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SOME CLASSIFICATIONS OF LORENTZIAN SURFACES WITH FINITE TYPE GAUSS MAP IN THE MINKOWSKI 4-SPACE

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Abstract

In this paper we study the Lorentzian surfaces with finite type Gauss map in the four-dimensional Minkowski space. First, we obtain the complete classification of minimal surfaces with pointwise 1-type Gauss map. Then, we get a classification of Lorentzian surfaces with nonzero constant mean curvature and of finite type Gauss map. We also give some explicit examples.

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1. Introductions

After the problem 'To what extent does the type of the Gauss map of a submanifold of $\mathbb{E}^m_r$ determine the submanifold?' was introduced by Chen and Piccini in [6], the study of submanifolds with finite type Gauss map became a very active research subject. Many affirmative partial solutions to this problem have appeared so far [3, 4, 6, 8, 17].

Let M be a semi-Riemannian submanifold in a semi-Euclidean space $\mathbb{E}^m_r$. A smooth mapping $\psi : M \to \mathbb{E}^N_S$ into another semi-Euclidean space is said to be of k-type if it can be expressed as a sum of eigenvectors corresponding to k distinct eigenvalues of the Laplace operator Δ of M. If such a k exists, then ψ is said to be of finite type. Many important results about finite type mappings have been obtained [1, 5, 13, 15, 16].

From the definition above, one can see that M has 1-type Gauss map if and only if the equation

$$\Delta\nu = \lambda(\nu + C) \tag{1.1}$$

is satisfied for a constant vector C and $\lambda \in \mathbb{R}$, where ν is the Gauss map of M. Similarly, a submanifold M is said to have pointwise 1-type Gauss map if the Laplacian of its

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Gauss map takes the form

$$\Delta\nu = f(\nu + C) \tag{1.2}$$

for a smooth function f and a constant vector C. The study of submanifolds with pointwise 1-type Gauss map or finite type Gauss map is nowadays a very active research subject (see for example [3, 9, 10, 17]). For example, in [17], the author obtained some classifications of quasi-minimal surfaces with finite type Gauss map in the Minkowski space–time $\mathbb{E}^4_1$ and in the de Sitter space–time $\mathbb{S}^4_1(1)$. Very recently, Dursun and Bektaş have studied the flat Lorentzian rotational surfaces in $\mathbb{E}^4_1$ with pointwise 1-type Gauss map [10].

In this paper we focus on Lorentzian surfaces with constant mean curvature in the Minkowski space–time $\mathbb{E}^4_1$ in terms of the finite type of their Gauss map. In Section 2, after we have described the notation that we will use in this paper, we give basic facts and definitions on the theory of submanifolds of semi-Euclidean spaces. In Section 3 we obtain the complete classification of Lorentzian minimal surfaces with pointwise 1-type Gauss map. In Section 4 we study Lorentzian surfaces with constant mean curvature in terms of the type of their Gauss map.

2. Preliminaries

2.1. Basic notation, formulas and definitions. Let $\mathbb{E}^m_t$ denote the semi-Euclidean m-space with the canonical semi-Euclidean metric tensor of index t given by

$$g = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{m} dx_j^2,$$

where $x_1, x_2, \dots, x_m$ are rectangular coordinates of the points of $\mathbb{E}^m_t$. We put

$$\mathbb{S}^{m-1}_t(r^2) = \{x \in \mathbb{E}^m_t : \langle x, x\rangle = r^{-2}\},$$
$$\mathbb{H}^{m-1}_{t-1}(-r^2) = \{x \in \mathbb{E}^m_t : \langle x, x\rangle = -r^{-2}\},$$

where $\langle\ ,\ \rangle$ is the indefinite inner product of $\mathbb{E}^m_t$.

A nonzero vector v in $\mathbb{E}^m_t$ is called space-like, time-like or light-like if $\langle v, v\rangle > 0$, $\langle v, v\rangle < 0$ or $\langle v, v\rangle = 0$, respectively. We will use the following well-known lemmas later [14].

Lemma 2.1. *Let U be a real vector space with a nondegenerate inner product $\langle\ ,\ \rangle$ with index 1. Then two light-like vectors v_1, v_2 are linearly dependent if and only if $\langle v_1, v_2\rangle = 0$.*

Lemma 2.2. *Let V be a subspace of a real vector space U and $\langle\ ,\ \rangle$ a nondegenerate inner product defined in U. Then $\langle\ ,\ \rangle|_V$ is nondegenerate if and only if $V \cap V^{\perp} = \{0\}$.*

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Let M be an n-dimensional semi-Riemannian submanifold of the semi-Euclidean space $\mathbb{E}^m_s$. We denote the Levi-Civita connections of $\mathbb{E}^m_s$ and M by $\widetilde{\nabla}$ and ∇, respectively. The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi \tag{2.2}$$

for any tangent vector field X, Y and any normal vector field ξ on M, where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively. On the other hand, the shape operator A and the second fundamental form h of M are related by

$$\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle. \tag{2.3}$$

The Gauss, Codazzi and Ricci equations are given, respectively, by

$$R(X, Y, Z, W) = \langle h(Y, Z), h(X, W)\rangle - \langle h(X, Z), h(Y, W)\rangle, \tag{2.4a}$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{2.4b}$$

$$\langle R^D(X, Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle, \tag{2.4c}$$

where R, R^D are the curvature tensors associated with the connections ∇ and D, respectively, and

