On Symmetric Determinants and Pfaffians.

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The name Pfaffian was given by Cayley to the rational function which is the square root of a zero-axial skew determinant of even order; thus the square root of the determinant

$$\begin{vmatrix} 0 & q_1 & q_2 & q_3 \\ -q_1 & 0 & q_4 & q_5 \\ -q_2 & -q_4 & 0 & q_6 \\ -q_3 & -q_5 & -q_6 & 0 \end{vmatrix}$$

is the Pfaffian $(q_1 q_6 - q_2 q_5 + q_3 q_4)$. A convenient notation for Pfaffians was devised by Sir Thomas Muir: thus the Pfaffian just mentioned is denoted by

$$\left| \begin{array}{ccc} q_1 & q_2 & q_3 \\ & q_4 & q_5 \\ & & q_6 \end{array} \right|$$

In the present paper we shall first consider zero-axial skew Pfaffians, such as

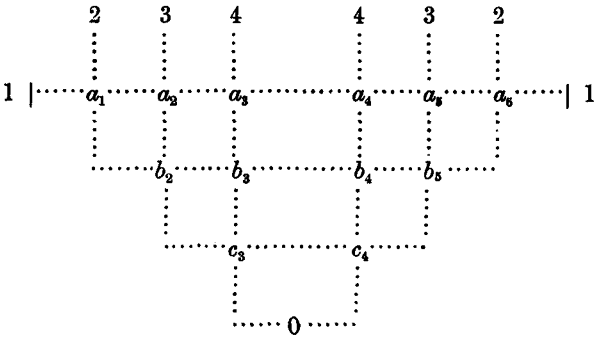
$$\left| \begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 \\ & b_2 & b_3 & b_4 & b_5 & 0 & -a_6 \\ & & c_3 & c_4 & 0 & -b_5 & -a_5 \\ & & & 0 & -c_4 & -b_4 & -a_4 \\ & & & & -c_3 & -b_3 & -a_3 \\ & & & & & -b_2 & -a_2 \\ & & & & & & -a_1 \end{array} \right|$$

which (by analogy with Muir's notation) we shall denote by

$$\left| \begin{array}{ccccccc} a_1 & a_2 & a_3 & & a_4 & a_5 & a_6 \\ & b_2 & b_3 & & b_4 & b_5 & \\ & & c_3 & & c_4 & & \\ & & & 0 & & & \\ & & & \overline{} & & & \end{array} \right|$$

An expression of this kind we shall call a Half-Pfaffian. A half-pfaffian will be denoted by P_n, where n is the number of lines.*

It will appear that the structure of a half-pfaffian depends on what may be called its double-frame-lines †: the double-frame-lines of the above half-pfaffian are marked in the following diagram :—



With these definitions, we may establish the following theorems :

1. *If the elements of a D.F. line are reversed in order, the H.P. is altered only in sign, and remains unaltered in magnitude.*

For if in a doubly zero-axial skew determinant of order $4m$, the r^{th} row is interchanged with the $(4m-r)^{\text{th}}$ row and the r^{th} column with the $(4m-r)^{\text{th}}$ column, the determinant still remains doubly zero-axial. Or if the r^{th} and the $(4m-r)^{\text{th}}$ frame-lines are interchanged, the zero-axial skew Pfaffian still remains zero-axial skew. By interchange of frame-lines I mean not only the interchange of the elements, but also the alteration in signs where it is necessary.‡

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* The only previous mention of these zero-axial skew Pfaffians which I have been able to discover in mathematical literature is in a paper by Sir Thomas Muir, *Quart. Journ. Math.*, **18** (1882), p. 175, where he remarks that they may be expressed as products of Pfaffians.

† So called because they are analogous to the frame lines of a Pfaffian.

‡ *Cf.* T. Muir, *Tran. Royal Soc. of S. Africa*, **14**. Part. 3 (1915), p. 210. Theorem 6.

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Thus $P_4 = \left| \begin{array}{ccccccc} a_1 & a_2 & a_3 & & a_4 & a_5 & a_6 \\ & b_2 & b_3 & & b_4 & b_5 & \\ & & c_3 & & c_4 & & \\ & & & \underline{0} & & & \end{array} \right.$

may be written in the form

$$- \left| \begin{array}{ccccccc} a_6 & a_5 & a_4 & & a_3 & a_2 & a_1 \\ & b_2 & b_3 & & b_4 & b_5 & \\ & & c_3 & & c_4 & & \\ & & & \underline{0} & & & \end{array} \right.$$

Now, equating the coefficients of a_6 in the first double frame-lines of these two H.P.'s in the above identity, we obtain the following theorem.

