1

The Gravitational Field

This chapter is aimed at introducing in an elementary yet rigorous way the mathematical properties of the Newtonian gravitational field – the basic entity – together with the Second Law of Dynamics, upon which stellar dynamics is founded. We begin from the gravitational field of a point mass, and then we move to consider the field of extended mass distributions by using the superposition principle. A direct proof of Newton's first and second theorems for homogeneous shells is worked out, followed by a different derivation based on the Gauss theorem.

1.1 The Gravitational Field of a Point Mass and of Extended Distributions

The understanding of the structure, equilibrium, and dynamical evolution of stellar systems (to be understood in a broad sense, ranging from small galactic open clusters up to giant clusters of galaxies) in terms of the fundamental physical laws, and in particular of classical gravity, is the main subject of stellar dynamics. Stellar systems are immense when compared to the human scale, and so a sense of the astonishing values of the masses, lengths, and times involved is absolutely necessary, especially for students of sister disciplines such as mathematics or physics, who may lack a specific background in observational astronomy.

So, let us start by considering our Sun, a yellow G-dwarf star (but not at all "dwarf" when compared to the masses and sizes of the vast majority of the $N_* \approx 3 \times 10^{11}$ stars present in our galaxy, the Milky Way), with a mass of $M_\odot \simeq 1.98 \times 10^{33}$ g and a radius of $R_\odot \simeq 7 \times 10^{10}$ cm. The shape of our galaxy is that of a quite flattened disk, with a radius of ≈ 26 kpc (1 kpc $= 10^3$ pc $= 3.08 \times 10^{21}$ cm), a central "bulge," and a disk thickness of ≈ 1 kpc. The Sun rotates around the galactic center on an almost circular orbit of radius ≈ 8 kpc within the galactic disk. Detailed information about our galaxy and other stellar systems can be found in books such as Bertin (2014), Bertin and Lin (1996), Binney and Merrifield (1998), Binney and Tremaine (2008), Cimatti et al. (2019), and Sparke and Gallagher (2007). To appreciate these figures in their full glory, it is useful to construct a scaled-down model of the Milky Way, such as one in which the Sun is imagined as a little sphere of $\simeq 0.7$ mm radius; in this model, all lengths are reduced by a factor of 10^{-12}.

https://doi.org/10.1017/9780511736117.003 Published online by Cambridge University Press

The Earth is now an invisible grain of dust, the radius of the Moon's orbit is $\simeq$0.4 mm, while the Earth's orbit is an (almost circular) ellipse with a semimajor axis of $\simeq$15 cm. The giant planets Jupiter, Saturn, Uranus, and Neptune rotate around the Sun at average distances of 0.78 m, 1.40 m, 2.87 m, and 4.50 m, respectively. On this scale, the *nearest* star (actually a multiple star, the Alpha Centauri system) is placed at $\approx$41 km from our Sun, the thickness of the Milky Way disk is $\approx 3 \times 10^4$ km, and its diameter is $\approx 1.6 \times 10^6$ km! The Sun would revolve around the galactic center on a roughly circular orbit with a radius of $\approx 2.5 \times 10^5$ km at a velocity of $\approx 2.1 \times 10^{-4}$ mm/s (i.e., $\approx$0.8 mm/hr) and a period[1] of $\approx$230 Myr (corresponding to a physical circular velocity of $\approx$220 km/s).

Therefore, it should not be surprising that, with extremely good approximation, in most (but not all) applications of stellar dynamics, stars can be considered *point masses* (see also Exercise 1.1). We then start our study quite naturally by considering the gravitational field of a point mass. Notice that the relevance of this case goes well beyond the point-mass approximation of stars in galaxies because, as the gravitational field produced by a generic mass distribution is given by the sum of the fields produced by each of its parts, the point masses that will be used in the next two chapters can also be interpreted as atoms or molecules when computing the gravitational field produced by a macroscopic material object!

