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STRONG AND UNIFORM CONTINUITY – THE UNIFORM SPACE CASE

DOUGLAS BRIDGES AND LUMINIŢA VÎŢĂ

Abstract

It is proved, within the constructive theory of apartness spaces, that a strongly continuous mapping from a totally bounded uniform space with a countable base of entourages to a uniform space is uniformly continuous. This lifts a result of Ishihara and Schuster from metric to uniform apartness spaces. The paper is part of a systematic development of computable topology using apartness as the fundamental notion.

To every uniform structure $\mathcal{U}$ on a set X there corresponds a relation $\bowtie$ of *apartness* between subsets S and T of X, defined by

$$S \bowtie T \iff \exists U \in \mathcal{U}\,(S \times T \subset {\sim}U)\,.$$

In turn, this provides us with a notion of *strong continuity* for a function $f : X \longrightarrow Y$ between uniform spaces, defined by

$$\forall S, T \subset X \;\; (f(S) \bowtie f(T) \Longrightarrow S \bowtie T)\,.$$

It is easy to show that a uniformly continuous map is strongly continuous. Our aim in this paper is to produce a proof of the following partial converse.

THEOREM 1. *Let X be a totally bounded uniform space with a countable base of entourages. Then every strongly continuous mapping from X into a uniform space is uniformly continuous.*

Our proof is entirely constructive in the sense of Errett Bishop: given a strongly continuous map $f : X \longrightarrow Y$ between uniform spaces, and an entourage V of Y, we show (at least in principle) how to find an entourage U of X such that

$$\forall x, y \in X\; ((x, y) \in U \Longrightarrow (f(x), f(y)) \in V).$$

To do this, instead of working with classical logic and a clearly specified notion of algorithm, such as a recursive one, we work with intuitionistic logic and an appropriate informal set theory. It follows that *all our results and proofs can easily be translated into any formalism for computable mathematics*, such as recursion theory [**1, 14**] and Weihrauch's type II effectivity theory [**19**]. They also hold in Brouwer's intuitionistic mathematics [**17**].

Our paper can be regarded as a contribution to computable topology based on a notion of apartness between subsets of the ambient space. In the general theory [**8**], strongly continuous mappings are important as the morphisms in the category of sets with apartness. Our result shows that in the subcategory of totally bounded uniform spaces with a countable

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base of entourages, these maps coincide with the usual morphisms. For related work see [18, 8, 12, 16].

Classically, every uniform space with a countable base of entourages is pseudometrisable (see [5, p. 142, Proposition 2]). It is not known whether (and it seems extremely unlikely that) this theorem, with its highly nonconstructive proof, holds constructively. Ishihara and Schuster have shown constructively that every strongly continuous mapping from a totally bounded metric space to a metric space is uniformly continuous [13]. Even if the foregoing pseudometrisation theorem turns out to be constructive, the Ishihara–Schuster result is extended by our Theorem 1, since the latter does not require that the codomain of the strongly continuous function be metrisable.

To read this paper you do not need any deep understanding of constructive mathematics (mathematics with intuitionistic logic – see [2, 3, 4, 6, 17]), but some intuitive feel for what is, and what is not, constructive will help. Nor do you need much background knowledge about uniform spaces. However, for the sake of clarity we now present some fundamental definitions in the theory of sets and uniform spaces.

By an *inequality* on a set X, we mean a binary relation $\neq$ that satisfies the following conditions:

$$\begin{aligned} x \neq y \quad &\Longrightarrow \quad y \neq x; \\ x \neq y \quad &\Longrightarrow \quad \neg (x = y) . \end{aligned}$$

Note that this notion of inequality is more general than the common classical one (denial of equality).

