\mathcal{D}(D_1, D_2) \subset \mathcal{C}(D_1, D_2)$;
- if $f \in \mathcal{D}(D_1, D_2)$, then D_1, D_2 are objects of $\mathcal{D}$;
- for all objects D of $\mathcal{D}$, the identity morphism 1_D is an element of $\mathcal{D}(D, D)$; and
- if $f \in \mathcal{D}(D_1, D_2)$, $g \in \mathcal{D}(D_2, D_3)$, then the composition of f and g in $\mathcal{C}$ satisfies $g \circ f \in \mathcal{D}(D_1, D_3)$.

Hence, a subcategory of a category is a subcollection of objects and morphisms of the category that is closed under composition, identity morphisms, and source and target. Note that a subcategory again forms a category.

Definition 1.4.2 A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called *full* if for all objects D, D' of $\mathcal{D}$

$$\mathcal{D}(D, D') = \mathcal{C}(D, D').$$

The category of abelian groups is a full subcategory of the category of groups. However, the category $\mathcal{I}$ of finite sets and injections is *not* a full subcategory of the category of finite sets. We restricted the morphisms.

Definition 1.4.3 A *functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$

- Assigns to every object C of $\mathcal{C}$ an object $F(C)$ of $\mathcal{D}$.
- For each pair of objects C, C' of $\mathcal{C}$, there is a function of sets

$$F : \mathcal{C}(C, C') \to \mathcal{D}(F(C), F(C')),\ f \mapsto F(f).$$

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- The following two axioms hold:

$$F(g \circ f) = F(g) \circ F(f) \quad \text{for all } f \in \mathcal{C}(C, C'), g \in \mathcal{C}(C', C''),$$

$$F(1_C) = 1_{F(C)}$$

for all objects C of $\mathcal{C}$.

Like for morphisms, we use the arrow notation $F: \mathcal{C} \to \mathcal{C}'$ to indicate a functor.

Examples 1.4.4

(1) The inclusion of a subcategory into its ambient category defines a functor.

(2) The identity map on objects and morphisms of a category $\mathcal{C}$ define the *identity functor*

$$\mathrm{Id}_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}.$$

(3) Let $(-)_{ab}: \mathsf{Gr} \to \mathsf{Ab}$ be the functor that assigns to a group G the factor group of G with respect to its commutator subgroup: $G/[G, G]$. The resulting group is abelian, and the functor is called the *abelianization*.

(4) Often, we will consider functors that forget part of some structure. These are called *forgetful functors*. For instance, we can consider the underlying set $U(X)$ of a topological space X, and this gives rise to the forgetful functor

$$U: \mathsf{Top} \to \mathsf{Sets}.$$

Similarly, if K is a field, then every K-vector space V has an underlying abelian group $U(V)$, and this gives rise to a forgetful functor

$$U: K\text{-}\mathsf{vect} \to \mathsf{Ab}.$$

Here, we used that continuous maps are in particular functions of sets and that K-linear maps are morphisms of abelian groups.

You should come up with at least five more examples of such forgetful functors.

(5) To a pair of topological spaces (X, A) and to a fixed $n \in \mathbb{N}_0$, you can assign the nth singular homology group of (X, A), $H_n(X, A)$. Then, this defines a functor from the category of pairs of topological spaces to abelian groups.

(6) If you consider topological spaces with a chosen basepoint and if you assign to such a space (X, x) its fundamental group with respect to the basepoint x, $\pi_1(X, x)$, then this defines a functor from Top_* to the category of groups

$$\pi_1: \mathsf{Top}_* \to \mathsf{Gr}.$$

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(7) A functor $F\colon [0] \to \mathcal{C}$ corresponds to a choice of an object in $\mathcal{C}$, namely $F(0)$.