The starting point of Newtonian gravity is the definition of the gravitational field $\mathbf{g}$, at position $\mathbf{x}$, produced by a material point of mass m at position $\mathbf{y}$ in some reference system S_0. Empirically, it is found that the field is radial, with

$$\mathbf{g}(\mathbf{x}) = -Gm \frac{\mathbf{x} - \mathbf{y}}{||\mathbf{x} - \mathbf{y}||^3}, \tag{1.1}$$

where $G \simeq 6.67 \times 10^{-8}$ cm^3/s^{2}g is the universal gravitational constant, $\|\ldots\| = \sqrt{\langle \ldots, \ldots \rangle}$ is the standard Euclidean norm, and $\langle \, , \, \rangle$ is the usual inner product over $\Re^3$ (see Appendix A.1); from now on we indicate vectors with bold letters, if not stated otherwise. Note that the field is not defined at the particle position and that $\mathbf{g}(\mathbf{x})$ depends on time through the position of the particle $\mathbf{y}(t)$: the classical gravitational field "propagates" instantaneously over all of the space. Equation (1.1), formally identical to that describing the electrostatic field produced by a point charge, encapsulates the two most important properties of the gravitational field of a point mass (i.e., the fact that the modulus of the field decreases radially as the inverse of the square distance from the particle and the fact that the force between two particles is attractive). The third fundamental property of classical gravity, confirmed by an enormous body of experimental evidence (a point unfortunately not always sufficiently stressed; however, see e.g. Feynman et al. 1977 among the notable exceptions), is the *superposition principle* (i.e., the fact that the gravitational field produced at the point $\mathbf{x}$ by two point masses of masses m_1 and m_2 placed at $\mathbf{y}_1$ and $\mathbf{y}_2$ is the vector

[1] For order-of-magnitude estimates, it is useful to recall that 1 yr $\simeq \pi \times 10^7$ s, and that a velocity of 1 km/s corresponds to $\simeq$1 pc/Myr.

sum $\mathbf{g}_1(\mathbf{x}) + \mathbf{g}_2(\mathbf{x})$ of the two fields). As a matter of fact, the *whole* theory of classical gravitation can be built from Eq. (1.1) and the superposition principle.

For example, the gravitational field produced at $\mathbf{x}$ by an extended mass distribution of density $\rho(\mathbf{y})$, such as a star, a planet, or a galaxy, can be immediately written as

$$\mathbf{g}(\mathbf{x}) = -G \int_{\Re^3} \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|^3} \rho(\mathbf{y}) d^3\mathbf{y}, \tag{1.2}$$

where $\rho(\mathbf{y})d^3\mathbf{y}$ is the mass element in the infinitesimal integration volume $d^3\mathbf{y}$ and the integral (a sum) embodies the superposition principle. Of course, in the special case of N point masses of mass m_i and position $\mathbf{x}_i$, one can define

$$\rho(\mathbf{y}) = \sum_{i=1}^{N} m_i \delta(\mathbf{y} - \mathbf{x}_i), \tag{1.3}$$

where δ is the so-called Dirac δ-function (actually a distribution; see Appendix A.2.5), and so Eq. (1.2) reduces to a standard sum, as expected from Eq. (1.1).

The student should appreciate that, in principle, Eq. (1.2) contains *all* of the information we need to determine the gravitational field of a given density distribution. In other words, Eq. (1.2) is the *general solution* of the problem of the calculation of the Newtonian gravitational field produced by an assigned mass distribution ρ: one could conclude that the only problem to be addressed is just how to calculate (analytically or numerically) the integral (1.2) in all of the cases of interest. However, this conclusion would be very wrong. In fact, the most profound properties of the gravitational field *cannot* be derived by simple evaluation of Eq. (1.2), and (as is common in physics) the defining *equations* of a problem invariably contain much more information than the solution itself. For these reasons, we now start from the "solution" given by Eqs. (1.1) and (1.2), and we look for the differential equations leading to this solution. Significant effort will then be spent in the study of the properties and implications of the obtained equations, and this effort will be repaid by the discovery of powerful mathematical methods that will lead us to a deep understanding of the gravitational field produced by mass distributions and, finally, as a useful by-product, to general techniques for the evaluation of Eq. (1.2).

