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SECOND ORDER OPERATORS WITH NON-ZERO ETA INVARIANT

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ABSTRACT. We give an example of an elliptic second order pseudodifferential operator with a non-zero eta invariant. The operator is constructed on homogeneous bundles over compact Lie groups and is formed by composing differential operators and an operator of class $\mathrm{OPS}^0_{\frac{1}{2},\frac{1}{2}}$. In general it is not elliptic but in the special case of even dimensional bundles over SU(2) it is elliptic. The eta invariant is calculated in the special case and in the non elliptic case a difference eta invariant is obtained.

1. **Introduction.** In this paper we shall give an example of a second order elliptic pseudodifferential operator on homogeneous bundles over a Lie group with non zero eta invariant. Previously, a non zero eta invariant had only been computed for first order operators of Dirac type.

Let M be a compact smooth Riemannian manifold of dimension m without boundary and let $\underset{\sim}{E}$ be a smooth vector bundle over M. Let $P\colon \Gamma(\underset{\sim}{E}) \longrightarrow \Gamma(\underset{\sim}{E})$ be an operator acting on the space of sections of $\underset{\sim}{E}$. If P has eigenvalues λ with multiplicty m_λ then the eta function of P is defined by

$$\eta_P(s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda) m_\lambda |\lambda|^{-s} \tag{1.1}$$

where

$$\operatorname{sign}\lambda = \begin{cases} 1 & \text{if } \lambda > 0 \\ -1 & \text{if } \lambda < 0\,. \end{cases} \tag{1.2}$$

If P is a self adjoint elliptic differential operator of degree d then this series converges absolutely for $\mathrm{Re}(s) > m/d$. Furthermore this defines a holomorphic function on $\mathrm{Re}(s) > m/d$ which has a meromorphic extension to the whole complex plane with simple poles. If P is only pseudodifferential we again have a holomorphic function defined by the series but the meromorphic extension may have poles which are not simple. However, there are no known examples where double or higher order poles actually occur. This and other related results are discussed in [7].

It is a deep theorem that $\eta_P(s)$ is regular at $s = 0$. We then define the eta invariant of P to be

$$\eta(P) = \eta_P(0). \tag{1.3}$$

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This differs from the more usual definition, which is $\eta(P) = \eta_P(0) + \dim \operatorname{Ker} P$, but is more convenient for this paper. Since $\eta_{-P}(s) = -\eta_P(s)$ the eta invariant is a measure of spectral asymmetry of P. This regularity if far from obvious. In [5] there are examples of operators whose local eta functions are not regular at $s = 0$. Of course these integrate to give global eta functions which are regular. However, the existence of these non regular local eta functions means that there is no apriori reason for $\eta_P(s)$ to be regular at $s = 0$. This deep theorem was first proved in [1] for the case m odd. It was extended to the case m even in [6]. Both proofs use global methods and are topological in nature. A proof using local formulae was offered in [14], but needs some additional global considerations to complete it, see [15].

If P is a first order operator then $\eta(P)$ plays an important role in the Atiyah-Patodi-Singer index theorem, see [1] and also [10] for further details. For operators of Dirac type $\eta(P)$ has been extensively studied and many examples are known. There are many other uses for the eta invariant: to name just one it can be used to detect exotic differentiable structures, see [11].

However, if m is odd and $d = 2$ there are no examples for which a non zero eta invariant has been computed previously. It is very difficult to compute the eta invariant of operators not of Dirac type. Here we compute the eta invariant of a pseudodifferential operator of the form $P = RLR$ where R is a first order differential operator. Our methods are group theoretic and do not rely on the index theorem.

In Section 2 we start by fixing some notation and defining the operators R and L. A very important step in this Section is the introduction of the difference eta function. This is given by the formula

$$\tilde{\eta}(s) = -2\sum_{\lambda}\langle\lambda, \mu\rangle^{-2s} d(\lambda + \rho)\, d(\lambda + \mu + \rho), \tag{1.4}$$

where the sum is over the dominant weights λ and d is the dimension polynomial. We also compute the symbol of R and show that R is elliptic on even dimensional irreducible bundles over SU(2) but non elliptic in the other cases.

In Section 3 we use the results of [12] to show that L is a pseudodifferential operator. Since R is only elliptic on the group SU(2) it is only necessary to describe L in this case. In fact L is closely related to the projection operator F of $L^2(M)$ onto $\mathcal{F} \subset C^\infty(M)$, the space of Fegan potentials, see [9] and Section 3 for more details on this. We show, that for $M = \mathrm{SU}(2)$, F is a pseudodifferential operator in the sense $F \in \mathrm{OPS}^0_{\frac{1}{2},\frac{1}{2}}$, see [12] for the definitions involved. The classical zero order pseudodifferential operators are denoted by $\mathrm{OPS}^0_{1,0}$ and both F and L are not of this type. Furthermore, the nature of F on groups other than SU(2) is not clear but, in light of the non-ellipticity of R on these groups, is not needed in our case. It is also very mysterious why these isospectral potentials should arise in the construction of second order operators with non zero eta invariant.