In the presence of an inequality, we define the *complement* of a subset S of X to be

$$\sim S = \{x \in X : \forall s \in S \ (x \neq s)\} .$$

Let X be a nonempty set, and let U and V be subsets of the Cartesian product $X \times X$. We define certain associated subsets as follows:

$$\begin{aligned} U^1 &= U, \\ U^{-1} &= \{(x, y) : (y, x) \in U\}, \\ U^{n+1} &= U \circ U^n \qquad (n = 1, 2, \ldots), \\ U \circ V &= \{(x, y) : \exists z \in X \ ((x, z) \in U \wedge (z, y) \in V)\}. \end{aligned}$$

We say that U is *symmetric* if $U = U^{-1}$. The *diagonal* of $X \times X$ is the set

$$\Delta = \{(x, x) : x \in X\}.$$

A family $\mathcal{U}$ of subsets of $X \times X$ is called a *uniform structure*, or *uniformity*, on X if the following conditions hold. (Note that condition U1, and the fact that – by U2 – each element of $\mathcal{U}$ is nonempty, show that $\mathcal{U}$ is a filter on $X \times X$.)

U1 (i) Every finite intersection of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.
(ii) Every subset of $X \times X$ that contains a member of $\mathcal{U}$ is in $\mathcal{U}$.

U2 Every member of $\mathcal{U}$ contains both the diagonal Δ and a symmetric member of $\mathcal{U}$.

U3 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subset U$.

U4 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$\forall \mathbf{x} \in X \times X \ (\mathbf{x} \in U \vee \mathbf{x} \notin V) .$$

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The elements of $\mathcal{U}$ are called the *entourages* of (the uniform structure on) X. A subfamily $\mathcal{B}$ of $\mathcal{U}$ is called a *base of entourages* if each element of U contains an element of $\mathcal{B}$.

Metric spaces and locally convex linear spaces are uniform spaces according to this definition.

The *uniform topology* on a uniform space $(X, \mathcal{U})$ is the topology $\tau_{\mathcal{U}}$ in which for each $x \in X$, the sets

$$U[x] = \{y \in X : (x, y) \in U\} \qquad (U \in \mathcal{U})$$

form a base of neighbourhoods of x.

We define the canonical inequality on a uniform space $(X, \mathcal{U})$ by

$$x \neq y \iff \exists U \in \mathcal{U}\ ((x, y) \notin U).$$

Note that, by axioms U1(ii) and U2, if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$. It follows that if $x \neq y$, then $y \neq x$. Moreover, since $\mathcal{U}$ contains Δ, if $x \neq y$, then $\neg(x = y)$. Thus $\neq$ is indeed an inequality relation on X. In turn, we define an associated inequality on $X \times X$ in the obvious way:

$$(x, y) \neq (x', y') \iff (x \neq x' \vee y \neq y').$$

It then follows from the axioms that for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subset U$ and

$$\forall \mathbf{x} \in X \times X\ (\mathbf{x} \in U \vee \mathbf{x} \in {\sim}V).$$

Even for the metric apartness space $\mathbf{R}$, the statements '$x \neq y$' and '$\neg(x = y)$' are not constructively equivalent unless we accept the following principle of unbounded search, known as *Markov's principle*.

> *For each binary sequence $(a_n)_{n=1}^{\infty}$, if it is impossible that $a_n = 0$ for all n, then there exists n such that $a_n = 1$.*

See [6, Chapter 1].

A mapping f of a uniform space $(X, \mathcal{U})$ into a uniform space $(Y, \mathcal{V})$ is said to be *uniformly continuous* if

$$\forall V \in \mathcal{V}\ \exists U \in \mathcal{U}\ \forall x, x' \in X\ ((x, x') \in U \Longrightarrow (f(x), f(x')) \in V).$$

A uniform space $(X, \mathcal{U})$ is *totally bounded* if, for each $U \in \mathcal{U}$, there exists a finitely enumerable subset $\{x_1, \ldots, x_n\}$ of X such that $X = \bigcup_{k=1}^{n} U[x_k]$. (A set S is *finitely enumerable* if, for some natural number n, there exists a mapping of the set of positive integers less than or equal to n onto S.)