1.2 Newton's First and Second Theorems

One of Newton's major accomplishments was the discovery of the *first* and *second theorems* concerning the gravitational fields produced by spherical and homogeneous material shells. Usually, the two theorems are proved by using the Gauss divergence theorem, as we will also do in Section 1.3. However, due to their importance, here we prove them from direct integration of Eq. (1.2), in the original spirit (while avoiding the subtleties of Newton's awe-inspiring geometric proof; e.g., see Binney and Tremaine 2008; Chandrasekhar 1995).

The first step is to show that the gravitational field produced by a generic spherical density distribution $\rho(r)$ is radial. This is accomplished by arbitrarily fixing a point of space $\mathbf{x}$, by defining the direction of the z-axis so that[2] $\mathbf{x} = (0, 0, r)$, and finally by changing the integration variables from Cartesian to spherical in Eq. (1.2). Integration over the angles (do it!) proves that the resulting field $\mathbf{g}$ is directed along the z-axis (i.e., along $\mathbf{x}$) and hence is radial. The second step is to specialize the density distribution ρ to the case of a homogeneous and infinitesimally thin shell of radius R and total mass M, so that from Eq. (A.97)

$$\rho(\mathbf{y}) = \frac{M\delta(r-R)}{4\pi r^2}, \quad r \equiv \|\mathbf{y}\|. \tag{1.4}$$

From the radial symmetry of the field, we can proceed to the explicit integration of the z-component of $\mathbf{g}$ at $\mathbf{x} = (0, 0, r)$, and after some algebra (see also Exercise 1.2), we finally obtain

$$\mathbf{g}(\mathbf{x}) = -\frac{GM}{2r^2}\left(1 + \frac{r-R}{|r-R|}\right)\mathbf{f}_r, \quad r \neq R, \tag{1.5}$$

where $\mathbf{f}_r$ is the radial unit vector in spherical coordinates (see Appendix A.8). It follows that for $r < R$ no field is present (Newton's first theorem), while for $r > R$ the field coincides with that produced by a material point of mass M, placed at the origin (Newton's second theorem).

A somewhat delicate situation arises when considering the field produced by the shell on itself (i.e., at $r = R$). In fact, it is clear from Eq. (1.5) that the function $\mathbf{g}(\mathbf{x})$, evaluated as a limit for $r \to R$, is discontinuous, with different left and right values. A natural question that arises is how to compute the field on the shell itself. Naively, a "reasonable" approach could be to fix $r = R$ before performing the integral over the surface of the shell and to pretend that for symmetry reasons the integral over the azimuthal angle φ is to be performed before the integral over the colatitude ϑ. With this approach, we deduce that a point *on* the shell experiences a *radial* force per unit mass given by

$$\mathbf{g}(\mathbf{x}) = -\frac{GM}{2R^2}\mathbf{f}_r, \tag{1.6}$$

the average value of Eq. (1.5) just inside and just outside the shell. However, a closer inspection of the integral reveals a more delicate situation (i.e., that the *tangential* component of $\mathbf{g}$ to the shell *cannot* be uniquely defined). For example, the student is encouraged to calculate the tangential component of $\mathbf{g}(0, 0, R)$ with a different order of integration from the "natural" one (i.e., by fixing the angle φ) and integrating first over $0 \leq \vartheta \leq \pi$: the tangential component of the field now diverges for $\vartheta \to 0$ (i.e., for the effect of the points of the material meridian line touching the point $\mathbf{x}$). Technically, the problem is due to the fact that, in the present case, the integral in Eq. (1.2) is not absolutely convergent, and the result depends on how the integral is computed. Here, the lesson is that, as a safety

[2] For simplicity, vectors are represented in the text as row vectors. See also Footnote 1 in Appendix A.

rule, symmetry arguments should be used to evaluate integrals only *after* the integrals are known to converge.[3]

1.3 The Gauss Theorem and the Gravitational Field

An alternative and more elegant derivation of Newton's first and second theorems can be obtained by using the Gauss divergence theorem (see Appendix A.4). However, before addressing this problem in Section 1.3.1, we must establish a few fundamental properties of the gravitational field.