The special case of the group $\mathrm{SU}(2) = S^3$ is studied in Section 4. Here the irreducible homogeneous bundles are indexed by their fiber dimension k. If k is even then the operators R and RLR are both elliptic. Applying the general theory of [7] and studying the difference eta function give the following result.

THEOREM 1.1. $\eta(RLR) = -(7k+1)/6$.

This is proved by the formula (4.5). This formula also confirms that RLR is not a $\mathbf{Z}_2$ admissible operator since the eta invariant of such operators consists entirely of 2-torsion, see [8] for the definitions involved. In the case when k is odd R is not elliptic and the general theory does not apply. Infact the series (1.1) for $\eta_{R^2}(s)$ never converges in this case. However, the difference eta function does converge to a holomorphic function with a meromorphic continuation to the whole plane. The value $\tilde{\eta}(0)$ is given by (4.5) for this case as well as the case k even.

The last Section, 5, is concerned with compact, simply connected, semi simple groups other than SU(2). Here the operator R is never elliptic. Thus the general theory does not apply and $\eta(R^2)$ does not exist. However, the difference eta function does exist and has a meromorphic continuation to the whole plane. Its value at $s = 0$ is given by the formula:

THEOREM 1.2. *$\tilde{\eta}(0) = \sum_{k=0}^{n}(-1)^{k+1}\frac{a_{2k+1}B_{k+1}}{(k+1)m^{4k+2}}$, where n is the integer part of $\frac{1}{2}\dim G - \frac{1}{2}$, B_k is the kth Bernoulli number and a_k and m are constants which depend upon G.*

This result and a description of both a_k and m are given in Section 5. The author would like to thank Peter Gilkey for raising the question of eta invariants of second order operators and for explaining many of the known results referred to in this paper. His constructive criticism of the early drafts of this paper have led to its great improvement. The author would also like to thank Michael Taylor for his help in describing $\mathrm{OPS}^0_{\frac{1}{2},\frac{1}{2}}$ and with the proof of Theorem 3.3. The author was supported by an Efroymson Memorial Lectureship.

2. **Description of the operators.** Let us start by fixing some notation. Let G be a compact semisimple Lie group of rank ℓ. Fix a maximal Torus T and a system of positive roots $\alpha_1, \ldots, \alpha_k$. Let $\rho = 1/2\sum \alpha_i$ be half the sum of the positive roots and $\langle\ ,\ \rangle$ be the killing form metric on G. On the Lie algebra $\mathcal{G}$, $\langle\ ,\ \rangle$ is the negative of the killing form. Define the dimension polynomial by

$$d(\lambda) = \Pi_i\langle\lambda, \alpha_i\rangle/\Pi_i\langle\rho, \alpha_i\rangle \tag{2.1}$$

with the product taken over all positive roots α_i, so that if $\pi_\lambda\colon G \longrightarrow \operatorname{Aut} V_\lambda$ is the irreducible representation with highest weight λ then $\dim V_\lambda = d(\lambda + \rho)$. Fix a nontrivial irreducible representation $\pi_\mu\colon G \longrightarrow \operatorname{Aut} E$ so $(E = V_\mu)$ and denote by $\underset{\sim}{E}$ the homogenous bundle over G associated to π_μ. This bundle $\underset{\sim}{E}$ can be trivialized by using left translation on G. Thus the sections of $\underset{\sim}{E}$ decompose

$$\Gamma(\underset{\sim}{E}) = C^\infty(G) \otimes E, \tag{2.2}$$

and using the Peter-Weyl theorem

$$\Gamma(\underset{\sim}{E}) = \sum_\lambda \dim V_\lambda \otimes E, \tag{2.3}$$

where the sum is over all λ which are dominant weights. Now decompose

$$V_\lambda \otimes E = \sum_\theta n_\mu(\lambda, \theta) V_\theta, \tag{2.4}$$

where the sum is over all dominant weights θ. The $n_\mu(\lambda, \theta)$ are the Clebsch-Gordan numbers and have the following properties.

1) If $n_\mu(\lambda, \theta) \neq 0$ then $\theta = \lambda + \nu$ where ν is a weight of π_μ.
2) $n_\mu(\lambda, \lambda + \mu) = 1$.
3) Generically (if λ is sufficiently far from the walls of the dominant Weyl chamber) $n_\mu(\lambda, \lambda + \nu) =$ multiplicity of weight ν in π_μ.
4) The multiplicity in 3) is often more complicated and for λ near the walls there is some extensive cancellation which may take place.