(8) A functor $F\colon [1] \to \mathcal{C}$ corresponds to the choice of two objects in $\mathcal{C}$, $F(0)$ and $F(1)$, and a morphism between them, $F(0 < 1)$:

$$F(0) \xrightarrow{F(0<1)} F(1).$$

(9) Let E be the category with two objects 0 and $0'$ and an isomorphism between them, that is, a morphism $f \in E(0, 0')$ and a morphism $g \in E(0', 0)$, such that $g \circ f = 1_0$ and $f \circ g = 1_{0'}$. Then, a functor F from E to any category $\mathcal{C}$ picks an isomorphism in $\mathcal{C}$ between $F(0)$ and $F(0')$. The category E is therefore often called the *wandering isomorphism*.

(10) A functor $F\colon [2] \to \mathcal{C}$ corresponds to the choice of a composable pair of morphisms in $\mathcal{C}$, so $F(0 < 2) = F(1 < 2) \circ F(0 < 1)$.

$$\begin{array}{ccc} F(0) & \xrightarrow{F(0<1)} & F(1) \\ & \searrow^{F(0<2)} & \downarrow{\scriptstyle F(1<2)} \\ & & F(2) \end{array}$$

(11) We can assign to a set S the free group generated by S, $\mathrm{Fr}(S)$. A function of sets $f\colon S \to T$ induces a group homomorphism

$$\mathrm{Fr}(f)\colon \mathrm{Fr}(S) \to \mathrm{Fr}(T)$$

and hence Fr is a functor from the category of Sets to the category of groups.

(12) Similarly, we can send a set S to the free abelian group generated by S, $\mathrm{Fra}(S)$. This assignment is a functor as well.

(13) An innocent-looking but very important example of a functor is the (covariant) *morphism functor*: For an arbitrary category $\mathcal{C}$ and any object C_0 of $\mathcal{C}$, we can consider the map

$$C \mapsto \mathcal{C}(C_0, C)$$

that sends an object C of $\mathcal{C}$ to the set of morphisms from C_0 to C in $\mathcal{C}$. This defines a functor

$$\mathcal{C}(C_0, -)\colon \mathcal{C} \to \mathsf{Sets}.$$

(14) Another important functor that is crucial for the discussion of limits and colimits later is the constant functor. Consider two arbitrary nonempty

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categories $\mathcal{C}$ and $\mathcal{D}$ and choose an object D of $\mathcal{D}$. The *constant functor from $\mathcal{C}$ to $\mathcal{D}$ with value D* is

$$\Delta_D : \mathcal{C} \to \mathcal{D}, \quad \Delta_D(C) = D, \quad \Delta_D(f) = 1_D$$

for all objects C in $\mathcal{C}$ and all $f \in \mathcal{C}(C_1, C_2)$.

(15) Let M be a smooth manifold and let $C(M)$ be the real vector space of all smooth real-valued functions on M. We denote by Sm the category of smooth manifolds and smooth maps. The assignment $M \mapsto C(M)$ defines a functor from the dual category of Sm to the category of real vector spaces

$$C : \mathsf{Sm}^o \to \mathbb{R}\text{-}\mathsf{vect}.$$

(16) Let X be a topological space, and let $\mathcal{C}$ be an arbitrary category. As discussed earlier, $\mathfrak{U}(X)$ denotes the category of open subsets of X. Its objects are the open subsets of X, and if U and V are objects of $\mathfrak{U}(X)$ with $U \subset V$, then there is a morphism $i_U^V : U \to V$.

A *presheaf* F on X is a functor $F : \mathfrak{U}(X)^{op} \to \mathcal{C}$.

Often, the morphisms $F(i_U^V) : F(V) \to F(U)$ are called *restriction maps* and are denoted by $\mathrm{res}_{V,U}$. The property of F to being a functor is then equivalent to requiring that $\mathrm{res}_{U,U} = \mathrm{id}_U$ for all objects U of $\mathfrak{U}(X)$, and for open subsets $U \subset V \subset W$ in X, it doesn't matter whether you restrict from W to V and then from V to U or you restrict directly from W to U:

$$\mathrm{res}_{W,U} = \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}.$$

Typical examples of presheaves are sets of functions on a topological space X, such as the continuous functions from X to the reals. If $p : E \to M$ is a smooth vector bundle on a smooth manifold M, then setting $F(U)$ to be the set of smooth sections of p on the open subset $U \subset M$ defines a presheaf.

