NOTE ON THE MODULAR REPRESENTATIONS OF SYMMETRIC GROUPS

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1. Let p be a fixed prime number. We denote by $k(n)$ the number of partitions of n and set

$$l(\lambda) = \sum_{\lambda_1,\dots,\lambda_p} k(\lambda_1)k(\lambda_2)\dots k(\lambda_p) \qquad \Big(\sum_1^p \lambda_i = \lambda,\ 0 \leqslant \lambda_i \leqslant \lambda\Big), \tag{1}$$

$$l^*(\lambda) = \sum_{\lambda_1,\dots,\lambda_{p-1}} k(\lambda_1)k(\lambda_2)\dots k(\lambda_{p-1}) \qquad \Big(\sum_1^{p-1} \lambda_i = \lambda,\ 0 \leqslant \lambda_i \leqslant \lambda\Big). \tag{2}$$

Recently it was shown by Nakayama and Osima [**8**] and Robinson [**12**] that the number of ordinary irreducible representations belonging to a p-block of weight β is equal to $l(\beta)$. For the number of modular irreducible representations Robinson [**12**] showed that it is independent on the p-core, and using this result, Osima [**10**] proved that it is actually equal to $l^*(\beta)$. In this note we shall give a direct computation of this number.

Now we mention some theorems necessary for the computation without proof. For Young's diagrams $[\alpha]$ and $[\alpha']$ of S_n and $S_{n'}$ $(n' < n)$, we set $r(\alpha, \alpha') = (-1)^r$ if $[\alpha]$ contains an $(n - n')$-hook of leg length r such that $[\alpha']$ can be obtained from $[\alpha]$ by removing it, otherwise we set $r(\alpha, \alpha') = 0$.

Denote by $\chi(\alpha; G)$ the ordinary irreducible character of S_n corresponding to $[\alpha]$, then Murnaghan-Nakayama's recurrence rule [**7**, p. 182; **6**; **15**] is as follows:

If G is an element of S_n containing a g-cycle P and $\bar{G}$ is the permutation of $n - g$ letters arising from G by removing this cycle, then

$$\chi(\alpha; G) = \sum_{[\alpha']} r(\alpha, \alpha')\chi(\alpha'; \bar{G}) \tag{3}$$

where $[\alpha']$ runs over all Young's diagrams of S_{n-g}.

Now let $[\alpha^{(0)}]$ be a p-core[1] with m nodes and $n = m + \beta p$. Then the number of Young's diagrams of $S_{m+\lambda p}$ with p-core $[\alpha^{(0)}]$ is equal [**8**; **9**; **12**; **13**] to $l(\lambda)$. We denote these diagrams by

$$[\alpha_1^{(\lambda)}], \dots, [\alpha_{l(\lambda)}^{(\lambda)}].$$

In case $\lambda \geqslant \mu > 0$ we set

$$\mathfrak{R}_j^{(\lambda,\mu)} = (r(\alpha_1^{(\lambda)}, \alpha_j^{(\lambda-\mu)}), \dots, r(\alpha_{l(\lambda)}^{(\lambda)}, \alpha_j^{(\lambda-\mu)})) \tag{4}$$

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[1]For the notion of p-cores, see [**7**], and for the relation between p-cores and p-blocks, see [**7**] and [**3**].

and

$$R_j^{(\lambda,\mu)}(G) = \sum_{k=1}^{l(\lambda)} r(\alpha_k^{(\lambda)}, \alpha_j^{(\lambda-\mu)})\chi(\alpha_k^{(\lambda)}; G) \qquad (j = 1, 2, \ldots, l(\lambda - \mu)). \tag{5}$$

Then we have [**10**, §1]

THEOREM.

$$\begin{aligned} R_j^{(\lambda,\mu)}(G) &= 0 && \textit{when } G \textit{ contains no } \mu p\textit{-cycle,} \\ &= \frac{n(G)}{n(\bar{G})}\chi(\alpha_j^{(\lambda-\mu)}; \bar{G}) && \textit{when } G \textit{ contains a } \mu p\textit{-cycle,} \end{aligned}$$

where $\bar{G}$ is the permutation arising from G by removing this cycle and $n(G)$, $n(\bar{G})$ are the orders of normalizers of G, $\bar{G}$ in $S_{m+\lambda p}$, $S_{m+(\lambda-\mu)p}$ respectively.