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Now let M be a Lorentzian surface in $\mathbb{E}^4_1$, $\{e_1, e_2; e_3, e_4\}$ a local orthonormal frame field on M such that $\langle e_1, e_1\rangle = -1$ and $\{f_1, f_2\}$ the pseudo-orthonormal base field of the tangent bundle of M given by $f_1 = (e_1 - e_2)/\sqrt{2}$ and $f_2 = (e_1 + e_2)/\sqrt{2}$. Then

$$H = -h(f_1, f_2), \tag{2.5a}$$

$$K = R(f_1, f_2, f_2, f_1) = R(e_1, e_2, e_2, e_1), \tag{2.5b}$$

$$K^D = R^D(f_1, f_2; e_3, e_4) = R^D(e_1, e_2; e_3, e_4), \tag{2.5c}$$

where H is the mean curvature vector and K and K^D stand for the Gaussian and normal curvatures of M, respectively. On the other hand, the Laplace operator of M is given by

$$\Delta = f_1 f_2 + f_2 f_1 - \nabla_{f_1} f_2 - \nabla_{f_2} f_1. \tag{2.6}$$

The relative null space at p of M is defined by

$$\mathcal{N}_p(M) = \{X \in T_p M | h(X, Y) = 0 \; \forall Y \in T_p M\}.$$

A Lorentzian surface M in $\mathbb{E}^4_1$ is said to have positive relative nullity if the dimension of $\mathcal{N}_p(M)$ is positive for all $p \in M$ [7]. We say that M has a degenerate relative null bundle if $(\mathcal{N}_p(M), \langle\, ,\, \rangle)$ is a degenerate inner product space for all $p \in M$.

We would like to state the following lemma obtained in [2] (see also [12, Proposition 2.1]).

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Lemma 2.3 [2]. *Let M be a Lorentzian surface in a semi-Euclidean space $\mathbb{E}^q_r$. Then there exist local coordinates (s,t) such that the induced metric is of the form*

$$g = -m^2(ds\,dt + dt\,ds), \quad s \in I_1, t \in I_2,$$

where $m = m(s,t)$ is a nonvanishing function and I_1, I_2 are some open intervals. Moreover, the Levi-Civita connection of M is given by

$$\nabla_{\partial_s}\partial_s = \frac{2m_s}{m}\partial_s, \quad \nabla_{\partial_s}\partial_t = 0, \quad \nabla_{\partial_t}\partial_t = \frac{2m_t}{m}\partial_t. \tag{2.7}$$

2.2. Gauss map. Let $G(n,m)$ denote the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m_r$ which is canonically imbedded in the vector space $\Lambda^n(\mathbb{E}^m_r)$ of n-vectors of $\mathbb{E}^m_r$. We note that there exists a linear isometry between $\Lambda^{m,n}$ and $\mathbb{E}^N_S$, where $\Lambda^{m,n}$ denotes the inner product space $(\Lambda^n(\mathbb{E}^m_r), \langle\ ,\ \rangle)$ given by

$$\langle X_1 \wedge X_2 \wedge \cdots \wedge X_n, Y_1 \wedge Y_2 \wedge \cdots \wedge Y_n\rangle = \det(\langle X_i, Y_j\rangle);$$

N and S are the dimension and index of $\Lambda^{m,n}$, respectively. Therefore, one can define the (tangent) Gauss map of a submanifold of a semi-Euclidean space as a $\mathbb{E}^N_S$-valued mapping. In fact, the (tangent) Gauss map of M is defined by

$$\begin{aligned} \nu : M &\to R^{N-1}_S(\varepsilon) \subset \mathbb{E}^N_S \\ p &\mapsto \nu(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p) \end{aligned} \tag{2.8}$$

for $\varepsilon \in \{-1, +1\}$, where $R^{N-1}_S(\varepsilon)$ denotes the complete semi-Riemannian manifolds, with constant sectional curvatures ε, and $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal base field of the tangent bundle of M. For a geometric interpretation of the Gauss map of M, see [6, 8, 10].

A submanifold M is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1.2) for a smooth function f and a constant vector C. More precisely, a pointwise 1-type Gauss map is called *of the first kind* if (1.2) is satisfied for $C = 0$, and *of the second kind* if $C \neq 0$. Moreover, if (1.2) is satisfied for a nonconstant function f, then M is said to have *proper* pointwise 1-type Gauss map.