We start by considering the application of the Gauss theorem to the gravitational field of a point mass, and so we evaluate first the *divergence* of the gravitational field in Eq. (1.1). It is a simple exercise (do it!) to show that for all $\mathbf{x} \neq \mathbf{y}$, the divergence of the field at $\mathbf{x}$ vanishes (the student may also wish to prove, by using the divergence operator in spherical coordinates, that the radial $1/r^2$ field is the *only* radial field in $\Re^3$ with this property). Therefore, from the Gauss theorem, it follows that the flux of the gravitational field produced by a point mass across a generic closed surface $\partial\Omega$, *not containing* the point mass, is zero. From the superposition principle, this conclusion immediately generalizes to the case of the flux of the field $\mathbf{g}$ produced by an arbitrary mass distribution placed outside the closed surface $\partial\Omega$.

Now, let us ask what happens to the flux if the point mass is *inside* the region Ω. Clearly, being $\operatorname{div}\mathbf{g} = 0$ for $\mathbf{x} \neq \mathbf{y}$, according to Eq. (A.112), the only possible nonzero contribution of the divergence to the flux integral can be due to what happens at the point $\mathbf{x} = \mathbf{y}$. We face two problems here. The first is that we cannot compute $\operatorname{div}\mathbf{g}$ for $\mathbf{x} = \mathbf{y}$, simply because the field is not defined there. The second is even more worrisome because from a naive interpretation of integration theory, one could expect that the volume integral evaluates to zero even if we pretend that the "value" of $\operatorname{div}\mathbf{g}$ at $\mathbf{x} = \mathbf{y}$ is infinite, being a point of a set of zero measure. This would be in stark contrast to the fact that at this stage we can certainly imagine a region containing the point mass with a shape such that the total flux is strictly negative! The only logical conclusion is that the flux is not zero and that integration theory is (obviously) correct, but that $\operatorname{div}\mathbf{g}$ for $\mathbf{x} = \mathbf{y}$ is a more "complicated" object than a function with infinite value: we now show that this is the case. The idea is borrowed from complex analysis, and it substitutes into our problem a different problem that we can solve. So, as is shown in Figure 1.1, we remove from Ω a spherical region of radius R centered on $\mathbf{y}$, and we produce a two-dimensional "cut" (the dashed line c in Figure 1.1) connecting the surface of the inner hole with the external surface $\partial\Omega$. With the exclusion of the spherical hole, our particle is now outside the resulting region, so that

[3] Perhaps the most famous such case is encountered in the computation of the Jeans mass (e.g., see Binney and Tremaine 2008; Shu 1992), when fixing to zero (the so-called Jeans swindle) the gravitational field produced by the unperturbed, infinite, and homogeneous three-dimensional density background. In fact, instead of integrating over spherical shells centered on the point of interest (with the result of a zero field from Newton's first theorem), we could integrate over parallel density slabs of unitary thickness, at the left and right of the point, obtaining a conditionally convergent alternating series (do it!); see also Exercise 1.8 for another example. A full discussion of the conditions for the existence of integral (1.2) can be found in Kellogg (1953).