We wish to define two operators on $\Gamma(\underset{\sim}{E})$. Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathcal{G}$ for $\langle\ ,\ \rangle$. By left translation regard $X_1, \ldots, X_n$ as vector fields on G. The operators are defined by

$$\begin{aligned} R &= \sum \nu(X_i) \otimes \pi_\mu(X_i) \text{ and} \\ L &= \begin{cases} -1 & \text{on } V_{\lambda+\mu} \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

Here $\nu(X_i)$ is the directional derivative in the direction of the vector field X_i and R acts via the decomposition (2.2). In each term of the decomposition (2.3) L acts via the further decomposition (2.4) as 1 or -1 as indicated. On each V_θ in (2.4) R is constant since

$$R = \frac{1}{2}\Big(\sum(\nu \otimes \pi_\mu)(X_i)^2 - \nu(X_i)^2 \otimes 1 - 1 \otimes \pi_\mu(X_i)^2\Big). \tag{2.6}$$

Thus if $\Omega = -\sum X_i^2$ is the Casimir operator

$$R = \frac{1}{2}\Big(-\nu \otimes \pi_\mu(\Omega) + \nu(\Omega) \otimes 1 + 1 \otimes \pi_\mu(\Omega)\Big). \tag{2.7}$$

Hence the constant on each V_θ in (2.4) is

$$C(\lambda, \theta) = \frac{1}{2}\Big(\|\lambda + \rho\|^2 + \|\mu + \rho\|^2 - \|\theta + \rho\|^2 - \|\rho\|^2\Big). \tag{2.8}$$

The multiplicity is then given by

$$\begin{aligned} m(\lambda, \theta) &= \dim V_\lambda n_\mu(\lambda, \theta) \dim V_\theta \\ &= n_\mu(\lambda, \theta)\, d(\lambda + \rho)\, d(\theta + \rho). \end{aligned} \tag{2.9}$$

Thus we have established the following results.

THEOREM 2.1. *The spectrum of R has eigenvalues $C(\lambda, \theta)$ with multiplicity $\sum m(\lambda, \theta)$ where the sum over all λ, θ that have the same value, $C(\lambda, \theta)$, under C.*

COROLLARY 2.2. *The spectrum of R^2 has eigenvalues $C(\lambda, \theta)^2$ with multiplicit y $\sum m(\lambda, \theta)$ and the sum as in Theorem 2.*

THEOREM 2.3. *The spectrum of RLR has eigenvalues $C(\lambda, \theta)^2$ (for $\theta \neq \lambda + \mu$) and $-C(\lambda, \lambda+\mu)^2$. The multiplicity in each case is $\sum m(\lambda, \theta)$ with the sum over all λ, θ giving rise to the same eigenvalue.*

We now calculate the eta functions for our operators R^2 and RLR. We see

$$\eta_{R^2}(s) = \sum_{\lambda,\theta} \operatorname{sign} C(\lambda, \theta) |C(\lambda, \theta)|^{-2s} m(\lambda, \theta). \tag{2.10}$$

Here the sum is over λ, θ dominant weights. Note that λ is no longer the eigenvalue and if θ does not occur in $V_\lambda \otimes E$ then $m(\lambda, \theta) = 0$. We define the difference eta function $\tilde{\eta}(s)$ by

$$\tilde{\eta}(s) = \eta_{R^2}(s) - \eta_{RLR}(s). \tag{2.11}$$

Then from Corollary 2.2 and Theorem 2.3 we see (for $\mu \neq 0$)

$$\tilde{\eta}(s) = 2 \sum_{\lambda} \operatorname{sign} C(\lambda, \lambda+\mu) m(\lambda, \lambda+\mu) |C(\lambda, \lambda+\mu)|^{-2s}. \tag{2.12}$$

This definition is valid even when the series (2.10) and its analogue for RLR do not converge. The computations in Sections 4 and 5 show that $\tilde{\eta}$ has an analytic continuation in all the cases considered in this paper. Now a simple calculation gives

$$C(\lambda, \lambda+\mu) = -\langle \lambda, \mu \rangle. \tag{2.13}$$

Thus we have proved the following:

THEOREM 2.4. $\tilde{\eta}(s) = -2 \sum_\lambda \langle \lambda, \mu \rangle^{-2s} d(\lambda+\rho) d(\lambda+\mu+\rho)$.

In both (2.12) and Theorem 2.4 the sum is over all dominant weights λ which are non-zero. The kernels of R^2 and RLR are the same and come from the sections corresponding to $\lambda = 0$ in the decomposition (2.3). These are the left invariant sections.

At this point it is convenient to calculate the symbol of R.

THEOREM 2.5. *The symbol of R is $\sigma_R(x, \xi) = \pi_\mu(\xi)$, where $x \in G$ and $\xi \in \mathcal{G}$ which is identified as an element of $T_x(G)$ by left translation.*

PROOF. For differential operator $D\colon \Gamma(\underset{\sim}{E}) \longrightarrow \Gamma(\underset{\sim}{F})$ between two bundles $\underset{\sim}{E}$ and $\underset{\sim}{F}$ we have the following expression in local coordinates:

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha. \tag{2.14}$$

Here $a_\alpha(x)\colon E_x \longrightarrow F_x$ and α is a multi index. The symbol of D is then

$$\sigma_D(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi_\alpha, \tag{2.15}$$

where $\xi = (\xi_1, \ldots, \xi_n)$ is an element of the tangent space $T_x(M)$. Applying this general definition in the case of R yields the result in the theorem.