Now let M be a Lorentzian surface in the Minkowski space $\mathbb{E}^4_1$ with a pseudo-orthonormal frame field $\{f_1, f_2; e_3, e_4\}$. Then the tangent Gauss map of M given by (2.8) becomes

$$\begin{aligned} \nu : M &\to \mathbb{H}^5_3(-1) \subset \mathbb{E}^6_3 \\ p &\mapsto \nu(p) = (f_1 \wedge f_2)(p). \end{aligned} \tag{2.9}$$

On the other hand, one may define the normal Gauss map of M by

$$\begin{aligned} \mu : M &\to \mathbb{S}^5(1) \subset \mathbb{E}^6 \\ p &\mapsto \nu(p) = (e_3 \wedge e_4)(p). \end{aligned} \tag{2.10}$$

We obtain the Laplacian of the Gauss map ν as follows (see [8, Lemma 3.2]).

Lemma 2.4. *Let M be a Lorentzian surface. Then ν and μ satisfy*

$$\Delta\nu = (2K + 4\langle H, H\rangle)\nu + 2K^D\mu - 2D_{f_1}H \wedge f_2 - 2f_1 \wedge D_{f_2}H. \tag{2.11}$$

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3. Minimal Lorentzian surfaces and their Gauss map

In this section we focus on the minimal Lorentzian surfaces in $\mathbb{E}^4_1$.
If M is minimal, that is, $H \equiv 0$ on M, then (2.11) becomes

$$\Delta\nu = 2K\nu + 2K^D\mu. \tag{3.1}$$

First, we want to give the following proposition.

PROPOSITION 3.1. *There exist two families of Lorentzian minimal surfaces in the Minkowski space $\mathbb{E}^4_1$ with pointwise 1-type Gauss map of the first kind:*

(i) *a minimal surface lying in a Lorentzian hyperplane of $\mathbb{E}^4_1$;*
(ii) *a surface with degenerate relative null bundle.*

Conversely, every Lorentzian minimal surface with pointwise 1-type Gauss map of the first kind in the Minkowski space $\mathbb{E}^4_1$ is congruent to an open portion of a surface obtained from these types of surfaces.

PROOF. A direct computation shows that the surfaces given in the proposition have pointwise 1-type Gauss map. Thus, we want to prove its converse.

Let M be a Lorentzian surface in $\mathbb{E}^4_1$ and s, t be the local coordinates mentioned in Lemma 2.3. Consider the pseudo-orthonormal basis $\{f_1, f_2\}$ given by

$$f_1 = \frac{1}{m}\partial_s \quad \text{and} \quad f_2 = \frac{1}{m}\partial_t.$$

If we suppose that M is minimal, that is, $H \equiv 0$, then (2.5a) implies that $h(f_1, f_2) = 0$. On the other hand, the Gauss map $\nu = f_1 \wedge f_2$ of M satisfies (3.1).

Now we assume that M has pointwise 1-type Gauss map of the first kind. Then (1.2) is satisfied for $C = 0$. From (1.2) and (3.1), we obtain $2K^D e_3 \wedge e_4 = 0$, from which we get $h(f_1, f_1) \wedge h(f_2, f_2) = 0$. Thus, $h(\partial_s, \partial_s)$ and $h(\partial_t, \partial_t)$ are linearly dependent.

Let I_1, I_2 be some open intervals and $x : I_1 \times I_2 \to \mathbb{E}^4_1$ an isometric immersion. Consider the functions

$$\begin{aligned} \psi_1 : I_1 \times I_2 &\to \mathbb{R} \\ (s_0, t_0) &\mapsto \langle h(\partial_s, \partial_s), h(\partial_s, \partial_s)\rangle|_{x(s_0,t_0)} \end{aligned}$$

and

$$\begin{aligned} \psi_2 : I_1 \times I_2 &\to \mathbb{R} \\ (s_0, t_0) &\mapsto \langle h(\partial_t, \partial_t), h(\partial_t, \partial_t)\rangle|_{x(s_0,t_0)}. \end{aligned}$$

Case 1: $\psi_1 \equiv 0$ or $\psi_2 \equiv 0$. In this case M has degenerate relative null bundle and we have the case (ii) of the proposition.

Case 2: $\psi_1 \neq 0$ and $\psi_2 \neq 0$. In this case the initial value problems

$$\phi_1' = \psi_1(\phi_2)^{-1/4}, \quad \phi_1(0) = s_0$$

and

$$\phi_2' = \psi_1(\phi_2)^{-1/4}, \quad \phi_2(0) = t_0$$

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admit unique solutions, say ϕ_1 and ϕ_2, respectively, where $s_0 \in I_1$ and $t_0 \in I_2$. Let S, T be local coordinates given by $S = \phi_1(s)$ and $T = \phi_2(t)$. Then we have $g = -\hat{m}^2(S,T)(dS\,dT + dT\,dS)$, where $\hat{m}(S,T) = m(\phi_1(s), \phi_2(t))$. Moreover, the normal vector fields $h(\partial_S, \partial_S)$ and $h(\partial_T, \partial_T)$ are linearly dependent and have unit length. Thus,

$$h(\partial_S, \partial_S) = \pm h(\partial_T, \partial_T). \tag{3.2}$$

Now let $\{e_3, e_4\}$ be an orthonormal base field of the normal bundle of M with $e_3 = h(\partial_S, \partial_S)$. From the Codazzi equation (2.4b), we obtain $D_{\partial_T} h(\partial_S, \partial_S) = D_{\partial_S} h(\partial_T, \partial_T) = 0$. Therefore, (3.2) implies that $De_3 = 0$, that is, e_3 is parallel. As M has codimension two, e_4 is also parallel. Moreover, by combining (2.3) and (3.2), we obtain $A_4 = A_{e_4} = 0$. Thus, we have $\widetilde{\nabla} e_4 = 0$, that is, e_4 is constant. Hence, M is contained in a hyperplane Π whose normal is e_4. Since e_4 is space-like, Π is Lorentzian. □

Next we obtain the following proposition.