Lemma 2. *A totally bounded uniform space with a countable base of entourages is separable.*

Proof. Let $(U_n)_{n=1}^{\infty}$ be a countable base of entourages for a totally bounded uniform space $(X, \mathcal{U})$. For each n there exists a finitely enumerable subset F_n of X such that $X = \bigcup_{x \in F_n} U_n[x]$. Let C be the countable set $\bigcup_{n=1}^{\infty} F_n$. Given $x \in X$ and $U \in \mathcal{U}$, find n such that $U_n \subset U$. Then there exists $y \in F_n \subset C$ such that $(x, y) \in U_n \subset U$, so $y \in U[x]$. It follows by the definition of the topology on X that C is dense in X. □

For convenience below, we define, for each positive integer n, an *n-chain of entourages* of X to be an n-tuple $(U_1, \ldots, U_n)$ of entourages such that for each k (where $1 < k \leqslant n$), we have U_k symmetric, $U_k^2 \subset U_{k-1}$, and

$$\forall \mathbf{x} \in X \times X \, (\mathbf{x} \in U_{k-1} \vee \mathbf{x} \in {\sim}U_k) \, .$$

Axiom U3 ensures that for each $U \in \mathcal{U}$ and each positive integer n there exists an n-chain $(U_1, \ldots, U_n)$ of entourages with $U_1 = U$.

The following lemma is crucial to the development of our argument, and is proved in [9, Lemma 1]. It is designed to take the sting out of a number of the succeeding proofs, and is necessitated by the constructive failure of what Bishop called the *limited principle of omniscience* (LPO):

> *for each binary sequence* $(\lambda_n)_{n=1}^{\infty}$ *either* $\lambda_n = 0$ *for all* n,
> *or else there exists* n *such that* $\lambda_n = 1$.

In its recursive interpretation, the limited principle of omniscience entails the decidability of the halting problem [6, Chapter 3].

Lemma 3. *Let X and Y be uniform spaces, let $f : X \longrightarrow Y$ be a strongly continuous function, and let V be an entourage of Y. Let $(\lambda_n)_{n=1}^{\infty}$ be an nondecreasing binary sequence, and let $(A_n)_{n=1}^{\infty}$ and $(S_n)_{n=1}^{\infty}$ be sequences of subsets of X such that:*

- *for each entourage U of X there exists ν such that for each $n \geqslant \nu$, either $A_n \times S_n = \emptyset$ or else $A_n \times S_n$ intersects U;*
- *if $\lambda_n = 0$, then $A_n = \emptyset$; and*
- *if $\lambda_n = 1 - \lambda_{n-1}$, then $A_n \neq \emptyset$, $S_n \neq \emptyset$, $f(A_n) \times f(S_n) \subset {\sim}V$, and $A_j = \emptyset$ for all $j > n$.*

Then there exists N such that $\lambda_n = \lambda_N$ for all $n \geqslant N$.

Two sequences $(x_n)_{n=1}^{\infty}$ and $(x'_n)_{n=1}^{\infty}$ in a uniform space $(X, \mathcal{U})$ are said to be *eventually close* if

$$\forall U \in \mathcal{U} \exists N \forall n \geqslant N \; ((x_n, x'_n) \in U).$$

A mapping f of X into a uniform space Y is *uniformly sequentially continuous* if the sequences $(f(x_n))_{n=1}^{\infty}$ and $(f(x'_n))_{n=1}^{\infty}$ are eventually close in Y whenever $(x_n)_{n=1}^{\infty}$ and $(x'_n)_{n=1}^{\infty}$ are eventually close in X.

A major step towards our main result is the following weak converse to the proposition that uniform continuity implies strong continuity.

Proposition 4 (see [9, Proposition 6]). *A strongly continuous mapping $f : X \longrightarrow Y$ between uniform spaces is uniformly sequentially continuous.*

We now establish a number of technical lemmas needed for the proof of our main result, Theorem 1.