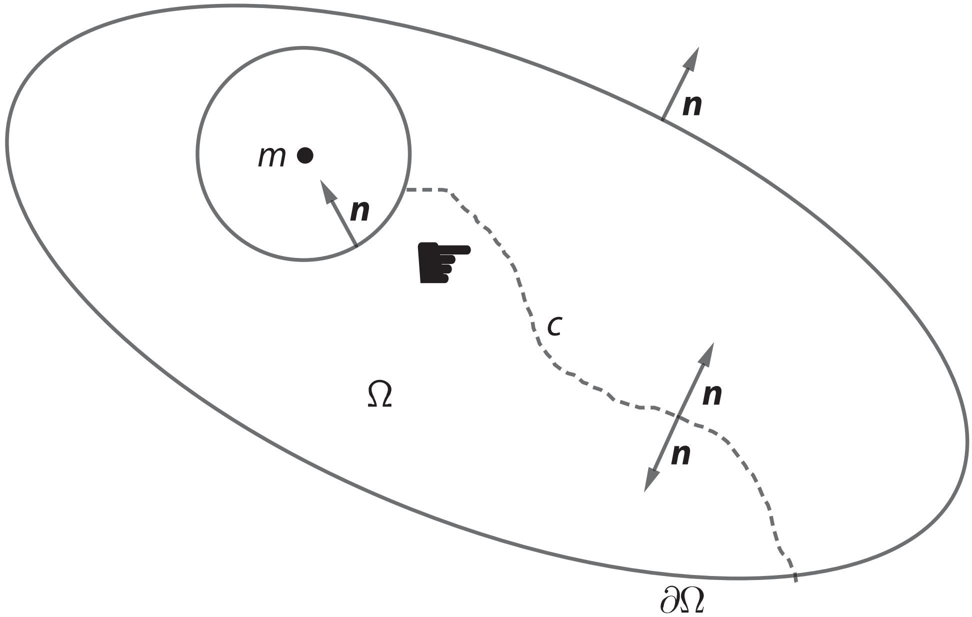

Figure 1.1 Schematic illustration of how, by using the divergence theorem, we can prove that the flux of the gravitational (or electrostatic) field of a particle of mass m (or charge q) contained in a closed and regular (but otherwise arbitrary) region $\Omega \subset \Re^3$ is $-4\pi Gm$ (or q/ϵ_0), independently of the position of the particle inside the volume. The dashed line indicates a generic two-dimensional "cut" with two opposite normals, one for each of the two sides of the cut.

the flux of **g** through the total boundary (made by the original $\partial\Omega$, by the surface of the spherical hole, and by the two geometrically coincident but analytically distinct surfaces of the cut c) evaluates to zero. Therefore, from the divergence theorem, the flux of **g** through $\partial\Omega$ is simply the negative of the flux on the inner spherical hole, because the flux on the two surfaces of the cut cancels out. By reversing the normal at the surface of the hole and performing the surface integral by exploiting the spherical symmetry of **g**, it is now simple to show (do it!) that the flux on the hole,[4] and so onto $\partial\Omega$, is $-4\pi Gm$.

In conclusion, by using the Gauss theorem applied to the special case of the three-dimensional radial $1/r^2$ field, we showed that for a point mass

$$\int_\Omega \operatorname{div}\mathbf{g}(\mathbf{x})d^3\mathbf{x} = \int_{\partial\Omega} \langle \mathbf{g}(\mathbf{x}), \mathbf{n}\rangle d^2\mathbf{x} = -4\pi Gm \times \begin{cases} 0, & \mathbf{y} \notin \Omega, \\ 1, & \mathbf{y} \in \Omega \end{cases}. \tag{1.7}$$

In other words, $\operatorname{div}\mathbf{g}$ behaves as the three-dimensional Dirac δ-function in Eq. (A.95) (technically a *distribution*), and we can formally write

[4] Quite obviously, even before computing it, we knew that the flux on the spherical hole is independent of the adopted value of R, which is consistent with the fact that the flux on $\partial\Omega$ cannot depend on our arbitrary choice of R!

