Canad. Math. Bull. Vol. **53** (1), 2010 pp. 23–36
doi:10.4153/CMB-2010-007-5

Boundedness From Below of Multiplication Operators Between α-Bloch Spaces

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Abstract. In this paper, the boundedness from below of multiplication operators between α-Bloch spaces $\mathcal{B}^\alpha$, $\alpha > 0$, on the unit disk D is studied completely. For a bounded multiplication operator $M_u \colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$, defined by $M_u f = uf$ for $f \in \mathcal{B}^\alpha$, we prove the following result:

(i) If $0 < \beta < \alpha$, or $0 < \alpha \le 1$ and $\alpha < \beta$, M_u is not bounded below;
(ii) if $0 < \alpha = \beta \le 1$, M_u is bounded below if and only if $\liminf_{z \to \partial D} |u(z)| > 0$;
(iii) if $1 < \alpha \le \beta$, M_u is bounded below if and only if there exist a $\delta > 0$ and a positive $r < 1$ such that for every point $z \in D$ there is a point $z' \in D$ with the property $d(z', z) < r$ and $(1 - |z'|^2)^{\beta - \alpha} |u(z')| \ge \delta$, where $d(\,\cdot\,,\,\cdot\,)$ denotes the pseudo-distance on D.

1 Introduction

Let D be the unit disk in the complex plane $\mathbb{C}$ and let $H(D)$ be the class of holomorphic functions on D. For $\alpha > 0$, a function $f \in H(D)$ is called an α-*Bloch function* if the semi-norm satisfies

$$\|f\|_\alpha := \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty,$$

and called a *little* α-*Bloch function* if $\lim_{z \to \partial D} (1 - |z|^2)^\alpha |f'(z)| = 0$. The class of all α-Bloch functions is called the α-*Bloch space*, denoted by $\mathcal{B}^\alpha$, which is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$, and the class of all little α-Bloch functions is called the *little* α-*Bloch space*, denoted by $\mathcal{B}_0^\alpha$. When $\alpha = 1$, we obtain Bloch functions, the Bloch space, and little Bloch space, and we denote $\mathcal{B} = \mathcal{B}^1$ and $\mathcal{B}_0 = \mathcal{B}_0^1$. For the general theory of Bloch functions and α-Bloch functions, see [2, 7].

For a holomorphic self-mapping ϕ of D and $u \in H(D)$, the weighted composition operator uC_ϕ on $H(D)$ is defined by $uC_\phi f = uf \circ \phi$ for $f \in H(D)$. If $\phi(z) \equiv z$ or $u \equiv 1$, the weighted composition operator becomes the multiplication operator or the composition operator and is denoted by M_u or C_ϕ, respectively. The boundedness and compactness of weighted composition operators have been studied completely. S. Ohno, K. Stroethoff, and R. Zhao [6] proved the following results.

Theorem 1.1 *Let $\beta > 0$. If $\alpha > 1$, then $uC_\phi \colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z \in D} \frac{|u(z)|(1 - |z|^2)^\beta |\phi'(z)|}{(1 - |\phi(z)|^2)^\alpha} < \infty \tag{1.1}$$

Received by the editors November 6, 2006.
Published electronically December 4, 2009.
Research supported by NSFC(China): 10171047, 10671093.
AMS subject classification: **32A18**, 30H05.
Keywords: α-Bloch function; multiplication operator.

and

$$\sup_{z\in D}\frac{(1-|z|^2)^\beta|u'(z)|}{(1-|\phi(z)|^2)^{\alpha-1}}<\infty. \tag{1.2}$$

If $\alpha=1$ *or* $0<\alpha<1$, *then* (1.2) *is replaced by*

$$\sup_{z\in D}(1-|z|^2)^\beta|u'(z)|\Big(1+\log\frac{1}{1-|\phi(z)|^2}\Big)<\infty \tag{1.3}$$

or

$$\sup_{z\in D}(1-|z|^2)^\beta|u'(z)|<\infty, \tag{1.4}$$

respectively.

For a multiplication operator, (1.1), (1.2), (1.3) become

$$\sup_{z\in D}(1-|z|^2)^{\beta-\alpha}|u(z)|<\infty, \tag{1.1$'$}$$

$$\sup_{z\in D}(1-|z|^2)^{\beta-\alpha+1}|u'(z)|<\infty, \tag{1.2$'$}$$

$$\sup_{z\in D}(1-|z|^2)^\beta|u'(z)|\Big(1+\log\frac{1}{1-|z|^2}\Big)<\infty, \tag{1.3$'$}$$

respectively.