2. First we shall remark that the following propositions are mutually equivalent:

(I) *The number of modular irreducible representations belonging to the block of weight β with p-core $[\alpha^0]$ is equal to $l^*(\beta)$.*

(II) *The rank of the vector module generated by*

$$\mathfrak{R}_j^{(\beta,\lambda)} \qquad (\lambda = 1, \ldots, \beta; j = 1, \ldots, l(\beta - \lambda))$$

is equal to $l(\beta) - l^*(\beta)$.

(III) *The rank of the module consisting of all solutions of the equation*

$$\sum_{\lambda=1}^{\beta} \sum_{j=1}^{l(\beta-\lambda)} x_j^{(\lambda)} \mathfrak{R}_j^{(\beta,\lambda)} = 0 \tag{6}$$

is equal to

$$\sum_{\lambda=1}^{\beta} l(\beta - \lambda) - (l(\beta) - l^*(\beta)).$$

(III′) *The rank of the module consisting of all solutions of the equation*

$$\sum_{\lambda=1}^{\beta} \sum_{j=1}^{l(\beta-\lambda)} x_j^{(\lambda)} R_j^{(\beta,\lambda)}(G) = 0, \qquad G \in S_n \tag{7}$$

is equal to

$$\sum_{\lambda=1}^{\beta} l(\beta - \lambda) - (l(\beta) - l^*(\beta)).$$

By Chung [**4**], Osima [**9**], and Littlewood [**5**] it was shown that

$$\mathfrak{R}_j^{(\beta,\lambda)} \qquad (\lambda = 1, \ldots, \beta; j = 1, \ldots, l(\beta - \lambda))$$

generate the module consisting of all vectors which are orthogonal with every column of the matrix of decomposition numbers corresponding to the p-block, namely, the module consisting of all solutions of the following equations:

$$\sum_{j=1}^{l(\beta)} x_j\,\chi(\alpha_j^{(\beta)};\,V) = 0$$

for every p-regular element V of $S_{m+\beta p}$.

Since the columns of the matrix of decomposition numbers are linearly independent, (I) and (II) are equivalent. The equivalence among (II), (III), and (III′) is almost evident.

In the following we shall prove the proposition (III′). Let

$$(x_j^{(\lambda)})$$

be a solution of (7). If $G = P_\lambda V$ in (7) with a λp-cycle P_λ and a p-regular permutation V of the $n - \lambda p$ letters not contained in P_λ, then since $\lambda \neq \mu$ implies

$$R_j^{(\beta,\mu)}(G) = 0$$

we obtain

$$\sum_{j=1}^{l(\beta-\lambda)} x_j^{(\lambda)}\, R_j^{(\beta,\lambda)}\,(P_\lambda\, V) = 0, \tag{8}$$

and hence, from the theorem in §1, we have

$$\sum_{j=1}^{l(\beta-\lambda)} x_j^{(\lambda)}\,\chi(\alpha_j^{(\beta-\lambda)};\,V) = 0 \tag{9}$$

for all p-regular elements of $S_{n-\lambda p}$.

If $\lambda = \beta$ then $l(0) = 1$ and $x^{(\beta)} = 0$, and if $\lambda < \beta$ then, from the result of Chung [4], Osima [9], and Littlewood [5] mentioned above, it turns out that

$$(x_j^{(\lambda)})_j$$

is a linear combination of

$$\mathfrak{R}_k^{(\beta-\lambda,\mu)} \qquad (\mu = 1, \ldots, \beta - \lambda;\, k = 1, \ldots, l(\beta - \lambda - \mu)).$$

Set

$$(x_j^{(\lambda)})_j = \sum_{\mu=1}^{\beta-\lambda}\ \sum_{k=1}^{l(\beta-\lambda-\mu)} x_k^{(\lambda;\mu)}\,\mathfrak{R}_k^{(\beta-\lambda,\mu)}. \tag{10}$$