2. *If we form a Pfaffian X_{n+2} by bordering a zero-axial skew Pfaffian P_n* (*i.e.*, by adding one row at the beginning and one column at the end of P_n), *and if we form another Pfaffian Y_{n+2}, this time by adding two columns at the end of P_n, and if the first n elements of the first row of X_{n+2}, altered in sign and reversed in order, form the last but one column of Y_{n+2}, and the elements of the last column of X_{n+2}, reversed in order, form the last column of Y_{n+2}, then $X_{n+2} = Y_{n+2}$.*

The following two properties of a H.P. are similar to known properties of a Pfaffian.*

3. *The value of a H.P. remains unaltered if the elements in the parts of the r^{th} D.F. line which are parallel to parts of the s^{th} D.F. line be interchanged with the corresponding elements of the latter, the two elements common to the two lines be changed in sign, and the remaining elements of the r^{th} line be altered in sign and interchanged with the remaining elements, altered in sign, of the s^{th} line, in order.*

By virtue of Theorem 1, the order of the elements of a D.F. line can begin from either of its two ends. For example,

$$P_4 = \left| \begin{array}{ccccccc} -b_2 & -a_2 & c_3 & & c_4 & -a_5 & -b_5 \\ & -a_1 & b_3 & & b_4 & -a_6 & \\ & & a_3 & & a_4 & & \\ & & & \underline{0} & & & \end{array} \right.$$

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* *Cf.* The same paper, Theorems 6 and 7.

or

$$= \left| \begin{matrix} -b_5 & -a_2 & c_4 & & c_3 & -a_5 & -b_2 \\ & -a_6 & b_3 & & b_4 & -a_1 & \\ & & a_4 & & a_3 & & \\ & & & 0 & & & \end{matrix} \right|$$

In the first case, the order of the elements in the first D.F. line begins with a_1, and in the second case the order begins with a_6, and in both cases, the order of the elements in the third D.F. begins with a_2.

4. *The value of a H.P. remains unaltered if the elements in the parts of the r^{th} D.F. line, which are parallel to parts of the s^{th} D.F. line be increased by m times the corresponding elements of the latter, the elements common to the two lines be left unchanged, and the remaining elements of the r^{th} line be diminished by m times the remaining elements of the s^{th} line, in order.*

5. *If the elements in the rows of a H.P. are in arithmetical progression, the H.P. reduces to a product of linear factors.*

Ex. $$P_4 = \left| \begin{matrix} a_1 & a_1+d_1 & a_1+2d_1 & a_1+3d_1 & a_1+4d_1 & a_1+5d_1 \\ & a_2 & a_2+d_2 & a_2+2d_2 & a_2+3d_2 & \\ & & a_3 & a_3+d_3 & & \\ & & & 0 & & \end{matrix} \right|$$

$$= \left| \begin{matrix} 5d_1 & 3d_1 & d_1 \\ & 3d_2 & d_2 \\ & & d_3 \end{matrix} \right| \times \left| \begin{matrix} 2a_1+5d_1 & 2a_1+5d_1 & 2a_1+5d_1 \\ & 2a_2+3d_2 & 2a_2+3d_2 \\ & & 2a_3+d_3 \end{matrix} \right|$$

$$= (5d_1 \times d_3)\,(2a_1+5d_1)\,(2a_3+d_3).$$

Similarly,

$$P_{2m} = d_1 d_3 \ldots d_{2m-1}\,(4m-3)\,(4m-7)\,(4m-11)\,\ldots\,5\times 1$$
$$\times \{2a_1+(4m-3)\,d_1\}\,\{2a_3+(4m-7)\,d_3\}\,\ldots\,(2a_{2m-1}+d_{2m-1}).$$

6. *A determinant which is symmetrical with respect to one principal diagonal, and persymmetric with respect to the other principal diagonal, can be expressed as a product of two Pfaffians.*

With the help of the operations

(1) $\text{col}_n - \text{col}_{n-1}$; $\text{col}_{n-1} - \text{col}_{n-2}$; ... ; $\text{col}_2 - \text{col}_1$

and

(2) $\text{row}_n + \text{row}_{n-1}$; $\text{row}_{n-1} + \text{row}_{n-2}$; ... ; $\text{row}_2 + \text{row}_1$

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we can transform the determinant into a bordered zero-axial skew determinant. The bordered determinant is persymmetric.