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COROLLARY 2.6. *The operator R is elliptic when π_μ is an even dimensional representation and $G = \mathrm{SU}(2)$. Furthermore R is not elliptic in any other case.*

PROOF. The operator D of (2.14) is elliptic if the symbol $\sigma_D(x, \xi)$ is invertible for all $\xi \neq 0$. In the case of R this gives R is elliptic if and only if $\pi_\mu(\xi)$ is invertible for all $\xi \neq 0$. In the case $G = \mathrm{SU}(2)$ and π_μ is an even dimensional representation it follows that 0 is not a weight of π_μ and so $\pi_\mu(\xi)$ is invertible. If $G = \mathrm{SU}(2)$ and π_μ is odd dimensional then 0 is a weight and $\pi_\mu(\xi)$ is never invertible. If G is any group other than SU(2) the Lie algebra $\mathcal{G}$ has non zero singular elements. Picking an element ξ in the Lie algebra of the maximal torus such that $\mu(\xi) = 0$ gives a non zero ξ such that $\pi_\mu(\xi)$ is not invertible.

If we had taken π_μ to be the trivial representation then R would have been the zero operator. The results of this section would either not make sense or be vacuously true in this case.

3. **The operator L.** To show that L is a pseudodifferential operator we need to consider the decomposition (2.3) in more detail. However, to begin let us define

$$\phi = \frac{1}{2}(1 - L). \tag{3.1}$$

Then using the decomposition (2.4) and the definition of (2.5) we see the following:

LEMMA 3.1. *The map $\phi \colon \Gamma(\underset{\sim}{E}) \longrightarrow \Gamma(\underset{\sim}{E})$ is projection onto the spaces $V_{\lambda+\mu}$.*

Clearly L is a pseudodifferential operator if and only if ϕ is pseudodifferential.

Let v_λ be the highest weight vector of V_λ, relative to a fixed maximal torus of G, and let $\langle\ ,\ \rangle$ be a G invariant innerproduct on V_λ which is normalized so that $\langle v_\lambda, v_\lambda\rangle = 1$. Complete v_λ to an orthonormal basis $\{v_\lambda, v_{\lambda_1}, \ldots, v_{\lambda_n}\}$ for V_λ where $n = \dim V_\lambda - 1$ depends on λ. In the case of the space E, which of course is V_μ, we denote this basis by $e_\mu, e_1, \ldots, e_n$, again with the appropriate value of n. The decomposition (2.3) now gives the following expression for $f \in \Gamma(\underset{\sim}{E})$:

$$f(x) = \sum_{\lambda,i,j,k} a_{\lambda,i,j,k}\langle v_i, \pi_\lambda(x)v_j\rangle e_k, \tag{3.2}$$

where the sum is over all λ which are dominant weights all i and j in the set $\{\lambda, \lambda_1, \ldots, \lambda_n\}$ and all k in $\{\mu, 1, \ldots, n\}$. The action of G on this section is then

$$g \cdot f(x) = \sum a_{\lambda ijk}\langle \pi_\lambda(g)v_i, \pi_\lambda(x)v_j\rangle \pi_\mu(g)e_k. \tag{3.3}$$

To identify the spaces $V_{\lambda+\mu}$ we observe that

$$f_{\lambda j}(x) = \langle v_\lambda, \pi_\lambda(x)v_j\rangle e_\mu, \tag{3.4}$$

for each j, is in the space $V_{\lambda+\mu}$ and is the highest weight vector of its copy of $V_{\lambda+\mu}$. Define the maps $p \colon E \longrightarrow E$ and $F \colon C^\infty(G) \longrightarrow C^\infty(G)$ by

$$p(e_k) = \begin{cases} e_\mu & \text{if } k = \mu \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

and

$$F(\langle v_i, \pi_\lambda(x)v_j\rangle) = \begin{cases} \langle v_\lambda, \pi_\lambda(x)v_j\rangle & i = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

From these we form $F \otimes p\colon \Gamma(E) \longrightarrow \Gamma(E)$ by

$$F \otimes p(f) = \sum_{\lambda,j} a_{\lambda,\lambda,j,\mu} f_{\lambda,j} \tag{3.6}$$

where f is given by (3.2), which defines the $a_{\lambda,\lambda,j,\mu}$, and $f_{\lambda j}$ is given in (3.4). Since $f_{\lambda j}$ is the highest weight vector of $V_{\lambda+\mu}$ there is a subset $G_{\lambda j}$ of G such that $\{gf_{\lambda j} : g\epsilon G_{\lambda j}\}$ is a basis for $V_{\lambda+\mu}$. Now define $S(F\otimes p)$ by

$$S(F\otimes p)\langle v_i, \pi_\lambda(x)v_j\rangle e_k = \sum_g g(F\otimes p)g^{-1}\langle v_i, \pi_\lambda(x)v_j\rangle e_k, \tag{3.7}$$

where the sum is over $g \in G_{\lambda j}$ and the action of g is given by (3.3).