Next suppose that $\lambda_1 + \lambda_2 \leqslant \beta$ and set $G = P_{\lambda_1}P_{\lambda_2}V$ in (7) where no two of P_{λ_1}, P_{λ_2}, and V have common letters. If $\lambda_1 \neq \lambda_2$ then

$$\begin{aligned} 0 &= \sum_j x_j^{(\lambda_1)}\, R_j^{(\beta,\lambda_1)}(G) + \sum_j x_j^{(\lambda_2)}\, R_j^{(\beta,\lambda_2)}(G) \\ &= \frac{n(G)}{n(P_{\lambda_2}\, V)}\sum_j x_j^{(\lambda_1)}\chi(\alpha_j^{(\beta-\lambda_1)};\, P_{\lambda_2}\, V) + \frac{n(G)}{n(P_{\lambda_1}\, V)}\sum_j x_j^{(\lambda_2)}\chi(\alpha_j^{(\beta-\lambda_2)};\, P_{\lambda_1}\, V) \end{aligned}$$

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$$= \frac{n(G)}{n(P_{\lambda_2} V)} \sum_{\mu} \sum_{k} x_k^{(\lambda_1;\mu)} R_k^{(\beta-\lambda_1,\mu)}(P_{\lambda_2} V)$$

$$+ \frac{n(G)}{n(P_{\lambda_1} V)} \sum_{\nu} \sum_{k} x_k^{(\lambda_2;\nu)} R_k^{(\beta-\lambda_2,\nu)}(P_{\lambda_1} V)$$

$$= \frac{n(G)}{n(V)} \sum_{k} (x_k^{(\lambda_1;\lambda_2)} + x_k^{(\lambda_2;\lambda_1)}) \chi(\alpha_k^{(\beta-\lambda_1-\lambda_2)}; V) = 0.$$

Hence, $x^{(\lambda_1;\lambda_2)} + x^{(\lambda_2;\lambda_1)} = 0$ if $\lambda_1 + \lambda_2 = \beta$, and if $\lambda_1 + \lambda_2 < \beta$ then

$$(x_k^{(\lambda_1;\lambda_2)} + x_k^{(\lambda_2;\lambda_1)})_k$$

is a linear combination of

$$\mathfrak{R}_j^{(\beta-\lambda_1-\lambda_2,\mu)} \qquad (\mu = 1, \ldots, \beta - \lambda_1 - \lambda_2; j = 1, \ldots, l(\beta - \lambda_1 - \lambda_2 - \mu)).$$

We set

$$(x_j^{(\lambda_1;\lambda_2)} + x_j^{(\lambda_2;\lambda_1)})_j = \sum_{\mu} \sum_{k} x_k^{(\lambda_1\lambda_2;\mu)} \mathfrak{R}_k^{(\beta-\lambda_1-\lambda_2,\mu)}. \tag{11}$$

When $\lambda_1 = \lambda_2$, by similar arguments as above, we have $x^{(\lambda_1;\lambda_2)} = 0$ if $2\lambda_1 = \beta$, and

$$x_j^{(\lambda_1;\lambda_1)} = \sum_{\mu} \sum_{k} x_k^{(\lambda_1\lambda_1;\mu)} \mathfrak{R}_k^{(\beta-2\lambda_1,\mu)} \qquad (2\lambda_1 < \beta). \tag{12}$$

Repeating the similar arguments, we have a set of coefficients

$$x_j^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)} \qquad \left(1 \leqslant \lambda_1 + \ldots + \lambda_t \leqslant \beta;\ j = 1, \ldots, l\left(\beta - \sum_1^t \lambda_i\right)\right)$$

which are independent of the order of $\lambda_1, \ldots, \lambda_{t-1}$, and the relations among these coefficients:

$$\left(\sum_i{}' x_j^{(\lambda_1\ldots\hat{\lambda}_i\ldots\lambda_t;\lambda_i)}\right)_j = 0 \qquad \beta = \sum_1^t \lambda_i, \tag{13}$$