Ex.

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_1 & a_2 & a_3 & a_4 \\ a_3 & a_2 & a_1 & a_2 & a_3 \\ a_4 & a_3 & a_2 & a_1 & a_2 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 + a_1 & a_3 + a_2 & a_4 + a_3 & a_5 + a_4 \\ a_2 - a_1 & 0 & a_1 - a_3 & a_2 - a_4 & a_3 - a_5 \\ a_3 - a_2 & a_3 - a_1 & 0 & a_1 - a_3 & a_2 - a_4 \\ a_4 - a_3 & a_4 - a_2 & a_3 - a_1 & 0 & a_1 - a_3 \\ a_5 - a_4 & a_5 - a_3 & a_4 - a_2 & a_3 - a_1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 - a_3 & a_2 - a_4 & a_3 - a_5 \\ & a_1 - a_3 & a_2 - a_4 \\ & & a_1 - a_3 \end{vmatrix} \times \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 & 2a_4 & -a_1 \\ & a_1 - a_3 & a_2 - a_4 & a_3 - a_5 & -a_2 \\ & & a_1 - a_3 & a_2 - a_4 & -a_3 \\ & & & a_1 - a_3 & -a_4 \\ & & & & -a_5 \end{vmatrix}$$

The determinant being centro-symmetric may also be expressed in the form

$$\begin{vmatrix} a_1 - a_5 & a_2 - a_4 \\ a_2 - a_4 & a_1 - a_3 \end{vmatrix} \times \begin{vmatrix} a_1 & 2a_2 & 2a_3 \\ a_2 & a_1 + a_3 & a_2 + a_4 \\ a_3 & a_2 + a_4 & a_1 + a_5 \end{vmatrix}$$

Similarly the determinant of the 6^{th} order,

$$= \begin{vmatrix} a_2 - a_1 & a_3 - a_2 & a_4 - a_3 & a_5 - a_4 & a_6 - a_5 \\ & a_3 - a_1 & a_4 - a_2 & a_5 - a_3 & a_6 - a_4 \\ & & a_3 - a_1 & a_4 - a_2 & a_5 - a_3 \\ & & & a_3 - a_1 & a_4 - a_2 \\ & & & & a_3 - a_1 \end{vmatrix}$$

$$\times \begin{vmatrix} a_1 + a_2 & a_2 + a_3 & a_3 + a_4 & a_4 + a_5 & a_5 + a_6 \\ & a_3 - a_1 & a_4 - a_2 & a_5 - a_3 & a_6 - a_4 \\ & & a_3 - a_1 & a_4 - a_2 & a_5 - a_3 \\ & & & a_3 - a_1 & a_4 - a_2 \\ & & & & a_3 - a_1 \end{vmatrix}$$

and also

$$= \begin{vmatrix} a_1 - a_6 & a_2 - a_5 & a_3 - a_4 \\ a_2 - a_5 & a_1 - a_4 & a_2 - a_3 \\ a_3 - a_4 & a_2 - a_3 & a_1 - a_2 \end{vmatrix} \times \begin{vmatrix} a_1 + a_2 & a_2 + a_3 & a_3 + a_4 \\ a_2 + a_3 & a_1 + a_4 & a_2 + a_5 \\ a_3 + a_4 & a_2 + a_5 & a_1 + a_6 \end{vmatrix}$$

The two Pfaffians, altered in sign, are respectively equal to these two axisymmetric determinants. We can write the two determinants respectively in the forms

$$(a_2 - a_1)\, y_1 + (a_3 - a_2)\, y_2 + (a_4 - a_3)\, y_3 + (a_5 - a_4)\, y_4 + (a_6 - a_5)\, y_5$$

and $(a_2 + a_1)\, y_1 + (a_3 + a_2)\, y_2 + (a_4 + a_3)\, y_3 + (a_5 + a_4)\, y_4 + (a_6 + a_5)\, y_5$.