Lemma 5. *Let X be a totally bounded uniform space with a countable base of entourages, and f a strongly continuous mapping of X into a uniform space Y. Let $(x_n)_{n=1}^{\infty}$ be a dense sequence in X (which exists by Lemma 2), and V any entourage of Y. Let $(\lambda_n)_{n=1}^{\infty}$ be an nondecreasing binary sequence, and let $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ be sequences of subsets of X, such that:*

- *if $\lambda_n = 0$, then $A_n = \emptyset$, and*
- *if $\lambda_n = 1 - \lambda_{n-1}$, then $A_n \neq \emptyset$, $\{x_1, \ldots, x_{n-1}\} \subset B_n$, $f(A_n) \times f(B_n) \subset {\sim}V$, and $A_j = \emptyset$ for all $j > n$.*

Then there exists N such that $\lambda_n = \lambda_N$ for all $n \geqslant N$.

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Proof. Take $S_n = \{x_1, \dots, x_{n-1}\}$. Given an entourage U of X, compute N such that $X = \bigcup_{i=1}^{N-1} U[x_i]$, and consider any $n \geqslant N$. If either $\lambda_n = 0$ or $\lambda_{n-1} = 1$, then $A_n \times S_n = \emptyset$. If $\lambda_n = 1 - \lambda_{n-1}$, then $A_n \neq \emptyset$ and for each $x \in A_n$ there exists i such that $1 \leqslant i \leqslant N - 1 < n$ and $x \in U[x_i]$; so $A_n \times S_n$ intersects U. Hence we can apply Lemma 3 to obtain the desired result. □

LEMMA 6. *If $f : X \longrightarrow Y$ is a strongly continuous mapping between uniform spaces, then $f(\bar{S}) \subset \overline{f(S)}$ for each $S \subset X$.*

Proof. Let $y = f(x)$, where $x \in \bar{S}$, and let V be any entourage of Y. It is enough to show that $f(S)$ intersects $V[y]$. To this end, construct a 3-chain (V_1, V_2, V_3) of entourages of Y with $V_1 = V$. We see from [16, Corollary 16 and Lemma 17] that

$$y \in V_3[y] \subset -{\sim}V_2[y] \subset V_1[y].$$

In particular, $\{y\} \bowtie {\sim}V_2[y]$, so, by the strong continuity of f,

$$\{x\} \bowtie f^{-1}({\sim}V_2[y]).$$

Choose a 2-chain (U_1, U_2) of entourages of X such that

$$\{x\} \times f^{-1}({\sim}V_2[y]) \subset {\sim}U_1.$$

By [16, Lemmas 13 and 15], we have $U_2[x] \subset -f^{-1}({\sim}V_2[y])$. Since $U_2[x]$ is a neighbourhood of x in X, it follows that there exists s in $S \cap -f^{-1}({\sim}V_2[y])$. Then $f(s) \notin {\sim}V_2[y]$, from which it follows that $f(s) \in V[y]$. □

LEMMA 7. *Let X be a totally bounded uniform space with a countable base of entourages. Let f be a strongly continuous mapping of X into a uniform space Y, let $(V_1, \dots, V_5)$ be a 5-chain of entourages of X, and let S be a finitely enumerable subset of X. Then:*

- *either for each $x \in X$ there exists $s \in S$ such that $(f(x), f(s)) \in V_1$; or else*
- *there exists $x \in X$ such that $\{f(x)\} \times f(S) \subset {\sim}V_5$.*

Proof. Write $S = \{s_1, \dots, s_M\}$. By Lemma 2, X is separable. Choosing a dense sequence $(x_n)_{n=1}^{\infty}$ in X, construct an nondecreasing binary sequence $(\lambda_n)_{n=1}^{\infty}$ such that for each n,

$$\lambda_n = 0 \Longrightarrow \forall k \leqslant n \exists i \leqslant M \ \ ((f(x_k), f(s_i)) \in V_4);$$
$$\lambda_n = 1 - \lambda_{n-1} \Longrightarrow \forall i \leqslant M \ \ ((f(x_n), f(s_i)) \in {\sim}V_5).$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $B_n = \emptyset$; if $\lambda_n = 1 - \lambda_{n-1}$, set $B_n = \{x_1, \dots, x_{n-1}\}$ and $B_k = \emptyset$ for all $k > n$. Let $B = \bigcup_{n=1}^{\infty} B_n$.