(14)

$$\left(\sum_i{}' x_j^{(\lambda_1\ldots\hat{\lambda}_i\ldots\lambda_t;\lambda_i)}\right)_j = \sum_{\lambda_{t+1}=1}^{\beta-\lambda_1-\ldots-\lambda_t} \sum_{k=1}^{l(\beta-\lambda_1-\ldots-\lambda_{t+1})} x_k^{(\lambda_1\ldots\lambda_t;\lambda_{t+1})} \mathfrak{R}_k^{(\beta-\lambda_1-\ldots-\lambda_t;\lambda_{t+1})},$$

$$\sum_1^t \lambda_i < \beta,$$

where $\{\lambda_1 \ldots \hat{\lambda}_i \ldots \lambda_t\}$ denotes the set of λ's arising from $\{\lambda_1 \ldots \lambda_t\}$ by removing λ_i, and $\sum_i{}'$ indicates the summation over all different $\{\lambda_1 \ldots \hat{\lambda}_i \ldots \lambda_t\}$.

Conversely, it is easily seen from the above arguments that if

$$\{x_j^{(\lambda)}; x_k^{(\lambda_1;\lambda_2)}; \ldots\}$$

satisfies the relations (13) and (14), $x_j^{(\lambda)}$ is a solution of (7).

The propositions (I)–(III′) are true for $\beta = 0$. We shall now assume that the

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propositions have already been shown for all numbers less than β and prove those for β by induction. Then the rank of the module generated by

$$\mathfrak{R}_k^{(\beta-\lambda,\mu)} \qquad (\mu = 1, \ldots, \beta - \lambda; k = 1, \ldots, l(\beta - \lambda - \mu))$$

is equal to $l(\beta - \lambda) - l^*(\beta - \lambda)$. Fix a basis for each λ $(1 \leqslant \lambda \leqslant \beta)$ that contains $\mathfrak{R}^{(\beta-\lambda,\beta-\lambda)}$, and set

$$x_k^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)} = 0 \tag{15}$$

when

$$\mathfrak{R}_k^{(\beta-\lambda_1-\ldots-\lambda_{t-1},\lambda_t)}$$

is not contained in the fixed basis. Then the systems of coefficients which satisfy the relations (13), (14), and (15) form a module isomorphic to the module of the solutions of (7), and hence it is sufficient to prove that the rank of this module is equal to

$$\sum_{\lambda=1}^{\beta} l(\beta - \lambda) - (l(\beta) - l^*(\beta)).$$

LEMMA. *Define the linear forms f, g, and h in $x^{(\lambda_1\ldots\lambda_t;\mu)}$ as follows:*

(i) *for $\lambda_1 + \ldots + \lambda_t = \beta$ we set*

$$f^{(\lambda_1\ldots\lambda_t)}(x) = \sum_i{}' x^{(\lambda_1\ldots\hat{\lambda}_i\ldots\lambda_t;\lambda_i)},$$

(ii) *for $\lambda_1 + \ldots + \lambda_t < \beta$ we set*

$$f_j^{(\lambda_1\ldots\lambda_t)}(x) = \sum_i{}' x_j^{(\lambda_1\ldots\hat{\lambda}_i\ldots\lambda_t;\lambda_i)} - \sum_\mu \sum_k x_k^{(\lambda_1\ldots\lambda_t;\mu)} r(\alpha_j^{(\beta-\lambda_1\ldots-\lambda_t)}; \alpha_k^{(\beta-\lambda_1-\ldots-\lambda_t-\mu)});$$

(iii) *when $1 \leqslant \lambda_1 + \ldots + \lambda_{t-1} \leqslant \beta - 1$ and*

$$\mathfrak{R}_j^{(\beta-\lambda_1-\ldots-\lambda_{t-1},\lambda_t)}$$

is not contained in a fixed basis we set

$$g_j^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)}(x) = x_j^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)},$$

(iv) *for $\mathfrak{R}^{(\beta;\lambda)}$ which is not contained in a fixed basis, we set*

$$h_k^{(\lambda)}(x) = x_k^{(\lambda)}.$$

Then the linear independence of f and g, and that of f, g, and h are both equivalent to the propositions (I)–(III′), *under the assumption that the propositions* (I)–(III′) *are true for all numbers less than β..*

Proof. We denote by A, B, C, and D the numbers of

$$x_j^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)}, f_j^{(\lambda_1\ldots\lambda_t)}, g_j^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)}, h_k^{(\lambda)}$$

respectively, and first compute A, B, and C.