PROPOSITION 3.2. *The operator* $\phi = S(F\otimes p)$.

PROOF. By construction both ϕ and $S(F\otimes p)$ are G invariant maps such that

$$\phi(f_{\lambda j}) = S(F\otimes p)(f_{\lambda j}) = f_{\lambda j}. \tag{3.8}$$

Thus $\phi | V_{\lambda+\mu} = S(F\otimes p) | V_{\lambda+\mu}$. Now if $f = \langle v_i, \pi_\lambda(x)v_j\rangle e_k \notin V_{\lambda+\mu}$ then

$$\phi(f) = S(F\otimes p)(f) = 0. \tag{3.9}$$

This completes the proof that $\phi = S(F\otimes p)$.

The main step in proving that L is a pseudodifferential operator is the following result.

THEOREM 3.3. *For* $G = \mathrm{SU}(2)$, $F \in \mathrm{OPS}^0_{\frac12,\frac12}$.

PROOF. We decompose the complexified Lie algebra:

$$\mathcal{G}_{\mathbf{C}} = H_{\mathbf{C}} \oplus \mathcal{G}_\alpha. \tag{3.10}$$

Then the range of F can be described as the set of functions on G which are annihilated by E_α, $\alpha > 0$, the "raising operators". Here E_α spans $\mathcal{G}_\alpha$. In the case $G = \mathrm{SU}(2)$ there is one such operator E_α which can be identified with $\overline{\partial}_b$ corresponding to the CR-structure of $\mathrm{SU}(2) = S^3$ as the unit sphere in $\mathbf{C}^2$. Thus in this case F is a Szegö projector and from the results in [12] (see also [3]) it follows that $F \in \mathrm{OPS}^0_{\frac12,\frac12}$.

COROLLARY 3.4. *On the group* $G = \mathrm{SU}(2)$ *the operators* ϕ *and* L *are pseudodifferential operators in* $\mathrm{OPS}^0_{\frac12,\frac12}$.

We also observe that F is a convolution operator. Let

$$\ell(x) = \sum_{\lambda,j} \langle v_\lambda, \pi_\lambda(x)v_j\rangle \tag{3.11}$$

Then ℓ is a distribution and the operator F is convolution with ℓ: $F(f) = \ell * f$. In the case of $G = \mathrm{SU}(2)$ it follows from Theorem 3.3 that $\hat{\ell} \in S^0_{\frac12,0}$.

As a further remark we note that in [10], there is an explicit calculation which shows that the operator on the circle S^1, which is analogous to F, is a pseudodifferential operator. The symbol of this operator on the circle is

$$\sigma_0(x,\xi) = \begin{cases} 1 & \xi > 0 \\ 0 & \xi < 0. \end{cases} \tag{3.12}$$

Thus we see explicitly that the parity condition $\sigma_0(x,-\xi) = (-1)^0\sigma_0(x,\xi)$ fails and the operator is not $\mathbf{Z}_2$ admissible in the sense of [8].

The operator F has an interesting interpretation. In [4], a space of complex valued potentials $\mathcal{F}$ was introduced. In [9], it was shown that these had the property:

$$\text{if } p \in \mathcal{F} \text{ then } \operatorname{spec}(\Delta + p) = \operatorname{spec}(\Delta), \tag{3.13}$$

where Δ is the Laplace operator on G and spec denotes the spectrum. The space $\mathcal{F}$ is the vector space spanned by the functions $\langle v_\lambda, \pi_\lambda(x)v_j\rangle$ and is called the *space of Fegan potentials*. Thus we have:

PROPOSITION 3.5. *The operator F: $C^\infty(G) \longrightarrow C^\infty(G)$ is projection onto the space $\mathcal{F}$.*

At this point we calculate the group theoretic symbol F. This is discussed in [12] and our calculation closely follows the procedure described there. In [12] this is just called the *symbol of F*. However, we use the term group theoretic symbol to distinguish this from the more traditional symbol of analysis of which the symbol in (3.12) is an example.