Proposition 3.2. *Let M be a Lorentzian minimal surface in $\mathbb{E}^4_1$. If M has pointwise 1-type Gauss map, then it is of the first kind.*

Proof. If M is a Lorentzian minimal surface, then (2.5a) implies that $h(f_1, f_2) = 0$, from which and (2.3) we have $\langle A_3 f_1, f_2\rangle = \langle A_4 f_1, f_2\rangle = 0$ for any pseudo-orthonormal frame field $\{f_1, f_2, e_3, e_4\}$. In addition, the Gauss map $\nu = f_1 \wedge f_2$ of M satisfies (3.1).

Now we assume that the Gauss map ν of M satisfies (1.2) for $C \neq 0$. From (1.2) and (3.1),

$$C = C_{12} f_1 \wedge f_2 + C_{34} e_3 \wedge e_4 \tag{3.3}$$

for some smooth functions C_{12} and C_{34}. Next we take into account that C is a constant vector and apply f_1 and f_2 separately to (3.3), to obtain

$$\begin{aligned} &f_1(C_{12}) f_1 \wedge f_2 + f_1(C_{34}) e_3 \wedge e_4 + C_{12} h(f_1, f_1) \wedge f_2 \\ &\quad + C_{34}(-A_3 f_1 \wedge e_4 + A_4 f_1 \wedge e_3) = 0, \end{aligned} \tag{3.4a}$$

$$\begin{aligned} &f_2(C_{12}) f_1 \wedge f_2 + f_2(C_{34}) e_3 \wedge e_4 + C_{12} f_1 \wedge h(f_2, f_2) \\ &\quad + C_{34}(-A_3 f_2 \wedge e_4 + A_4 f_2 \wedge e_3) = 0, \end{aligned} \tag{3.4b}$$

from which we see that C_{12}, C_{34} are constant. From (3.4), we also have

$$\begin{aligned} &C_{12}^2 \langle h(f_1, f_1) \wedge f_2, f_1 \wedge h(f_2, f_2)\rangle \\ &\quad = C_{34}^2 \langle -A_3 f_1 \wedge e_4 + A_4 f_1 \wedge e_3, -A_3 f_2 \wedge e_4 + A_4 f_2 \wedge e_3\rangle. \end{aligned} \tag{3.5}$$

By a direct computation, we get

$$\langle h(f_1, f_1) \wedge f_2, f_1 \wedge h(f_2, f_2)\rangle = \langle h(f_1, f_1), h(f_2, f_2)\rangle, \tag{3.6a}$$

$$\langle A_3 f_1 \wedge e_4, A_4 f_2 \wedge e_3\rangle = 0, \tag{3.6b}$$

$$\langle A_4 f_1 \wedge e_3, A_3 f_2 \wedge e_4 = 0, \tag{3.6c}$$

$$\langle A_3 f_1 \wedge e_4, A_3 f_2 \wedge e_4\rangle = -\langle h(f_1, f_1), e_3\rangle\langle h(f_2, f_2), e_3\rangle, \tag{3.6d}$$

$$\langle A_4 f_1 \wedge e_3, A_4 f_2 \wedge e_3\rangle = -\langle h(f_1, f_1), e_4\rangle\langle h(f_2, f_2), e_4\rangle. \tag{3.6e}$$

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By combining (3.5)–(3.6), we obtain

$$(C_{12}^2 + C_{34}^2)\langle h(f_1, f_1), h(f_2, f_2)\rangle = 0.$$

Since $C \neq 0$ by the assumption, the above equation implies that $h(f_1, f_1)$ and $h(f_2, f_2)$ are orthogonal.

Consider the open subset $\mathcal{U} = \{p \in M | h(f_1, f_1) \neq 0 \text{ and } h(f_2, f_2) \neq 0\}$ of M and let $\{e_3, e_4\}$ be a local orthonormal base field of the normal bundle of M such that $h(f_1, f_1) = \alpha_3 e_3$ and $h(f_2, f_2) = \alpha_4 e_4$ on $\mathcal{U}$, where α_3 and α_4 are some functions. From (3.4),

$$C_{12}\alpha_3 f_2 \wedge e_3 = -C_{34}(-A_3 f_1 \wedge e_4 + A_4 f_1 \wedge e_3),$$
$$C_{12}\alpha_4 f_1 \wedge e_4 = -C_{34}(-A_3 f_2 \wedge e_4 + A_4 f_2 \wedge e_3)$$

on $\mathcal{U}$. From these equations, we have $A_3 f_1 = A_4 f_2 = 0$ on $\mathcal{U}$, which imply that $h|_{\mathcal{U}} = 0$, because of (2.3). However, this is a contradiction if $\mathcal{U}$ is not empty.