Next, construct an nondecreasing binary sequence $(\mu_n)_{n=1}^{\infty}$ such that, for each n,

$$\mu_n = 0 \Longrightarrow \forall k \leqslant n \exists i \leqslant M \ \ ((f(x_k), f(s_i)) \in V_2);$$
$$\mu_n = 1 - \mu_{n-1} \Longrightarrow \forall i \leqslant M \ \ ((f(x_n), f(s_i)) \in {\sim}V_3).$$

We may assume that $\mu_1 = 0$. If $\mu_n = 0$, set $A_n = \emptyset$; if $\mu_n = 1 - \mu_{n-1}$, set $A_n = \{x_n\}$ and $A_k = \emptyset$ for all $k > n$. Let $A = \bigcup_{n=1}^{\infty} A_n$. We show that

$$f(A) \times f(B) \subset {\sim}V_5. \tag{1}$$

To this end, let $x \in A$ and $y \in B$, and suppose that $(f(x), f(y)) \in V_4$. Choosing n such that $y \in B_n$, we see that $\lambda_n = 1 - \lambda_{n-1}$, $B = B_n = \{x_1, \dots, x_{n-1}\}$, $y = x_j$ for some $j \leqslant n - 1$, and there exists $i \leqslant M$ such that $(f(y), f(s_i)) \in V_4$. Hence

$$(f(x), f(s_i)) \in V_4^2 \subset V_3. \tag{2}$$

On the other hand, since $x \in A$, there exists k such that $\mu_k = 1 - \mu_{k-1}$, $x = x_k$, and $(f(x), f(s_i)) \in {\sim}V_3$, which contradicts inclusion (2). Hence $\neg((f(x), f(y)) \in V_4)$, and therefore $(f(x), f(y)) \in {\sim}V_5$. Since x and y are arbitrary elements of A and B respectively, we now obtain inclusion (1).

It follows from this and the strong continuity of f that there exists an entourage U of X such that $A \times B \subset {\sim}U$. Since X is totally bounded, there exists N such that $X = \bigcup_{n=1}^{N} U[x_n]$. If $\lambda_N = 1$, then there exists $j \leqslant N$ such that $\{f(x_j)\} \times f(S) \subset {\sim}V_5$. So, without loss of generality, we assume that $\lambda_N = 0$. Suppose that $\mu_n = 1 - \mu_{n-1}$ for some n. Then for each $i \leqslant M$, we have $(f(x_n), f(s_i)) \in {\sim}V_3$, and therefore $(f(x_n), f(s_i)) \notin V_4$. Hence $\lambda_m \neq 0$, and therefore $\lambda_m = 1 - \lambda_{m-1}$, for some $m \leqslant n$; clearly, $N < m$. We now have $A = \{x_n\}$, $B = \{x_1, \ldots, x_{m-1}\}$, and

$$\{x_n\} \times \{x_1, \ldots, x_{m-1}\} \subset {\sim}U,$$

which is absurd since, by our choice of N, there exists $j \leqslant N < m$ such that $(x_n, x_j) \in U$. It follows, for all n, that $\mu_n = 0$, and therefore there exists an $i \leqslant M$ such that we have $(f(x_n), f(s_i)) \in V_2$.

Finally, consider any $x \in X$. Since x is in the closure of the set $\{x_n : n \geqslant 1\}$, we see from Lemma 6 that $f(x)$ is in the closure of $\{f(x_n) : n \geqslant 1\}$; whence there exists n such that $f(x_n) \in V_2[f(x)]$. Choosing $i \leqslant M$ such that $(f(x_n), f(s_i)) \in V_2$, we have $(f(x), f(s_i)) \in V_2^2 \subset V_1$. □

The following proposition, the last link in the chain connecting us to the proof of Theorem 1, is of interest in its own right and also has an important corollary.