The operator F is a convolution operator and so, following [12], its symbol is

$$\sigma_F(x,\pi_\lambda) = \pi_\lambda(\ell), \tag{3.14}$$

where π_λ is an irreducible representation of G and ℓ is the distribution given in (3.10). Notice that in our case the symbol is independent of x. The right hand side of (3.14) is defined as

$$\pi_\lambda(\ell) = \int_G \ell(y)\pi_\lambda(y)\,dy \tag{3.15}$$

and is a map

$$\pi_\lambda(\ell)\colon E_\lambda \longrightarrow E_\lambda. \tag{3.16}$$

We now calculate this map

$$\pi_\lambda(\ell)v = \sum_j \int \langle \pi_\lambda(y)v, v_j\rangle \ell(y)\,dy, \tag{3.17}$$

where the sum is over j so that $\{v_j\}$ is the orthonormal basis of E_λ described at the beginning of this section. Thus by the Schur orthogonality relations we have

$$\pi_\lambda(\ell)v = \langle v, v_\lambda\rangle v_\lambda, \tag{3.18}$$

that is $\pi_\lambda(\ell)$ is projection onto the highest weight vector of E_λ. This gives us the following result.

PROPOSITION 3.6. *The group theoretic symbol of F, $\sigma_F(x,\pi_\lambda)\colon E_\lambda \longrightarrow E_\lambda$, is projection onto the highest weight vector of E_λ.*

From this result the symbol of L is implicitly determined. We wish to show that L is elliptic but this is not clearly seen from these symbol calculations. However, there is a more direct approach which can be used.

LEMMA 3.7. *The operator L is elliptic.*

PROOF. From (2.5) we see that $L^2 = 1$. Thus if $\sigma_L(x,\xi)$ is the symbol of L we have $\sigma_L(x,\xi)^2 = 1$. Thus $\sigma_L(x,\xi)$ is invertible and L is elliptic.

4. **The case of** SU(2). In the case $G = \mathrm{SU}(2)$ we can obtain an expression for $\tilde{\eta}$ in terms of the classical Riemann zeta function. For SU(2) there is one positive root which we normalize to be 1. Then $\rho = \frac{1}{2}$ and the non-zero dominant weights are $\lambda = \frac{1}{2}n$ where n is a positive integer. The killing form is then given by

$$\langle\lambda,\mu\rangle = \frac{1}{2}\lambda\mu \tag{4.1}$$

and the dimension formula is

$$d(\lambda+\rho) = 2\lambda + 1. \tag{4.2}$$

Thus

$$\begin{aligned}\tilde{\eta}(s) &= -2^{4s+1}\mu^{-2s}\sum_{n=1}^{\infty} n^{-2s}(n+1)(n+2\mu+1)\\ &= -2^{4s+1}\mu^{-2s}\big(\zeta(2s-2) + (2\mu+2)\zeta(2s-1) + (2\mu+1)\zeta(2s)\big).\end{aligned} \tag{4.3}$$

Here $n = 2\lambda$ and ζ is the classical Riemann zeta function. Hence

$$\tilde{\eta}(0) = -2\big(\zeta(-2) + (2\mu+2)\zeta(-1) + (2\mu+1)\zeta(0)\big). \tag{4.4}$$

From [13] we find $\zeta(0) = -1/2$, $\zeta(-1) = -1/12$ and $\zeta(-2) = 0$. Thus

$$\tilde{\eta}(0) = \frac{7\mu+4}{3}. \tag{4.5}$$

To compute the eta invariant of RLR we first consider the operator R. The representation π_μ with highest weight μ has weights:

$$\mu, \mu-1, \mu-2, \ldots, -\mu, \tag{4.6}$$

each with multiplicity 1. Now it is easy to calculate that the tensor product of two representations decomposes into irreducibles as follows

$$\pi_\lambda \otimes \pi_\mu = \pi_{\lambda+\mu} \otimes \pi_{\lambda+\mu-1} \otimes \cdots \otimes \pi_{|\lambda-\mu|}. \tag{4.7}$$

Thus if $\frac{1}{2}m$ is a weight of π_μ the representation with highest weight $\lambda + \frac{1}{2}m$ occurs in the decomposition (4.7) providing $\lambda \geq \frac{1}{2}\mu - \frac{1}{4}m$. Let $\mu = \frac{1}{2}k$, $\lambda = \frac{1}{2}n$ and then we have from (2.8) the following.

LEMMA 4.1. *The eigenvalues of R are $-nm/4 + (k^2 + 2k - m^2 - 2m)/8$ which occur with multiplicity $(n + 1)(n + m + 1)$ where k is a fixed positive integers, $m \in \{k, k-2, \ldots, -k\}$ and n is any integer such that $n \geq \frac{1}{2}k - \frac{1}{2}m$.*

There are now two cases for the eta invariant of R^2. If k is odd then R is elliptic since its symbol $\pi_\mu(\xi)$ is invertible if $\xi \neq 0$. Since R^2 is positive definite the eta function of R^2 agrees with the zeta function. Now $\eta_{R^2}(s)$ is given by a local formula and since $\dim \mathrm{SU}(2) = 3$ is odd this vanishes. Therefore $\eta_R^2(0) = 0$ and we have the result.