Therefore, we have $h(f_1, f_1) = 0$ or $h(f_2, f_2) = 0$, which yields that M has degenerate relative null bundle. Thus, Proposition 3.1 implies that M has pointwise 1-type Gauss map of the first kind, which yields a contradiction. □

Lemma 3.3. *Let M be a Lorentzian surface in $\mathbb{E}^4_1$. Then M has degenerate relative null bundle if and only if it is congruent to the surface given by*

$$x(s, t) = s\eta_0 + \beta(t), \tag{3.7}$$

where η_0 is a constant light-like vector and β is a null curve in $\mathbb{E}^4_1$ with $\langle \eta_0, \beta(t)\rangle \neq 0$.

Proof. Let M be a Lorentzian surface in $\mathbb{E}^4_1$, x its position vector and s, t the local coordinates mentioned in Lemma 2.3 satisfying (2.7). Consider the tangent vector fields $f_1 = (1/m)\partial_s$ and $f_2 = (1/m)\partial_t$.

Now assume that $\mathcal{N}_p(M)$ is degenerate for all $p \in M$. Because of Lemma 2.2, we may assume that $\mathcal{N}_p(M) = \text{span}\{f_1\}$, which implies that $h(f_1, f_1) = h(f_1, f_2) = 0$. From these equations and (2.7), we have $\widetilde{\nabla}_{\partial_s}\partial_s = \nabla_{\partial_s}\partial_s$ and $\widetilde{\nabla}_{\partial_s}\partial_t = 0$, from which we obtain $x_{ss} = 2(m_s/m)x_s$ and $x_{st} = 0$. By integrating these equations and re-defining s suitably, we obtain that M is congruent to the surface given by (3.7). □

By combining all the results given in this section, we state the following result.

Theorem 3.4. *Let M be a Lorentzian minimal surface in $\mathbb{E}^4_1$. Also, suppose that no open part of M is contained in a hyperplane of $\mathbb{E}^4_1$. Then, the following conditions are equivalent:*

(i) *M has pointwise 1-type Gauss map;*
(ii) *M has pointwise 1-type Gauss map of the first kind;*
(iii) *M has degenerate relative null bundle;*
(iv) *M is congruent to the surface given by (3.7) for a constant light-like vector $\eta_0 \in \mathbb{E}^4_1$ and a null curve β in $\mathbb{E}^4_1$ satisfying $\langle \eta_0, \beta(t)\rangle \neq 0$.*

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We also want to state the following corollary of this theorem.

COROLLARY 3.5. *A Lorentzian minimal surface in $\mathbb{E}^4_1$ has proper pointwise 1-type Gauss map if and only if it lies in an Lorentzian hyperplane of $\mathbb{E}^4_1$ and it has nonconstant Gaussian curvature.*

4. Lorentzian surfaces with constant mean curvature

In this section we focus on Lorentzian surfaces with nonzero constant mean curvature in the four-dimensional Minkowski space $\mathbb{E}^4_1$.

4.1. Pointwise 1-type Gauss map of the first kind. By using (2.11), one can obtain the following theorem and its corollary, which are similar to the characterization of surfaces in the Euclidean space $\mathbb{E}^4$ in terms of their Gauss map [6, 9].

THEOREM 4.1. *Let M be a nonminimal Lorentzian surface in $\mathbb{E}^4_1$. Then M has pointwise 1-type Gauss map if and only if its mean curvature vector is parallel. In that case, (1.2) is satisfied for $f = 2K + 4\langle H, H\rangle$ and $C = 0$.*

Note that if H is parallel, then $\langle H, H\rangle$ is constant. Therefore, we have the following result.

COROLLARY 4.2. *Let M be a nonminimal Lorentzian surface in $\mathbb{E}^4_1$. Then M has (global) 1-type Gauss map if and only if it has parallel mean curvature vector and constant Gaussian curvature.*

REMARK 4.3. See [11] for a complete classification of Lorentzian surfaces with parallel mean curvature vector field.

4.2. Pointwise 1-type Gauss map of the second kind. In this subsection we obtain a classification of Lorentzian surfaces with constant mean curvature in terms of the type of their Gauss map.

REMARK 4.4. In the previous subsection we obtained the classification of Lorentzian surfaces with parallel mean curvature vector in terms of the type of their Gauss map. Therefore, throughout this subsection we assume that DH does not vanish on any point of M.

LEMMA 4.5. *Let M be a nonminimal Lorentzian surface in $\mathbb{E}^4_1$ with flat normal bundle and constant mean curvature. If M has pointwise 1-type Gauss map of the second kind, then its shape operators can be diagonalized simultaneously. Moreover, there exists an orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ such that*

$$\widetilde{\nabla}_{e_1} e_1 = \widetilde{\nabla}_{e_1} e_2 = 0, \quad \widetilde{\nabla}_{e_1} e_3 = \widetilde{\nabla}_{e_1} e_4 = 0, \tag{4.1a}$$

$$\widetilde{\nabla}_{e_2} e_1 = 0, \quad \widetilde{\nabla}_{e_2} e_2 = \kappa e_3, \tag{4.1b}$$

$$\widetilde{\nabla}_{e_2} e_3 = \varepsilon\kappa e_2 + \tau e_4, \quad \widetilde{\nabla}_{e_2} e_4 = -\tau e_3 \tag{4.1c}$$

for some constants κ, τ, where $\varepsilon = \langle e_1, e_1\rangle$.