https://doi.org/10.1017/9780511736117.003 Published online by Cambridge University Press

$$\mathrm{div}_{\mathbf{x}} \frac{\mathbf{x}-\mathbf{y}}{||\mathbf{x}-\mathbf{y}||^3} = 4\pi\delta(\mathbf{x}-\mathbf{y}), \tag{1.8}$$

where the subscript indicates the coordinates used to compute the divergence. From the superposition principle, we now conclude that, given an arbitrary mass distribution $\rho(\mathbf{x})$ and an arbitrary closed (and sufficiently regular) region $\Omega \subset \Re^3$, the flux of $\mathbf{g}$ produced by ρ through the boundary $\partial\Omega$ is $-4\pi GM$, where $M = \int_\Omega \rho d^3\mathbf{x}$ is the mass contained in the volume. Highlighting an elementary but (perhaps) not always appreciated point is in order here. In fact, it is clear that a *null* flux over a closed surface $\partial\Omega$ does not imply a *null* field inside the region Ω, as is obvious for the case of a point mass external to a given volume. In turn, this also means that, in general, one *cannot* prove that the gravitational (or electrostatic) field inside an empty cavity is zero simply by considering the null flux through some closed surface contained in the cavity. A naturally related question is then why inside an electrically charged, conducting, and closed surface of arbitrary shape at equilibrium not only the flux but also the field *is* zero (the so-called *Faraday's cage*), while inside a material surface of similar shape the gravitational field *is not* zero, even though the force law in the two cases is mathematically identical. To properly answer this question, we must move to the next chapter, where we encounter the concept of *gravitational potential*, which is fundamental in stellar dynamics.

We will now use Eq. (1.8) to obtain the differential equation relating the gravitational field of a mass distribution to its density. In practice, we compute[5] the divergence under the sign of the integral of Eq. (1.2), and from Eq. (1.8), we obtain the fundamental identity

$$\mathrm{div}\,\mathbf{g}(\mathbf{x}) = -4\pi G\rho(\mathbf{x}). \tag{1.9}$$

If we now insert into Eq. (1.9) the expression for the field of a point mass from Eq. (1.1), we deduce that

$$\rho(\mathbf{x}) = m\delta(\mathbf{x}-\mathbf{y}), \tag{1.10}$$

or, in other words, that *physically* the Dirac δ-function can be imagined as the "density distribution" of a point mass. This shows in the most apparent way that the δ-function is a *dimensional* object with the inverse volume units of the space under consideration, and that, from a mathematical point of view, a point mass is "more" than just a simple point of "infinite density."

1.3.1 Newton's First and Second Theorems Again

We are now in the position to prove again, using a different approach, Newton's first and second theorems, and to show the power of the Gauss theorem in Eq. (1.7). In fact, from the property that the gravitational field of a spherically symmetric mass distribution $\rho(r)$ is

[5] Of course, this step can be rigorously justified (e.g., see Kellogg 1953).

https://doi.org/10.1017/9780511736117.003 Published online by Cambridge University Press

radial (i.e., $\mathbf{g} = g_r(r)\mathbf{f}_r$; see Section 1.2), and using a spherical control volume of radius r centered on the origin, Eq. (1.7) reduces immediately to

$$4\pi r^2 g_r(r) = -4\pi GM(r), \quad M(r) = 4\pi \int_0^r \rho(t)t^2 dt, \tag{1.11}$$

where the (still unknown) component g_r has been "factorized" out of the integral thanks to its geometric properties and to a wise choice for the integration surface. In practice, the gravitational (or electrostatic) field of a spherical mass (charge) distribution at distance r from the center is the same field of a point mass (charge) of mass $M(r)$ or charge $Q(r)$. Newton's two theorems are immediately recovered from Eq. (1.11) for the case of ρ given the homogeneous shell in Eq. (1.4). The student is encouraged to follow the same line of reasoning and deduce the expressions analogous to Eq. (1.11) for cylindrical densities $\rho(R)$ and planar densities $\rho(z)$, taking into account the precautionary notes at the end of Section 1.2.

A final warning to enthusiastic students: unfortunately, the powerful Gauss theorem is not a magic tool that can, elegantly and effortlessly, give the gravitational field of an arbitrary mass distribution while sparing the tedious work of integration. In fact, even though the Gauss theorem holds for arbitrary volumes and mass distributions, it should be clear that the derived expressions for **g** of spherical, cylindrical, and planar distributions are based on the *exceptional* circumstance that the geometry of the surfaces over which the flux integrand can be factorized is known *before* **g** is actually calculated. For generic mass distributions, the geometric properties of **g** are not known in advance, such that the question of what kind of surface one could/should use to repeat the treatment of special geometries is as difficult as the problem of determining **g** itself!