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(a) *Computation of A.* The number of

$$x_j^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)}$$

with $\lambda_1 + \ldots + \lambda_t = \lambda$ is equal to

$$l(\beta - \lambda)\left\{\sum_{\mu=1}^{\lambda} k(\lambda - \mu)\right\},$$

and hence

$$A = \sum_{\lambda=1}^{\beta}\sum_{\mu=1}^{\lambda} l(\beta - \lambda)k(\lambda - \mu). \tag{16}$$

(b) *Computation of B.* The number of

$$f_j^{(\lambda_1\ldots\lambda_t)}$$

with $\lambda_1 + \ldots + \lambda_t = \lambda$ is equal to $l(\beta - \lambda)k(\lambda)$, and hence

$$B = \sum_{\lambda=1}^{\beta} k(\lambda)l(\beta - \lambda).$$

(c) *Computation of C.* The number of

$$g_j^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)}$$

with $\lambda_1 + \ldots + \lambda_{t-1} = \lambda$ $(1 \leqslant \lambda \leqslant \beta - 1)$ is equal to

$$k(\lambda)\left\{\sum_{\mu=1}^{\beta-\lambda} l(\beta - \lambda - \mu) - l(\beta - \lambda) + l^*(\beta - \lambda)\right\},$$

and hence

$$\begin{aligned} C &= \sum_{\lambda=1}^{\beta-1}\sum_{\mu=1}^{\beta-\lambda} k(\lambda)l(\beta - \lambda - \mu) - \sum_{\lambda=1}^{\beta-1} k(\lambda)l(\beta - \lambda) + \sum_{\lambda=1}^{\beta-1} k(\lambda)l^*(\beta - \lambda) \\ &= \sum_{\lambda=2}^{\beta}\sum_{\mu=1}^{\lambda-1} k(\lambda - \mu)l(\beta - \lambda) - \sum_{\lambda=1}^{\beta-1} k(\lambda)l(\beta - \lambda) + \sum_{\lambda=1}^{\beta-1} k(\lambda)l^*(\beta - \lambda). \end{aligned} \tag{17}$$

From these computations we have

$$\begin{aligned} A - (B + C) &= l(\beta - 1)k(0) + \sum_{\lambda=2}^{\beta} l(\beta - \lambda)k(0) - l(0)k(\beta) \\ &\qquad - \sum_{\lambda=1}^{\beta-1} l^*(\beta - \lambda)k(\lambda) \\ &= \sum_{\lambda=1}^{\beta} l(\beta - \lambda) - (l(\beta) - l^*(\beta)). \end{aligned} \tag{18}$$

It is easily seen that

$$l(\beta) - l^*(\beta) = \sum_{\lambda-1}^{\beta} l^*\,(\beta-\lambda)\,k(\lambda).$$

Assume now the linear independence of the linear forms f and g. Since the rank of the module consisting of all solutions of the system of linear equations

$$(19) \qquad f_j^{(\lambda_1\ldots\lambda_t)}(x) = 0, \quad g_j^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)}(x) = 0$$

coincides with that of the module of solutions of (7), and since f and g are linearly independent, the rank is equal to $A - (B + C)$; this proves the proposition (III′) from (18).

Conversely, suppose that the propositions (I)–(III′) are true. Then from the proposition (II), D is equal to

$$\sum_{\lambda=1}^{\beta} l(\beta - \lambda) - (l(\beta) - l^*(\beta)).$$

It is easily seen that the system of linear equations (19) and $h_j^{(\lambda)}(x) = 0$ has only the trivial solution. Since $A - (B + C + D) = 0$, f, g, and h are linearly independent.