For $a\in D$, let ϕ_a denote the Möbius transformation of D onto itself which exchanges 0 and a. We have $\phi_a=\phi_a^{-1}$, *i.e.*, $\phi_a\circ\phi_a$ is the identity mapping, and for $z\in D$,

$$\frac{|\phi_a'(z)|}{1-|\phi_a(z)|^2}=\frac{1}{1-|z|^2}, \tag{1.5}$$

$$\frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2}=1-|\phi_a(z)|^2. \tag{1.6}$$

It follows from (1.5) that for $f\in H(D)$, we have

$$(1-|z|^2)|(f\circ\phi_a)'(z)|=\big(1-|\phi_a(z)|^2\big)\,|f'(\phi_a(z))|\quad\text{for } z\in D. \tag{1.7}$$

Equation (1.7) is used in this paper quite often without mention.

The pseudo-distance on D is defined by

$$d(z_1,z_2)=|\phi_{z_1}(z_2)|=\frac{|z_1-z_2|}{|1-\bar{z}_1z_2|}\quad\text{for } z_1,z_2\in D.$$

It is invariant under Möbius transformations of D onto itself. For a holomorphic self-mapping ϕ, denote

$$\tau_\phi(z)=\frac{(1-|z|^2|)\phi'(z)|}{1-|\phi(z)|^2}\quad\text{for } z\in D,$$

which is the dilation of ϕ with respect to the hyperbolic metric. The classical Schwarz–Pick lemma asserts that $\tau_\phi(z) \leq 1$ for $z \in D$ (see [1]), and it follows from (1.5) that $\tau_{\phi_a}(z) \equiv 1$.

A bounded weighted composition operator $uC_\phi\colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is said to be bounded below from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$, if there exists a $\delta > 0$ such that $\|uC_\phi f\|_{\mathcal{B}^\beta} \geq \delta \|f\|_{\mathcal{B}^\alpha}$ for $f \in \mathcal{B}^\alpha$. For the boundedness from below of composition operators on the Bloch space $\mathcal{B}$, the following result is known, see [3, 5].

Theorem 1.2 *The following conditions are equivalent:*

(i) *C_ϕ is bounded below on $\mathcal{B}$;*
(ii) *C_ϕ is bounded below on the subset $\{\phi_a : a \in D\}$ of $\mathcal{B}$;*
(iii) *there exist a $\delta > 0$ and an $r \in (0,1)$ such that for any $w \in D$ there is a $z' \in D$ with the property that $d(\phi(z'), w) \leq r$ and $\tau_\phi(z') \geq \delta$.*

Recently, the above result was generalized to composition operators on $\mathcal{B}^\alpha$ for $\alpha > 1$ by H. Chen and P. Gauthier [4].

Theorem 1.3 *If $\alpha > 1$, then $C_\phi\colon \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is bounded below if and only if there exist a $\delta > 0$ and an $r \in (0,1)$ such that for any $w \in D$ there is a $z' \in D$ with the property that $d(\phi(z'), w) < r$, $\tau_\phi(z') \geq \delta$ and $(1 - |z'|^2)/(1 - |\phi(z')|^2) \geq \delta$.*

In this paper, the boundedness from below of multiplication operators between α-Bloch spaces is studied completely. We prove the following result. Let $M_u\colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$ be a bounded multiplication operator. If $0 < \beta < \alpha$, or $0 < \alpha \leq 1$ and $\alpha < \beta$, M_u is not bounded below. If $0 < \alpha = \beta \leq 1$, M_u is bounded below if and only if $\liminf_{z \to \partial D} |u(z)| > 0$. If $1 < \alpha \leq \beta$, M_u is bounded below if and only if there exist a $\delta > 0$ and a positive $r < 1$ such that for every point $z \in D$ there is a point $z' \in D$ with the property that $d(z', z) < r$ and $(1 - |z'|^2)^{\beta - \alpha} |u(z')| \geq \delta$.

2 Some Lemmas

Lemma 2.1 *For $z_1, z_2 \in D$, we have*

$$\frac{1 - |z_2|^2}{1 - |z_1|^2} \leq \frac{1 + d(z_1, z_2)}{1 - d(z_1, z_2)}. \tag{2.1}$$

Proof Applying (1.6), we have

$$1 - |z_2|^2 = 1 - |\phi_{z_1}(\phi_{z_1}(z_2))|^2 = \frac{(1 - |\phi_{z_1}(z_2)|^2)(1 - |z_1|^2)}{|1 - \bar{z}_1 \phi_{z_1}(z_2)|^2}.$$