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PROOF. Let $\{f_1, f_2; e_3, e_4\}$ be a local orthonormal frame field such that $H = -h(f_1, f_2) = ce_3$ for a constant $c \neq 0$. Then

$$A_3 f_1 = cf_1 - h_1^3 f_2, \quad A_3 f_2 = -h_2^3 f_1 + cf_2, \quad A_4 f_1 = -h_1^4 f_2, \quad A_4 f_2 = -h_2^4 f_1, \tag{4.2}$$

where $h_i^\beta = \langle h(f_i, f_i), e_\beta \rangle$. In addition, since $K^D = 0$, the Ricci equation (2.4c) implies that

$$h_1^3 h_2^4 - h_2^3 h_1^4 = 0. \tag{4.3}$$

On the other hand, from (2.11),

$$\Delta \nu = (2K + 4c^2)\nu + 2c\omega_{34}(f_1) f_2 \wedge e_4 - 2c\omega_{34}(f_2) f_1 \wedge e_4, \tag{4.4}$$

where ω_{34} is the connection form defined by $\omega_{34}(X) = \langle D_X e_3, e_4 \rangle$.

Now suppose that M has pointwise 1-type Gauss map of the second kind, that is, (1.2) is satisfied for a smooth function f and a nonzero constant vector C. From (1.2) and (4.4),

$$\langle C, f_1 \wedge e_3 \rangle = \langle C, f_2 \wedge e_3 \rangle = \langle C, \mu \rangle = 0 \tag{4.5}$$

and

$$f(1 - C_{12}) = 2K + 4c^2, \tag{4.6a}$$

$$fC_{14} = -2c\omega_{34}(f_1), \tag{4.6b}$$

$$fC_{24} = 2c\omega_{34}(f_2), \tag{4.6c}$$

where we put $C_{14} = \langle C, f_1 \wedge e_4 \rangle$, $C_{24} = \langle C, f_2 \wedge e_4 \rangle$ and $C_{12} = \langle C, \nu \rangle$.

By applying f_1 and f_2 to each equation in (4.5) and using (4.5) again,

$$-cC_{14} + h_1^3 C_{24} = 0, \quad h_2^3 C_{14} - cC_{24} = 0, \tag{4.7a}$$

$$h_1^3 C_{12} + \omega_{34}(f_1) C_{14} = 0, \quad -cC_{12} + \omega_{34}(f_2) C_{14} = 0, \tag{4.7b}$$

$$cC_{12} + \omega_{34}(f_1) C_{24} = 0, \quad -h_2^3 C_{12} + \omega_{34}(f_2) C_{24} = 0. \tag{4.7c}$$

Since $c \neq 0$, if $C_{14}C_{24} = 0$ at a point $p \in M$, then (4.7a) implies that $C_{14} = C_{24} = 0$ at p. In this case, from (4.6), we have $DH|_p = 0$. However, this is a contradiction. Therefore, we see that $C_{14}C_{24}$ does not vanish on M. Thus, (4.7) implies that

$$h_1^3 h_2^3 - c^2 = 0, \tag{4.8a}$$

$$h_2^3 \omega_{34}(f_1) + c\omega_{34}(f_2) = 0. \tag{4.8b}$$

Note that the Gauss equations (2.4a) and (4.8a) imply that $K = \det A_4$.

Now we consider the pair of two orthogonal tangent vector fields of M given by $X_1 = h_2^3 f_1 + cf_2$ and $X_2 = h_2^3 f_1 - cf_2$. By a direct calculation using (4.8),

$$h(X_1, X_2) = 0 \quad \text{and} \quad \omega_{34}(X_1) = \langle h(X_1, X_1), e_3 \rangle = 0.$$

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Thus, the matrix representation of the shape operators $A_3 = A_{e_3}, A_4 = A_{e_4}$ with respect to the orthonormal base field $\{e_1, e_2\}$ of the tangent bundle of M becomes

$$A_3 = \mathrm{diag}(0, 2c), \quad A_4 = \mathrm{diag}(\varepsilon\zeta, -\varepsilon\zeta) \tag{4.9}$$

for a smooth function ζ, where $e_i = X_i/|\langle X_i, X_i\rangle|^{1/2}, i = 1, 2$, and $\varepsilon = \langle e_1, e_1\rangle$, that is, the shape operators can be diagonalized simultaneously. Moreover, the connection form ω_{34} satisfies $\omega_{34}(e_1) = 0$. Therefore, (4.4) becomes

$$\Delta\nu = (2\zeta^2 + 4c^2)\nu - 2\varepsilon c\omega_{34}(e_2)e_1 \wedge e_4. \tag{4.10}$$