Exercises

1.1 With this exercise, we quantify the possibility of considering stars as point masses in real stellar systems. Consider an idealized stellar system of radius R and homogeneously filled with N identical stars of radius $R_\odot$. We define a *geometric collision* as the situation realized when the centers of two stars are separated by a distance $d \leq 2R_\odot$. By considering the co-volume of N "cylinders" of length λ (a rough estimate of the *mean free path*; e.g., see Born 1969) and radius $2R_\odot$ (why?), argue that an estimate of λ can be obtained by imposing that the total volume of the cylinders equals the volume of the sphere, i.e., formally

$$\frac{\lambda}{2R} = \frac{R^2}{6NR_\odot^2}. \tag{1.12}$$

Estimate λ for an elliptical galaxy and for a globular cluster.

1.2 Let

$$\mathbf{g}(\mathbf{x}) = -Gm\frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|^{1+\alpha}}, \quad \alpha < 3, \tag{1.13}$$

a natural generalization of the classical gravitational field produced by a particle of mass m, where G is some universal constant (e.g., see Di Cintio and Ciotti 2011). Show that the radial component of the gravitational field produced at radius r by the homogeneous spherical shell of total mass M and radius R given in Eq. (1.4) is

$$g_r(r) = -\frac{GM}{4Rr^2} \times \begin{cases} 2Rr - (r^2 - R^2)\ln\dfrac{r_-}{r_+}, & \alpha = 1, \\ \dfrac{r_+^{3-\alpha} - r_-^{3-\alpha}}{3-\alpha} - (r^2 - R^2)\dfrac{r_+^{1-\alpha} - r_-^{1-\alpha}}{\alpha - 1}, & \alpha \neq 1, \end{cases} \tag{1.14}$$

where $r_+ \equiv r + R$ and $r_- \equiv |r - R|$. Discuss the relevant case of $\alpha = -1$ (the harmonic oscillator) and show that for $\alpha = 2$ we reobtain Eq. (1.5). What happens to the field at $r = R$?

1.3 For $\mathbf{x} \neq \mathbf{y}$, calculate the divergence of the field given in Eqs. (1.13) and (1.14) and discuss the result as a function of α by using the Gauss theorem applied to concentric spherical control surfaces of increasing radius. Explain geometrically why inside a uniform shell a mass point is attracted toward the shell for $\alpha > 2$ and toward the center for $\alpha < 2$.

1.4 Consider a homogeneous sphere of constant density ρ_0, and from Eq. (1.11) show that the field inside the sphere is that of the three-dimensional isotropic harmonic oscillator

$$g_r(r) = -\frac{4\pi G\rho_0}{3}r. \tag{1.15}$$

By using the superposition principle and the Newton's second theorem, determine the field inside a spherical hole – not necessarily concentric – carved in the sphere: isn't the result beautiful? *Hint*: Imagine the hole is produced by the superposition of a sphere of negative density $-\rho_0$ and calculate its repulsive field from Eq. (1.15).

1.5 Point particles, rings, and disks are often encountered in problems of stellar dynamics. Prove that the three-dimensional density of a *particle* of mass m placed on the z-axis at distance a from the origin, in spherical and cylindrical coordinates, can be written respectively as

$$\rho = m\frac{\delta(r-a)\delta(\vartheta)}{2\pi r^2 \sin\vartheta} = m\frac{\delta(z-a)\delta(R)}{2\pi R}. \tag{1.16}$$

Moreover, show that for a *homogeneous ring* of total mass M, radius a, linear density $\lambda = M/(2\pi a)$, placed on the $z = 0$ plane, and with the center at the origin,

$$\rho = M\frac{\delta(r-a)\delta(\vartheta - \pi/2)}{2\pi r^2 \sin\vartheta} = M\frac{\delta(R-a)\delta(z)}{2\pi R}. \tag{1.17}$$