By the lemma and the hypothesis of induction, f, g, and h for weight λ less than β are linearly independent. We now prove the linear independence of f and g for weight β.

Let

$$(20) \qquad \sum a_i^{(\lambda_1\ldots\lambda_t)} f_i^{(\lambda_1\ldots\lambda_t)}(x) + \sum b_j^{(\mu_1\ldots\mu_{s-1};\mu_s)} g_j^{(\mu_1\ldots\mu_{s-1};\mu_s)}(x) = 0$$

be a linear relation among f and g. The coefficient of $x_j^{(\lambda)}$ in the left-hand side is equal to $a_j^{(\lambda)}$ and hence $a_j^{(\lambda)} = 0$. Take a λ $(1 \leqslant \lambda \leqslant \beta - 1)$ and set

$$x_j^{(\lambda\mu_1\ldots\mu_{t-1};\mu_t)} = y_j^{(\mu_1\ldots\mu_{t-1};\mu_t)}, \quad x_j^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)} = 0$$

when $\{\lambda_1 \ldots \lambda_{t-1}\}$ does not contain λ. Then

$$f_j^{(\lambda\mu_1\ldots\mu_t)}(x), \quad g_j^{(\lambda\mu_1\ldots\mu_{t-1};\mu_t)}(x) \qquad (t \geqslant 1)$$

are transferred to the linear forms, f, g, and h in

$$y_j^{(\mu_1\ldots\mu_{t-1};\mu_t)}$$

in the case of weight $\beta - \lambda$, and (20) becomes a linear relation among these linear forms. Thus it follows that

$$a_j^{(\lambda\mu_1\ldots\mu_t)} = 0, \quad b_k^{(\lambda\mu_1\ldots\mu_{t-1};\mu_t)} = 0$$

for any λ, and

$$f_j^{(\lambda_1\ldots\lambda_t)}(x), \quad g_k^{(\lambda_1\ldots\lambda_{t-1};\lambda_t)}(x)$$

are linearly independent. This proves the propositions (I)–(III′) by the lemma.

References

1. R. Brauer and C. Nesbitt, *On the modular representations of groups of finite order*, University of Toronto Studies, Math. Ser., *4* (1937).
2. ———, *On the modular characters of groups*, Ann. Math., *42* (1941), 556–590.
3. R. Brauer and G. de B. Robinson, *On a conjecture by Nakayama*, Trans. Royal Soc. Canada, Sec. III, *40* (1947), 11–25.

4. J. H. Chung, *Modular representations of the symmetric group*, Can. J. Math., *3* (1951), 309–327.
5. D. E. Littlewood, *Modular representations of symmetric groups*, Proc. Royal Soc. London (A), *209* (1951), 333–352.
6. F. D. Murnaghan, *On the representations of the symmetric group*, Amer. J. Math., *59* (1937), 437–488.
7. T. Nakayama, *Some modular properties of irreducible representations of a symmetric group* I, II, Jap. J. Math., *17* (1948), 165–184, 277–294.
8. T. Nakayama and M. Osima, *Note on blocks of symmetric group*, Nagoya Math. J., *2* (1951), 111–117.
9. M. Osima, *On some character relations of symmetric groups*, Okayama Math. J., *1* (1952), 63–68.
10. ———, *Some remarks on the characters of the symmetric groups*, Can. J. Math., *5* (1953).
11. G. de B. Robinson, *On the modular representations of the symmetric group*, Proc. Nat. Acad. Sci., *38* (1952).
12. ———, *On a conjecture by J. H. Chung*, Can. J. Math. *4* (1952), 373–380.
13. R. A. Staal, *Star diagrams and the symmetric group*, Can. J. Math., *2* (1950), 79–92.
14. R. M. Thrall and G. de B. Robinson, *Supplement to a paper by G. de B. Robinson*, Amer. J. Math., *73* (1951), 721–724.
15. H. Weyl, *The classical groups* (Princeton, 1939).

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