THEOREM 4.2. *Let $\underset{\sim}{E}$ be an even dimensional bundle over* SU(2) *associated to the irreducible representation π_μ. Then $\eta(RLR) = -(7\mu + 4)/3$.*

The second case is when k is even. Now $m = 0$ is a weight of π_μ and so the eigenvalue $(k^2 + 2k)/8$ of R occurs for each representation space with $n \geqq \frac{1}{2}k$. Thus this non zero eigenvalue occurs with infinite multiplicity. Consequently the series for the eta function of R^2 does not converge for any value of s. This is the case when R is not elliptic. The presence of a zero weight of π_μ means that the symbol of R, $\pi_\mu(\xi)$, is never invertible. However, even though there are no eta functions for R^2 and RLR, the formal series for $\tilde{\eta}(s)$ does converge for $\mathrm{Re}(s)$ sufficiently large to a function with a metamorphic continuation to the whole plane and the value $\tilde{\eta}(0)$ is given by (4.5).

5. **The difference eta function for other groups.** If we proceed as in the previous section, but with the group SU(3), then using appropriate normalizations we obtain:

(5.1)
$$\underset{\sim}{\eta}(s) = \frac{-1}{2(18)^{2s}} \sum_{m,n=0}^{\infty}{}' \frac{(m+1)(m+1)(m+n+2)(n+a+1)(m+b+1)(n+m+a+b+2)}{\big((2a+b)n + (a+2b)m\big)^{2s}}.$$

Here the prime with the summation sign indicates that the term with $m = n = 0$ is omitted. This expression is not readily identifiable in terms of special functions. However, the problem is not to describe $\tilde{\eta}(s)$ where but to evaluate $\tilde{\eta}(0)$ where

$$\tilde{\eta}(s) = -2 \sum_{\lambda} \langle \lambda, \mu \rangle^{-2s} d(\lambda + \rho)\, d(\lambda + \mu + \rho), \tag{5.2}$$

for $\mathrm{Re}(s)$ large enough and $\tilde{\eta}$ is defined by analytic continuation for other values of s.

Using the Mellin transform gives the result

$$\langle \lambda, \mu \rangle^{-2s} = \frac{1}{\Gamma(2s)} \int_0^\infty t^{2s-1} e^{-\langle \lambda, \mu \rangle t}\, dt. \tag{5.3}$$

Thus for $\mathrm{Re}(s)$ large enough

$$\tilde{\eta}(s) = -\frac{2}{\Gamma(2s)} \int_0^\infty \sum_{\lambda} d(\lambda + \rho)\, d(\lambda + \mu + \rho) t^{2s-1} e^{-\langle \lambda, \mu \rangle t}\, dt, \tag{5.4}$$

where we have interchanged the integral and sum. Now we shall use the Hankel contour integral. Let C be a contour of Hankel's type, see [13], that is one which starts at $+\infty$

goes round the origin anticlockwise and returns to $+\infty$. The other condition is that C misses the points $n\pi i$ on the imaginary axis. There is then Hankel's formula:

$$\int_C (-z)^{2s-1} e^{-\langle\lambda,\mu\rangle z}\, dz = -2i\sin(2\pi s)\int_0^\infty t^{2s-1} e^{-\langle\lambda,\mu\rangle t}\, dt. \tag{5.5}$$

Using this and (4.4) gives the formula

$$\tilde\eta(s) = \frac{1}{\Gamma(2s) i \sin(2\pi s)}\int_C (-z)^{2s-1}\sum_\lambda d(\lambda+\rho)\, d(\lambda+\rho+\mu) e^{-\langle\lambda,\mu\rangle z}\, dz. \tag{5.6}$$

this has to be interpreted in the following way. For $\mathrm{Re}(z)$ sufficiently large set

$$\theta(z) = \sum_\lambda d(\lambda+\rho)\, d(\lambda+\mu+\rho) e^{-\langle\lambda,\mu\rangle z} \tag{5.7}$$

and now define θ for other z by analytic continuation. Then equation (5.6) becomes

$$\tilde\eta(s) = \frac{\Gamma(1-2s)}{i\pi}\int_C (-z)^{2s-1}\theta(z)\, dz. \tag{5.8}$$

This is now valid for all s so setting $s=0$ gives

$$\tilde\eta(0) = \frac{-1}{i\pi}\int_C z^{-1}\theta(z)\, dz. \tag{5.9}$$

and by Cauchy's formula we have

$$\tilde\eta(0) = -2\,\mathrm{Res}\big(z^{-1}\theta(z)\big) \text{ at } z=0, \tag{5.10}$$

where Res denotes the residue. Now θ has a pole at $z=0$ so it is not sufficient to take $\theta(0)$ as the value of this residue. What we need is the coefficient of z° in the Laurent series for θ.