In addition, by using (4.7b) and (4.7c), we obtain $2cC_{12} = \omega_{34}(e_2)\langle C, e_1 \wedge e_4\rangle$. By combining this equation and (1.2),

$$\phi C = \omega_{34}(e_2)\nu - 2\varepsilon ce_1 \wedge e_4, \tag{4.11a}$$
$$f = \omega_{34}(e_2)^2 + 2\zeta^2 + 4c^2, \tag{4.11b}$$

where ϕ is the smooth function given by

$$\phi = \frac{f}{\omega_{34}(e_2)}. \tag{4.11c}$$

On the other hand, from (4.10), we have $\langle C, e_2 \wedge e_4\rangle = 0$. By applying e_2 to this equation, we get $\omega_{12}(e_2) = 0$, where ω_{12} is the connection form defined by $\omega_{12}(X) = \langle\nabla_X e_3, e_4\rangle$.

By combining this equation with (4.9),

$$\widetilde{\nabla}_{e_1}e_1 = -\varepsilon\omega_{12}(e_1)e_2 + \zeta e_4, \quad \widetilde{\nabla}_{e_1}e_2 = -\varepsilon\omega_{12}(e_1)e_1, \tag{4.12a}$$
$$\widetilde{\nabla}_{e_2}e_1 = 0, \quad \widetilde{\nabla}_{e_2}e_2 = -2\varepsilon ce_3 + \zeta e_4, \tag{4.12b}$$
$$\widetilde{\nabla}_{e_1}e_3 = 0, \quad \widetilde{\nabla}_{e_2}e_3 = -2ce_2 + \omega_{34}(e_2)e_4, \tag{4.12c}$$
$$\widetilde{\nabla}_{e_1}e_4 = -\varepsilon\zeta e_1, \quad \widetilde{\nabla}_{e_2}e_4 = \varepsilon\zeta e_2 - \omega_{34}(e_2)e_3. \tag{4.12d}$$

Moreover, from the Gauss equation (2.4a), for $X = W = e_1, Y = Z = e_2$, and the Codazzi equation (2.46b), for $X = e_1, Y = Z = e_2$,

$$e_2(\omega_{12}(e_1)) = \varepsilon\omega_{12}(e_1)^2 + \zeta^2, \tag{4.13a}$$
$$e_2(\zeta) = 2\varepsilon\zeta\omega_{12}(e_1), \tag{4.13b}$$
$$\zeta\omega_{34}(e_2) = 2c\omega_{12}(e_1). \tag{4.13c}$$

Now we want to show that $\zeta \equiv 0$ on M. Consider the open subset $\mathcal{M} = \{p|\zeta(p) \neq 0\}$ and assume that it is not empty. By applying e_2 to (4.13c) and using (4.13),

$$e_2(\omega_{34}(e_2)) = 2c\zeta - \varepsilon\omega_{12}(e_1)\omega_{34}(e_2) \tag{4.13d}$$

on $\mathcal{M}$.

By applying e_2 to (4.11a) and using (4.12b), (4.12c), (4.13c) and (4.13d), we obtain $e_2(\phi)C = -\varepsilon\omega_{12}(e_1)C$, which implies that

$$e_2(\phi) = -\varepsilon\omega_{12}$$

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on $\mathcal{M}$. Next we compute the left-hand side of this equation by using (4.13) to get

$$(2\varepsilon + 1)\omega_{34}(e_2)^2 + 12\varepsilon\omega_{12}(e_1)^2 = 2\zeta^2 + 4c^2 \tag{4.14}$$

on $\mathcal{M}$, which implies that $\varepsilon = 1$. Next we apply e_2 to (4.14) and use (4.13) to obtain

$$\omega_{12}(e_2)(-3\omega_{34}(e_2)^2 + 12\omega_{12}(e_1)^2 + 8\zeta^2 + 12c^2) = 0 \tag{4.15}$$

on $\mathcal{M}$. However, (4.13c), (4.14) and (4.15) imply that $\zeta = 0$ on $\mathcal{M}$, which is a contradiction. Therefore, we have proved that $\zeta \equiv 0$ on M. In addition, (4.13c) implies that $\omega_{12} = 0$ and, from the Ricci equation (2.4c) and (4.13d), we get $\omega_{34}(e_2) = \tau$ for a constant τ. By combining all results of this subsection, we see that the frame field $\{e_1, e_2; e_3, e_4\}$ satisfies (4.1) for the constant $\kappa = -2\varepsilon c$. □

Next, by considering the surfaces satisfying the conditions obtained in Lemma 4.5, we get the following classification theorem.