Proposition 8. *Let X and Y be uniform apartness spaces, and f a strongly continuous mapping of X into Y such that $f(X)$ is totally bounded. Then f is uniformly continuous.*

Proof. Given an entourage V of Y, construct a 5-chain $(V_1, V_2, V_3, V_4, V_5)$ of entourages of Y such that $V_2^3 \subset V_1 = V$, and $V_4^3 \subset V_3$. Choose $x_1, \ldots, x_m$ in X such that $Y = Y_1 \cup \cdots \cup Y_m$, where $Y_i = V_4[f(x_i)]$; then set $X_i = f^{-1}(Y_i)$. For $1 \leqslant i, j \leqslant m$ construct c_{ij} such that

$$c_{ij} = 0 \Longrightarrow (f(x_i), f(x_j)) \in V_2;$$
$$c_{ij} = 1 \Longrightarrow (f(x_i), f(x_j)) \in {\sim}V_3.$$

For each (i, j) with $c_{ij} = 1$, we have $Y_i \bowtie Y_j$. To see this, consider such i, j and an element (y, y') of $Y_i \times Y_j$, and suppose that $(y, y') \in V_4$. Then $(f(x_i), y) \in V_4$ and $(y', f(x_j)) \in V_4$, so $(f(x_i), f(x_j)) \in V_4^3 \subset V_3$, a contradiction. Hence $(y, y') \in {\sim}V_5$. It follows that $Y_i \times Y_j \subset {\sim}V_5$, and therefore that $Y_i \bowtie Y_j$. By the strong continuity of f, $X_i \bowtie X_j$, and therefore there exists an entourage E_{ij} of X such that $X_i \times X_j \subset {\sim}E_{ij}$. Let

$$E = \bigcap \{E_{ij} : c_{ij} = 1\},$$

which also is an entourage of X. Consider points x and y of X with $(x, y) \in E$. Choose i and j such that $f(x) \in Y_i$ and $f(y) \in Y_j$. If $c_{ij} = 1$, then

$$(x, y) \in X_i \times X_j \subset {\sim}E_{ij} \subset {\sim}E,$$

a contradiction.

Hence $c_{ij} = 0$, and so $(f(x_i), f(x_j)) \in V_2$. Since $(f(x), f(x_i)) \in V_4 \subset V_2$ and $(f(y), f(x_j)) \in V_4 \subset V_2$, it follows that $(f(x), f(y)) \in V_2^3$ and therefore that $(f(x), f(y)) \in V$. □

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For those readers prepared to explore the general constructive theory of apartness spaces, we digress briefly to introduce a corollary of Proposition 8.

Let X be any apartness space – that is, a set equipped with a binary relation $\bowtie$ between subsets, for which the axioms in [10] hold. We say that a uniformity $\mathcal{U}$ on X is *compatible* with the given apartness $\bowtie$ if

$$S \bowtie T \iff \exists U \in \mathcal{U}\,(S \times T \subset \sim U).$$

COROLLARY 9. *A given apartness space has at most one compatible uniformity that is totally bounded.*

Proof. Let $(X, \bowtie)$ be an apartness space, and suppose that there are two totally bounded uniformities $\mathcal{U}$ and $\mathcal{U}'$ that are compatible with the apartness on X. Denote the apartness relations corresponding to $\mathcal{U}$ and $\mathcal{U}'$ by $\bowtie_{\mathcal{U}}$ and $\bowtie_{\mathcal{U}'}$ respectively. Then the identity mapping from $(X, \bowtie)$ onto $(X, \bowtie_{\mathcal{U}})$ is strongly continuous, as is its inverse; likewise, the identity mapping from $(X, \bowtie)$ onto $(X, \bowtie_{\mathcal{U}'})$ is strongly continuous, as is its inverse. Hence the identity mapping from $(X, \bowtie_{\mathcal{U}})$ to $(X, \bowtie_{\mathcal{U}'})$ is strongly continuous, as is its inverse. It follows from Proposition 8 that the uniformities $\mathcal{U}$ and $\mathcal{U}'$ are equivalent. □

At last we are in a position to give the proof of Theorem 1.