Now $\langle\lambda,\mu\rangle \in 1/m\mathbf{Z}$ where m depends on μ, see [2]. Set $\wedge_n = \{\lambda \in P : \langle\lambda,\mu\rangle = n/m\}$ and define a function p by

$$p(n) = \sum_{\lambda\in\wedge n} d(\lambda+\rho)\, d(\lambda+\mu+\rho). \tag{5.11}$$

Then p is a polynomial and $\deg p = \dim G - 1$. Thus the function of interest is now

$$\theta(z) = \sum_{n=1}^\infty p(n) e^{-nz/m}. \tag{5.12}$$

LEMMA 5.1.

$$\theta(z) = p\Big(\frac{-1}{m}\frac{d}{dz}\Big)\Big(\frac{e^{-z/m}}{1-e^{-z/m}}\Big).$$

PROOF. We calculate:

$$\sum_1^\infty e^{-nz/m} = \frac{e^{-z/m}}{1-e^{-z/m}} \tag{5.13}$$

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and so

$$\sum_1^n n^k e^{-nz/m} = \frac{(-1)^k}{m^k}\frac{d^k}{dz^k}\Big(\frac{e^{-z/m}}{1-e^{-z/m}}\Big). \tag{5.14}$$

Hence the result follows.

Notice that a consequence of this lemma is to provide a formula for θ which is valid for all z except $mn\pi i$. Thus the complex analysis done earlier is valid. Now from [13], we find:

$$\frac{e^{-z/m}}{1-e^{-z/m}} = \sum_{n=1}^{\infty}(-1)^n\frac{\phi_n'(1)}{n!}\Big(\frac{z}{m}\Big)^{n-2}, \tag{5.15}$$

where ϕ_n is the nth Bernoulli polynomial and $\phi_n'(z) = \frac{d}{dz}\phi_n(z)$. We also find the following values for $\phi_n'(1)$:

$$\phi_{2k}'(1) = 0, \quad \phi_{2k+1}'(1) = (-1)^{k-1}(2k+1)B_k, \tag{5.16}$$

where B_k is the kth Bernoulli number. Now let C_k be the coefficient of z° in the expansion of $\frac{(-1)^k}{m^k}\frac{d^k}{dz^k}(\frac{e^{-z/m}}{1-e^{-z/m}})$ and let a_k be the coefficients of the polynomial $p: p(n) = \sum a_k n^k$. Then the coefficient of z° in the Laurent expansion of $\theta(z)$ is $\sum a_k C_k$.

LEMMA 5.2. $C_k = \frac{\phi_{k+2}'(1)}{(k+2)(k+1)}m^{2k}$.

PROOF. We need to differentiate (5.14) k times. The coefficient of z° then occurs when $n = k+2$.

COROLLARY 5.3. *i)* $C_{2k} = 0$ *and*
ii) $C_{2k+1} = \frac{(-1)^k B_{k+1}}{2(k+1)}m^{4k+2}$.

Substituting these values into equation (4.10) gives the value of $\widetilde{\eta}(0)$.

THEOREM 5.4. $\widetilde{\eta}(0) = \sum_{k=0}^{n}\frac{(-1)^{k+1}a_{2k+1}B_{k+1}}{(k+1)m^{4k+2}}$. *Here* $n = [\frac{1}{2}\dim G - \frac{1}{2}]$ *is the integer part of* $\frac{1}{2}\dim G - \frac{1}{2}$, a_k *is given above before Lemma 5.2,* B_k *is the kth Bernoulli number and m is such that* $m\langle\lambda,\mu\rangle \in \mathbf{Z}$ *for all* $\lambda \in P$. *Note further that the coefficients* a_k *depend on* μ. *This completes the proof of Theorem 1.2.*

We are working with compact, semisimple and simply connected groups other than SU(2). As noted in Section 2 the operator R is never elliptic on these groups. Thus the general theory of elliptic operators is not available to guarantee the existence of $\eta(R^2)$.

If 0 is a weight of π_μ then we see that

$$\frac{1}{2}\big(\|\mu+\rho\|^2 - \|\rho\|^2\big) \tag{5.17}$$

is an eigenvalue for infinitely many representation spaces π_λ in (2.3). Thus the formal series for $\eta_{R^2}(s)$ never converges. However, the formal series for $\widetilde{\eta}$ does converge for $\mathrm{Re}(s)$ sufficiently large and the value $\widetilde{\eta}(0)$ is given by Theorem 5.4.

In general the eigenvalues of R are given by

$$-2\langle\lambda,\phi\rangle\big(\|\mu\|^2+2\langle\mu-\phi,\rho\rangle\big) \tag{5.18}$$

where ϕ is a weight of π_μ. This follows by a simple computation form (2.8). Since we can always find a weight of π_μ which is orthogonal to infinitely many dominant weights we see that

$$\|\mu\|^2+2\langle\mu-\phi,\rho\rangle \tag{5.19}$$

is an eigenvalue with infinite multiplicity. Thus the series for the eta invariant of R^2 never converges.

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