Theorem 4.6 (The classification theorem). *Let M be a nonminimal Lorentzian surface in $\mathbb{E}^4_1$ with normal flat bundle and constant mean curvature. Then M has pointwise 1-type Gauss map of the second kind if and only if it is congruent to one of the following surfaces:*

(i) *a surface given by* $x(s,t) = (s, (a/\lambda)\cos\lambda t, (a/\lambda)\sin\lambda t, \sqrt{1-a^2}t), 0 < a < 1$;
(ii) *a surface given by* $x(s,t) = ((a^2/3)t^3 + t, \sqrt{2}at, (a^2/3)t^3, s)$;
(iii) *a surface given by* $x(s,t) = ((\sqrt{a^2-1}/\lambda)\cosh\lambda t, (\sqrt{a^2-1}/\lambda)\sinh\lambda t, at, s)$, $a > 1$;
(iv) *a surface given by* $x(s,t) = ((\sqrt{a^2+1}/\lambda)\sinh\lambda t, (\sqrt{a^2+1}/\lambda)\cosh\lambda t, at, s)$;
(v) *a surface given by* $x(s,t) = (\sqrt{1+a^2}t, (a/\lambda)\cos\lambda t, (a/\lambda)\sin\lambda t, s)$

for a nonzero constant λ.

Proof. Let $\{e_1, e_2; e_3, e_4\}$ be the orthonormal frame field given in Lemma 4.5 and $\varepsilon = \langle e_1, e_1\rangle$. Since $\nabla_{e_i}e_j = 0, i, j = 1, 2$, there exist local coordinates (s, t) on M such that the induced metric is $g = \varepsilon(ds^2 - dt^2)$ and $e_1 = \partial_s, e_2 = \partial_t$. Moreover, the first equation in (4.1b) gives $x_{st} = 0$, where $x : M \to \mathbb{E}^4$ is an isometric immersion. Therefore,

$$x(s,t) = \alpha(s) + \beta(t) \tag{4.16}$$

for some curves α and β satisfying

$$\langle\alpha'(s), \alpha'(s)\rangle = \varepsilon, \quad \langle\alpha'(s), \beta'(t)\rangle = 0, \quad \langle\beta'(t), \beta'(t)\rangle = -\varepsilon. \tag{4.17}$$

Obviously, all integral curves of the vector fields e_1 and e_2 are congruent to the curves $\alpha(s)$ and $\beta(t)$, respectively.

On the other hand, since $\widetilde{\nabla}_{e_i}e_1 = 0, i = 1, 2$, e_1 is a constant vector. Thus, up to isometries of $\mathbb{E}^4_1$, we may assume that either $e_1 = (1,0,0,0)$ or $e_1 = (0,0,0,1)$ subject to $\varepsilon = -1$ or $\varepsilon = 1$, respectively.

Case 1: $\varepsilon = -1$. In this case, we have $e_1 = \partial_s = (1,0,0,0)$. Thus, up to translations, we may assume that $\alpha(s) = (s,0,0,0)$. Because of (4.17), $\beta(t)$ is a curve lying in the

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Euclidean hyperplane Π whose normal is e_1. In addition, from (4.1b) and (4.1c), we see that β has constant curvature κ and constant torsion τ, that is, it is the position vector of a right circular helix. Hence, we have the case (i) of the theorem.

Case 2: $\varepsilon = 1$. In this case, we have $e_1 = \partial_s = (0, 0, 0, 1)$. Thus, up to translations, we may assume that $\alpha(s) = (0, 0, 0, s)$. Because of (4.17), $\beta(t)$ is a Lorentzian curve lying in the Lorentzian hyperplane Π_2 whose normal is e_1. In addition, from (4.1b) and (4.1c), we see that β is the position vector of a helix with constant curvature κ and constant torsion τ. Thus, $e_2 = \beta'(t)$ satisfies $e_2''' = (\kappa^2 - \tau^2)e_2'$.

Case 2a: $\kappa = \tau$. In this case, up to congruency, we can assume that

$$\beta(t) = \Big(\frac{a^2}{3}t^3 + t,\ \sqrt{2}at, \frac{a^2}{3}t^3, 0\Big).$$

Hence, we get the case (ii) of the theorem.

Case 2b: $\kappa > \tau$. In this case,

$$\beta(t) = \Big(\frac{\sqrt{a^2-1}}{\lambda}\cosh\lambda t, \frac{\sqrt{a^2-1}}{\lambda}\sinh\lambda t, at, 0\Big), \quad a > 1$$

or

$$\beta(t) = \Big(\frac{\sqrt{a^2+1}}{\lambda}\sinh\lambda t, \frac{\sqrt{a^2+1}}{\lambda}\cosh\lambda t, at, 0\Big)$$

for a nonzero constant λ. Hence, we have the case (iii) or (iv) of the theorem, respectively.

Case 2c: $\kappa < \tau$. Similarly,

$$\beta(t) = \Big(\sqrt{1+a^2}t, \frac{a}{\lambda}\cos\lambda t, \frac{a}{\lambda}\sin\lambda t, 0\Big),$$

which gives the case (v) of the theorem.

The converse follows from a direct computation. □

Since a surface with positive relative nullity satisfies the conditions of Theorem 4.6, we have the following result.

Corollary 4.7. *Let M be a nonminimal Lorentzian surface in $\mathbb{E}^4_1$ with positive relative nullity. Then M has pointwise 1-type Gauss map of the second kind if and only if it is congruent to one of the surfaces given in Theorem 4.6.*

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