If $\mathcal{C}$ is a concrete category (see 5.1.12), then the elements of $F(U)$ are called *sections of F on U*, and $F(X)$ are the *global sections*. Sometimes, these notions are also used for general $\mathcal{C}$.

Remark 1.4.5 If $F : \mathcal{C} \to \mathcal{D}$ is a functor, then of course you can collect all objects of the form $F(C)$ for objects C of $\mathcal{C}$ and all morphisms $F(f)$ for f a morphism in $\mathcal{C}$, but beware that the image of a functor is not a subcategory of $\mathcal{D}$ in general.

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Take, for instance, $\mathcal{C}$ as the category that consists of two disjoint copies of the poset [1]:

$$\mathcal{C}: \quad C_0 \xrightarrow{f} C_1 \quad C_0' \xrightarrow{f'} C_1'$$

and let $\mathcal{D}$ be the category [2]. Then, we can define a functor from $\mathcal{C}$ to $\mathcal{D}$ by declaring that $F(f) = (0 < 1)$, $F(f') = (1 < 2)$. As f cannot be composed with f' in $\mathcal{C}$, the composition $(0 < 2)$ is not in the image of $\mathcal{C}$ under F [**Sch70**, I.4.1.4].

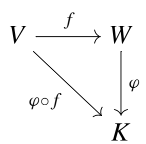
Functors $F: \mathcal{C} \to \mathcal{D}$ are often called *covariant functors*, whereas functors $F: \mathcal{C}^o \to \mathcal{D}$ are called *contravariant functors from* $\mathcal{C}$ *to* $\mathcal{D}$. Thus, these are assignments from the class of objects in $\mathcal{C}$ to the class of objects in $\mathcal{D}$, so that on the level of morphism sets, we get

$$F: \mathcal{C}(C, C') \to \mathcal{D}(F(C'), F(C))$$

with $F(g \circ f) = F(f) \circ F(g)$ and $F(1_C) = 1_{F(C)}$.

Singular cochains (or singular cohomology groups) define a contravariant functor from the category of topological spaces to the category of cochain complexes (or graded abelian groups).

If you assign to a vector space its dual vector space, then for every K-linear map $f: V \to W$, you get a K-linear map $f^*: W^* \to V^*$, which is defined as $\varphi \mapsto \varphi \circ f$



This turns the process of building the dual of a vector space into a contravariant functor from the category of K-vector spaces to itself.

Example 1.4.6 Contravariant functors from the fundamental groupoid of a space X, $\Pi(X)$, to the category of abelian groups, $\mathcal{G}: \Pi(X)^o \to \mathsf{Ab}$, are called (abelian) bundles of groups on X or a system of local coefficients on X. This can be used to define homology with local coefficients (see, for instance, [**Ste43**], [**Wh78**, Chapter VI], or [**DK01**, Chapter 5]).

For every point $x \in X$, we get an abelian group $\mathcal{G}(x)$, and for every homotopy class $[w]$ of a path from x to y, there is a group homomorphism $\mathcal{G}([w]): \mathcal{G}(y) \to \mathcal{G}(x)$. Note that the $\mathcal{G}([w])$s are automatically isomorphisms with inverse $\mathcal{G}([\bar{w}])$, where $\bar{w}$ is the time-reversed path of w.

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Exercise 1.4.7 Consider two groups G, G' and the corresponding categories $\mathcal{C}_G$ and $\mathcal{C}_{G'}$ with one object and morphisms G and G'. Show that functors $F\colon \mathcal{C}_G \to \mathcal{C}_{G'}$ correspond to group homomorphisms $f\colon G \to G'$.