Proof. Let X be a totally bounded uniform space with a countable base $(U_n)_{n=1}^{\infty}$ of entourages. We may assume that $U_1 \supset U_2 \supset \cdots$. Let $(x_k)_{k=1}^{\infty}$ be a dense sequence in X. Then there exists a strictly increasing sequence $(k_n)_{n=1}^{\infty}$ of positive integers such that

$$X = \bigcup_{j=1}^{k_n} U_n[x_j]$$

for each n. Let $F_n = \{x_1, \ldots, x_{k_n}\}$. Consider a strongly continuous mapping f of X into a uniform space Y. In view of Proposition 8, it is enough to prove that $f(X)$ is totally bounded. Accordingly, given an entourage V of Y, construct a 5-chain $(V_1, \ldots, V_5)$ of entourages of Y with $V_1 = V$. Using Lemma 7, construct a nondecreasing binary sequence $(\lambda_n)_{n=1}^{\infty}$ such that

$$\lambda_n = 0 \Longrightarrow \exists x \in X \forall k \leqslant k_n\,((f(x), f(x_k)) \in \sim V_5);$$
$$\lambda_n = 1 - \lambda_{n-1} \Longrightarrow \forall x \in X \exists k \leqslant k_n\,((f(x), f(x_k)) \in V_1).$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $A_n = B_n = \emptyset$. If $\lambda_n = 1 - \lambda_{n-1}$, then, as $\lambda_{n-1} = 0$, there exists $\zeta \in X$ such that $(f(\zeta), f(x_k)) \in \sim V_5$ for all $k \leqslant k_{n-1}$; set $A_n = \{\zeta\}$, $B_n = F_{n-1}$, and $A_j = B_j = \emptyset$ for all $j \geqslant n$; then $f(A_n) \times f(B_n) \subset \sim V_5$. We can now apply Lemma 5 to find N such that if $\lambda_n = \lambda_N$ for all $n \geqslant N$.

Assume that $\lambda_N = 0$. Then for each n, $\lambda_n = 0$ and so there exists $z_n \in X$ such that

$$f(z_n) \times f(F_n) \subset \sim V_5. \tag{3}$$

On the other hand, for each n there exists $\zeta_n \in F_n$ such that $(z_n, \zeta_n) \in U_n$. It follows that the sequences $(z_n)_{n=1}^{\infty}$ and $(\zeta_n)_{n=1}^{\infty}$ are eventually close: for if U is any entourage of X and we choose m such that $U_m \subset U$, then we have $(z_n, \zeta_n) \in U_n \subset U_m \subset U$ for all $n \geqslant m$. By Proposition 4, the sequences $(f(z_n))_{n=1}^{\infty}$ and $(f(\zeta_n))_{n=1}^{\infty}$ are eventually close, so there exists n such that $(f(z_n), f(\zeta_n)) \in V_5$. However, since $\zeta_n \in F_n$, this contradicts (3). Thus $\lambda_N \neq 0$, so $\lambda_N = 1$ and therefore $f(X) = \bigcup_{j=1}^{k_N} V[x_j]$. Since V is an arbitrary entourage of Y, we conclude that $f(X)$ is totally bounded. □

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Douglas Bridges d.bridges@math.canterbury.ac.nz
http://www.math.canterbury.ac.nz/∼mathdsb
Luminiţa Vîţă Luminita@Math.net
http://www.math.canterbury.ac.nz/∼mathlsv

Department of Mathematics & Statistics
University of Canterbury
Private Bag 4800
Christchurch
New Zealand

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