Thus,

$$\frac{1 - |z_2|^2}{1 - |z_1|^2} = \frac{1 - |\phi_{z_1}(z_2)|^2}{|1 - \bar{z}_1 \phi_{z_1}(z_2)|^2} \leq \frac{1 - |\phi_{z_1}(z_2)|^2}{(1 - |\phi_{z_1}(z_2)|)^2} = \frac{1 + |\phi_{z_1}(z_2)|}{1 - |\phi_{z_1}(z_2)|}.$$

Since $|\phi_{z_1}(z_2)| = d(z_1, z_2)$, the lemma is proved. ∎

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Lemma 2.2 *Let $f \in \mathcal{B}^\alpha$. If $\alpha = 1$, then*

$$|f(z) - f(0)| \le \frac{\|f\|_1}{2} \log \frac{1+|z|}{1-|z|}$$

and

$$|f(z)| \le \|f\|_{\mathcal{B}} \Big(1 + \log \frac{1}{1-|z|^2} \Big) \quad \textit{for } z \in D. \tag{2.2}$$

If $\alpha > 1$, then

$$|f(z) - f(0)| \le \frac{C_\alpha \|f\|_\alpha}{(1-|z|^2)^{\alpha-1}}$$

and

$$|f(z)| \le \frac{C_\alpha \|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}} \quad \textit{for } z \in D. \tag{2.3}$$

If $0 < \alpha < 1$, then

$$|f(z) - f(0)| \le C_\alpha \|f\|_\alpha$$

and

$$|f(z)| \le C_\alpha \|f\|_{\mathcal{B}^\alpha} \quad \textit{for } z \in D. \tag{2.4}$$

Throughout this paper C_α denotes a positive constant depending on α only, which may have different values at different places. Lemma 2.2 is easy to prove.

Lemma 2.3 *For $\alpha > 0$ and $a \in D \setminus \{0\}$, define*

$$f_{\alpha,a}(z) = \frac{1}{\alpha \bar{a}} \frac{(1-|a|^2)}{(1-\bar{a}z)^\alpha} \quad \textit{for } z \in D.$$

Then

$$1 \le \|f_{\alpha,a}\|_\alpha \le 2^{|\alpha-1|}. \tag{2.5}$$

Proof If $\alpha > 1$, for $z \in D$, by (1.6),

$$\begin{aligned}(1-|z|^2)^\alpha |f'_{\alpha,a}(z)| &= \frac{(1-|z|^2)^\alpha (1-|a|^2)}{|1-\bar{a}z|^{\alpha+1}} \\ &= \frac{(1-|z|^2)^{\alpha-1}}{|1-\bar{a}z|^{\alpha-1}} \big(1 - |\phi_a(z)|^2\big) \le 2^{\alpha-1} \quad \text{for } z \in D.\end{aligned}$$

By the same reasoning, if $\alpha \le 1$, we have $(1-|z|^2)^\alpha |f'_{\alpha,a}(z)| \le 2^{1-\alpha}$ for $z \in D$. On the other hand, $(1-|a|^2)^\alpha |f'_{\alpha,a}(a)| = 1$. This shows the lemma. ∎

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Lemma 2.4 *Let $a_n \in D$ and $a_n \to \partial D$. If $0 < \alpha < 1$, $\beta > 0$, and $u \in \mathcal{B}^\beta$, then*

$$\sup_{z\in D}(1-|z|^2)^\beta |u'(z) f_{\alpha,a_n}(z)| \to 0 \quad \text{as } n \to \infty. \tag{2.6}$$

If $u \in \mathcal{B}_0^\beta$, (2.6) holds for $\alpha = 1$ also.

Proof Let $\alpha < 1$ and $u \in \mathcal{B}^\beta$ and denote $h_n(z) = (1-|z|^2)^\beta |u'(z) f_{\alpha,a_n}(z)|$. Then,

$$\sup_{z\in D} h_n(z) \le \|u\|_\beta \sup_{z\in D} \frac{(1-|a_n|^2)}{\alpha|a_n||1-\overline{a}_n z|^\alpha} \le \frac{2}{\alpha|a_n|}(1-|a_n|)^{1-\alpha}\|u\|_\beta.$$

Equation (2.6) follows. If $u \in \mathcal{B}_0^\beta$, for $\epsilon > 0$, there exists an $r' < 1$ such that $(1-|z|^2)^\beta |u'(z)| < \epsilon$ for $|z| > r'$. Note that $|f_{1,a_n}(z)| < (1+|a_n|)/|a_n| < 4$ for $z \in D$, if $|a_n| > 1/2$. Thus, $\sup_{|z|>r'} h_n(z) < 4\epsilon$ for sufficiently large n. On the other hand, $\sup_{|z|\le r'} h_n(z) \to 0$ as $n \to \infty$, since $f_{1,a_n}(z) \to 0$, as $n \to \infty$, uniformly for $|z| \le r'$. This shows (2.6), since ϵ may be small arbitrarily. The lemma is proved. ∎