Exercise 1.4.8 Let $(S, \leq)$ and $(T, \leq)$ be two posets. A morphism of posets is an order-preserving function $f\colon S \to T$, that is, if $s_1 \leq s_2$ in S, then $f(s_1) \leq f(s_2)$ in T. Show that functors from the category S to the category T are precisely morphisms of posets.

Let $\mathcal{C}$ be a category, and let C be an object in $\mathcal{C}$. We can use the functors $\mathcal{C}(C, -)$ and $\mathcal{C}(-, C)$ for testing whether C is projective or injective. The following criterion is a direct consequence of the definitions, bearing in mind that epimorphisms in the category Sets are precisely surjective functions.

Proposition 1.4.9

- *The object C is projective if and only if $\mathcal{C}(C, -)\colon \mathcal{C} \to$ Sets preserves epimorphisms.*
- *The object C is injective if and only if $\mathcal{C}(-, C)\colon \mathcal{C}^o \to$ Sets sends monomorphisms to epimorphisms.*

Functors can be used to compare categories.

Definition 1.4.10

- A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *isomorphism of categories* if there exists a functor $G\colon D \to \mathcal{C}$ with the properties $F \circ G = \mathrm{Id}_{\mathcal{D}}$ and $G \circ F = \mathrm{Id}_{\mathcal{C}}$.

  In particular, F induces a bijection between the classes of objects of $\mathcal{C}$ and $\mathcal{D}$ and on the morphism sets.
- A functor is *full* if the assignment

$$F\colon \mathcal{C}(C, C') \to \mathcal{D}(F(C), F(C')),\ f \mapsto F(f) \tag{1.4.1}$$

  is surjective for all pairs of objects C, C' of $\mathcal{C}$.
- A functor is *faithful* if the assignment in (1.4.1) is injective for all pairs of objects C, C' of $\mathcal{C}$.
- A functor is *fully faithful* if the assignment in (1.4.1) is a bijection for all pairs of objects C, C' of $\mathcal{C}$.
- A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for all objects D of $\mathcal{D}$, there is an object C of $\mathcal{C}$, such that $F(C)$ is isomorphic to D.

Exercise 1.4.11 Let R be an associative ring with unit and denote by R^{op} the ring that has the same underlying abelian group as R but whose multiplication is reversed. Show that the categories of left R-modules and of right R^{op}-modules are isomorphic.

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Exercise 1.4.12 Recall the join of categories from Definition 1.1.7. Show that there is an isomorphism of categories between $(\mathcal{C} * \mathcal{D})^o$ and $\mathcal{D}^o * \mathcal{C}^o$.

Prove that there is an isomorphism between the categories $[0] * [0]$ and $[1]$. Show that $[i] * [j]$ is isomorphic to $[i + j + 1]$.

Exercise 1.4.13 Let $F \colon \mathcal{C} \to \mathcal{D}$ be a faithful functor. Show that F detects monomorphisms, that is, if $F(f)$ is a monomorphism, then f is a monomorphism. Do full functors detect epimorphisms?

Remark 1.4.14

- Faithful functors can forget structure. For instance, the forgetful functor from groups to sets is faithful.
- There are examples of functors that are fully faithful but not isomorphisms of categories. The important point is that full faithfulness does not imply that the functor is essentially surjective, and it does not rule out that the functor maps different objects to the same image.
- If $\mathcal{D}$ is a subcategory of $\mathcal{C}$, then there is an inclusion functor $I \colon \mathcal{D} \to \mathcal{C}$. The category $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ if the inclusion functor is a full functor.

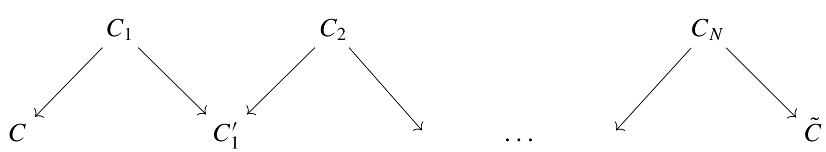
We can compose functors, and we have identity functors; thus, categories behave like objects in a category. This can be made precise.