7. Another kind of determinant, the arrangement of whose elements will be evident from the example, *can be expressed as a product of two Pfaffians.* A case of this determinant occurs in a paper by Sir Thomas Muir.*

Let us take a determinant of the 6th order, viz.,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ -a_1 & a_1 & a_2 & a_3 & a_4 & a_5 \\ -a_2 & -a_1 & a_1 & a_2 & a_3 & a_4 \\ -a_3 & -a_2 & -a_1 & a_1 & a_2 & a_3 \\ -a_4 & -a_3 & -a_2 & -a_1 & a_1 & a_2 \\ -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & a_1 \end{vmatrix}$$

Performing on this determinant the operations

$$\text{col}_6 + \text{col}_5\,;\ \text{col}_5 + \text{col}_4\,;\ \text{col}_4 + \text{col}_3\,;\ \text{col}_3 + \text{col}_2\,;\ \text{col}_2 + \text{col}_1\,;$$

we obtain the determinant

$$\begin{vmatrix} a_1 & a_1+a_2 & a_2+a_3 & a_3+a_4 & a_4+a_5 & a_5+a_6 \\ -a_1 & 0 & a_1+a_2 & a_2+a_3 & a_3+a_4 & a_4+a_5 \\ -a_2 & -(a_1+a_2) & 0 & a_1+a_2 & a_2+a_3 & a_3+a_4 \\ -a_3 & -(a_2+a_3) & -(a_1+a_2) & 0 & a_1+a_2 & a_2+a_3 \\ -a_4 & -(a_3+a_4) & -(a_2+a_3) & -(a_1+a_2) & 0 & a_1+a_2 \\ -a_5 & -(a_4+a_5) & -(a_3+a_4) & -(a_2+a_3) & -(a_1+a_2) & 0 \end{vmatrix}$$

which

$$= \begin{vmatrix} a_1+a_2 & a_2+a_3 & a_3+a_4 & a_4+a_5 & a_5+a_6 \\ & a_1+a_2 & a_2+a_3 & a_3+a_4 & a_4+a_5 \\ & & a_1+a_2 & a_2+a_3 & a_3+a_4 \\ & & & a_1+a_2 & a_2+a_3 \\ & & & & a_1+a_2 \end{vmatrix}$$

$$\times \begin{vmatrix} -a_1 & -a_2 & -a_3 & -a_4 & -a_5 \\ & a_1+a_2 & a_2+a_3 & a_3+a_4 & a_4+a_5 \\ & & a_1+a_2 & a_2+a_3 & a_3+a_4 \\ & & & a_1+a_2 & a_2+a_3 \\ & & & & a_1+a_2 \end{vmatrix}$$

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* On circulants of odd order. *Quart. Jour. of Math.*, **18** (1882), p. 262.

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If we first perform on the same determinant the operations

$$\text{row}_6 + \text{row}_2\,;\;\; \text{row}_5 + \text{row}_3$$

and then on the resulting determinant the operations

$$\text{col}_1 + \text{col}_6\,;\;\; \text{col}_2 + \text{col}_5\,;\;\; \text{col}_3 + \text{col}_4$$

we see that the determinant

$$= \begin{vmatrix} a_1+a_6 & a_2+a_5 & a_3+a_4 \\ a_5-a_1 & a_1+a_4 & a_2+a_3 \\ a_4-a_2 & a_3-a_1 & a_1+a_2 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2-a_1 & a_1+a_3 & a_2+a_4 \\ a_3-a_2 & a_4-a_1 & a_1+a_5 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+a_2+a_3+a_4+a_5+a_6 & a_2+a_3+a_4+a_5 & a_3+a_4 \\ a_2+a_3+a_4+a_5 & a_1+a_2+a_3+a_4 & a_2+a_3 \\ a_3+a_4 & a_2+a_3 & a_1+a_2 \end{vmatrix}$$

$$\times \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1+a_2+a_3 & a_2+a_3+a_4 \\ a_3 & a_2+a_3+a_4 & a_1+a_2+a_3+a_4+a_5 \end{vmatrix}$$