Finally, show that for a *razor-thin disk* of surface density $\Sigma(R)$ on the $z = 0$ plane,

$$\rho = \Sigma(r)\frac{\delta(\vartheta - \pi/2)}{r\sin\vartheta} = \Sigma(R)\delta(z), \tag{1.18}$$

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so that from Eqs. (1.17) and (1.18) it follows that the "surface density" of the homogeneous ring can be written as

$$\Sigma_{\rm ring} = \frac{M\delta(r-a)}{2\pi r} = \frac{M\delta(R-a)}{2\pi R}. \qquad (1.19)$$

1.6 What happens to Eq. (1.7) in the special case of the point mass placed on the boundary $\partial\Omega$? *Hint*: Suppose the tangent plane to $\partial\Omega$ exists at the particle's position. Draw a little semisphere of radius ϵ centered on the particle and use Eq. (1.7) to evaluate the total flux on the new resulting surface made by $\partial\Omega$, minus the disk of radius ϵ, plus the semisphere of radius ϵ. Conclude that, no matter whether the semisphere includes or excludes the particle, the limit of the total flux through $\partial\Omega$ for $\epsilon \to 0$ is $-2\pi Gm$, so that from Eq. (1.8) it follows that integration of the Dirac δ-function on a regular point of a boundary of a closed volume evaluates to 1/2.

1.7 By using Newton's second theorem and Eq. (1.2) twice, prove that two nonoverlapping spheres of masses M_1 and M_2 and centers at $\mathbf{x}_1$ and $\mathbf{x}_2$ attract each other as two material points of masses M_1 and M_2 placed at $\mathbf{x}_1$ and $\mathbf{x}_2$.

1.8 This problem illustrates the danger of the naive use of symmetry arguments when calculating integrals. It is commonly said that the gravitational (or electrostatic) field of a homogeneous, infinite, razor-thin density distribution of surface density σ is constant and directed normally to the plane "because the field components parallel to the plane vanish by obvious symmetry arguments." Actually, it is easy to prove that within Newtonian gravity (or electrostatics) the horizontal field is *undetermined* (see also Footnote 3). Consider in the (x, y) plane four material squares of surface density σ, with a common vertex at the origin. Let a be the length of the side of the square in the first quadrant, and by direct integration show that the gravitational field components at the point $(0, 0, z)$ above the origin are

$$g_x = g_y = G\sigma\left(\operatorname{arcsinh}\frac{a}{z} - \operatorname{arcsinh}\frac{a}{\sqrt{a^2+z^2}}\right), \qquad (1.20)$$

where $\operatorname{arcsinh}(x) = \ln(x + \sqrt{1+x^2})$, and

$$g_z = -G\sigma\arctan\frac{a^2}{z\sqrt{a^2+z^2}}, \qquad (1.21)$$

(e.g., see Kellogg 1953; McMillan 1958). Consider now the behavior of the total field $\mathbf{g}$ produced by the four squares of side length a, b, c, and d, when the sides are independently extended to infinity. Show that g_z reduces to the value obtained from the Gauss theorem. What happens to the tangential components?

1.9 As is well known, Newton's first theorem can be seen as the result of a perfect cancellation at a generic point inside a homogeneous spherical shell of the r^{-2} force produced by the two infinitesimal mass elements determined by the intersection of the shell itself and the two sides of the conus with an infinitesimal opening angle and a vertex in the considered point, followed by a rotation of the conus's axis over the whole solid angle (e.g., see Binney and Tremaine 2008; Chandrasekhar 1995). Repeat this wonderful geometric argument for the case of a generic point inside a homogenous ring and conclude that the resulting attractive force is directly radially toward the nearest point of the ring (see also Exercise 2.33): therefore, when computing the gravitational field of a disk, its external regions cannot be ignored, a fact of great importance for the interpretation of the rotation of disk galaxies.