Definition 1.4.15 We denote by cat the category whose objects are all small categories and whose morphisms between a category $\mathcal{C}$ and a category $\mathcal{D}$ are all functors from $\mathcal{C}$ to $\mathcal{D}$.

Why do we restrict to small categories? We insisted on the morphisms between two objects forming a set. Take, for instance, the category of sets. The functors from Sets to itself contain the constant functors, so for each set, there is a constant functor, with that set as its value. This would already be a proper class of functors.

For a small category, we can define a suitable notion of connectedness.

Definition 1.4.16 Let $\mathcal{C}$ be a small category. Two objects C_1 and C_2 are said to be equivalent if there is a morphism in $\mathcal{C}$ between C_1 and C_2. We consider the equivalence relation generated by this relation. Thus, two objects $C, \tilde{C}$ are equivalent if there is a finite zigzag of morphisms of $\mathcal{C}$ connecting C and $\tilde{C}$:



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If every object of $\mathcal{C}$ is connected to any other object in $\mathcal{C}$, that is, if there is just one such equivalence class, then we call the category $\mathcal{C}$ *connected*.

1.5 Terminal and Initial Objects

Some categories possess special objects.

Definition 1.5.1

- An object t of a category $\mathcal{C}$ is called *terminal* if there exists a unique morphism $f_C: C \to t$ in $\mathcal{C}$ from every object C of $\mathcal{C}$ to t.
- Dually, an object s is called *initial* if there exists a unique morphism $f^C: s \to C$ in $\mathcal{C}$ from s to every object C of $\mathcal{C}$.
- An object 0 is a *zero object* in $\mathcal{C}$ if it is terminal and initial.

Remark 1.5.2 If t is terminal, then the endomorphisms of t consist only of the identity map of t, and dually, the set of endomorphisms of an initial object is $\{1_s\}$.

A small category that possesses an initial or a terminal object is connected.

Terminal and initial objects are unique up to isomorphism. If $\mathcal{C}$ has a zero object 0, then for all pairs of objects C, C' in $\mathcal{C}$, there is the unique morphism

$$C \xrightarrow{f_C} 0 \xrightarrow{f^{C'}} C'.$$

This is often called the *zero morphism* and is denoted by $0: C \to C'$.

Exercise 1.5.3

- Show that any composite of a morphism with the zero morphism is zero.
- If $f: C \to C'$ is a monomorphism and if the composition $f \circ g$ is the zero morphism, then $g = 0$.

Examples 1.5.4

- In the category of sets Sets, the empty set is the initial object and any set with one element is terminal. There is no zero object in Sets. This changes if we consider the category of pointed sets, Sets_*. The objects of Sets_* are sets with a chosen basepoint, and morphisms are functions of sets that map the basepoint in the source to the basepoint in the target. In this example, every set with one element is a zero object.
- Let R be an associative ring with unit. The category of left R-modules has a zero object, and this is the zero module 0. For $R = \mathbb{Z}$, we obtain that the trivial group is a zero object in the category of abelian groups Ab.

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- This also applies to the category of groups: the trivial group is an initial object, and it is also terminal.
- Let G be a group. In the translation category $\mathcal{E}_G$, every object is initial and terminal.

Exercise 1.5.5 Let X be a partially ordered set. What does it say about the partial order relation on X if X has a terminal or initial object? When does X possess a zero object?

Exercise 1.5.6 Let $\mathcal{C}$ be an arbitrary category. Show that the join of $\mathcal{C}$ with [0], $\mathcal{C} * [0]$, has 0 as a terminal object and that $[0] * \mathcal{C}$ has 0 as an initial object.

Definition 1.5.7 The category $\mathcal{C} * [0]$ is the *inductive cone with base* $\mathcal{C}$, and $[0] * \mathcal{C}$ is the *projective cone with base* $\mathcal{C}$.

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