These two determinants can be written respectively in the forms

$$(a_1+a_2)f_1 + (a_2+a_3)f_2 + (a_3+a_4)f_3 + (a_4+a_5)f_4 + (a_5+a_6)f_6$$

and
$$-a_1f_1 - a_2f_2 - a_3f_3 - a_4f_4 - a_5f_5\,.$$

We obtain similar results where the order of the determinant is odd. Some of the elements in the determinant are positive and the others are negative, but if all the elements are of the same sign still we get results of a similar nature. For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_1 & a_2 & a_3 \\ a_3 & a_2 & a_1 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 & a_1 \end{vmatrix} = \begin{vmatrix} a_3-a_2 & a_2-a_1 \\ a_4-a_1 & a_3-a_1 \end{vmatrix} \times \begin{vmatrix} a_3 & a_4+a_2 & a_5+a_1 \\ a_2 & a_3+a_1 & a_4+a_1 \\ a_1 & a_2+a_1 & a_3+a_2 \end{vmatrix}$$

and also

$$= \begin{vmatrix} a_2-a_1 & a_3-a_2 & a_4-a_3 \\ & a_2-a_1 & a_3-a_2 \\ & & a_2-a_1 \end{vmatrix} \times \begin{vmatrix} a_2-a_1 & a_3-a_2 & a_4-a_3 & a_5-a_4 & -a_1 \\ & a_2-a_1 & a_3-a_2 & a_4-a_3 & -a_1 \\ & & a_2-a_1 & a_3-a_2 & -a_2 \\ & & & a_2-a_1 & -a_3 \\ & & & & -a_4 \end{vmatrix}$$

In the case of Muir's determinant we see that

$$\left|\begin{array}{cccccc} \alpha & \beta & \gamma & 0 & -\gamma & -\beta \\ -\alpha & \alpha & \beta & \gamma & 0 & -\gamma \\ -\beta & -\alpha & \alpha & \beta & \gamma & 0 \\ -\gamma & -\beta & -\alpha & \alpha & \beta & \gamma \\ 0 & -\gamma & -\beta & -\alpha & \alpha & \beta \\ \gamma & 0 & -\gamma & -\beta & -\alpha & \alpha \end{array}\right|$$

$$= \left|\begin{array}{ccccc} \alpha+\beta & \beta+\gamma & \gamma & -\gamma & -\beta-\gamma \\ & \alpha+\beta & \beta+\gamma & \gamma & -\gamma \\ & & \alpha+\beta & \beta+\gamma & \gamma \\ & & & \alpha+\beta & \beta+\gamma \\ & & & & \alpha+\beta \end{array}\right|$$

$$\times \left|\begin{array}{ccccc} \alpha & \beta & \gamma & 0 & -\gamma \\ & \alpha+\beta & \beta+\gamma & \gamma & -\gamma \\ & & \alpha+\beta & \beta+\gamma & \gamma \\ & & & \alpha+\beta & \beta+\gamma \\ & & & & \alpha+\beta \end{array}\right|$$

$$= \left|\begin{array}{ccccc} \alpha & \beta & \gamma & 0 & -\gamma \\ & \alpha+\beta & \beta+\gamma & \gamma & -\gamma \\ & & \alpha+\beta & \beta+\gamma & \gamma \\ & & & \alpha+\beta & \beta+\gamma \\ & & & & \alpha+\beta \end{array}\right|^2$$

8. Let

$$D_n = \left|\begin{array}{cccccccc} x_1 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots & a_{n-1} \\ -a_1 & -x_1 & a_2 & a_3 & a_4 & a_5 & \dots & a_{n-1} \\ \alpha_1 & \alpha_1 & x_2 & b_3 & b_4 & b_5 & \dots & b_{n-1} \\ \beta_1 & \beta_1 & -b_3 & -x_2 & b_4 & b_5 & \dots & b_{n-1} \\ \gamma_1 & \gamma_1 & \gamma_2 & \gamma_2 & x_3 & c_5 & \dots & c_{n-1} \\ \delta_1 & \delta_1 & \delta_2 & \delta_2 & -c_5 & -x_3 & \dots & c_{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}\right|$$

By the operations $\text{row}_1 - \text{row}_2$; $\text{col}_2 - \text{col}_1$, we obtain $(a_1^2 - x_1^2)$ as a factor, and the determinant reduces to a similar determinant of order $n-2$. In this way we can reduce the whole determinant to a product of linear factors; in fact,

$$D_{2m} = (a_1^2 - x_1^2)(b_3^2 - x_2^2)(c_5^2 - x_3^2) \dots \text{ to } m \text{ factors.}$$

$$D_{2m+1} = \{(a_1^2 - x_1^2)(b_3^2 - x_2^2)(c_5^2 - x_3^2) \dots \text{ to } m \text{ factors}\} \times x_{m+1}.$$

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