Lemma 2.5 *If $0 < \alpha < 1$, $\alpha < \beta$ and $u \in \mathcal{B}^\beta$, then*

$$\lim_{z\to\partial D}(1-|z|^2)^{\beta-\alpha}|u(z)| = 0, \tag{2.7}$$

As a consequence of (2.7), for any sequence $a_n \in D$, which tends to ∂D, we have

$$\sup_{z\in D}(1-|z|^2)^\beta |u(z) f'_{\alpha,a_n}(z)| \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

If $\beta > \alpha = 1$ and $u \in \mathcal{B}_0^\beta$, (2.7) and (2.8) also hold.

Proof Under the former assumption, (2.7) is a direct consequence of Lemma 2.2. To prove (2.7) under the latter assumption, let $\epsilon > 0$. There exists an $r_0 < 1$ such that $(1-|z|^2)^\beta |u'(z)| < \epsilon$ for $|z| > r_0$. For $z = re^{i\theta}$ with $r > r_0$, we have

$$\begin{aligned} |u(z)| &\le |u(r_0 e^{i\theta})| + \int_{r_0}^r |u'(\rho e^{i\theta})| d\rho \le |u(r_0 e^{i\theta})| + \epsilon \int_{r_0}^r \frac{d\rho}{(1-\rho^2)^\beta} \\ &\le |u(r_0 e^{i\theta})| + \frac{\epsilon}{(\beta-1)(1-r)^{\beta-1}}, \end{aligned}$$

$$(1-|z|^2)^{\beta-1}|u(z)| \le (1-|z|^2)^{\beta-1} M + \frac{2^{\beta-1}\epsilon}{\beta-1},$$

where $M = \max\{|u(r_0 e^{i\theta})| : 0 \le \theta \le 2\pi\}$. Thus,

$$\limsup_{z\to\partial D}(1-|z|^2)^{\beta-1}|u(z)| \le \frac{2^{\beta-1}\epsilon}{\beta-1}.$$

Equation (2.7) is proved, since ϵ may be arbitrarily small.

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It follows from (2.7) that for $\epsilon > 0$, there exists an $r' < 1$ such that

$$(1-|z|^2)^{\beta-\alpha}|u(z)| < \epsilon \text{ for } |z| > r'.$$

Denote $k_n(z) = (1-|z|^2)^{\beta}|u(z)f'_{\alpha,a_n}(z)|$. Then,

$$\sup_{|z|>r'} k_n(z) \leq \|f_{\alpha,a_n}\|_\alpha \sup_{|z|>r'} (1-|z|^2)^{\beta-\alpha}|u(z)| \leq \epsilon\|f_{\alpha,a_n}\|_\alpha \leq 2^{1-\alpha}\epsilon.$$

It is obvious that $\sup_{|z|\leq r'} k_n(z) \to 0$ as $n \to \infty$, since $f_{\alpha,a_n}(z) \to 0$, as $n \to \infty$, uniformly for $|z| \leq r'$. Equation (2.8) is proved since ϵ may be arbitrarily small. The proof is complete. ■

Lemma 2.6 *Let $a_n \in D$ be a sequence such that $a_n \to \partial D$. If $u \in \mathcal{B}_0$, then for any positive number $r < 1$,*

$$\sup_{d(z,a_n)\leq r} |u(z) - u(a_n)| \to 0 \quad \textit{as } n \to \infty.$$

Proof Let $r < 1$ be given. For $\epsilon > 0$, there exists an $r' < 1$ such that $(1-|z|^2)|u'(z)| < \epsilon$ for $|z| > r'$. Since $a_n \to \partial D$, there is an N such that the pseudo-disk $\overline{\Delta}_n = \{z : d(z,a_n) \leq r\}$ is contained in the annulus $\{z : r' < |z| < 1\}$, and consequently, $(1-|z|^2)|u'(z)| < \epsilon$ for $z \in \overline{\Delta}_n$ provided that $n > N$. For $n > N$ and $z' \in \overline{\Delta}_n$, letting $u_n = u \circ \phi_{a_n}$ and $\zeta' = \phi_{a_n}(z')$, we have

$$\begin{aligned}|u(z') - u(a_n)| = |u_n(\zeta') - u_n(0)| &= \int_0^{\zeta'} |u_n'(\zeta)||d\zeta| \\ &\leq \frac{1}{1-|\zeta'|^2}\int_0^{\zeta'} (1-|\zeta|^2)|u_n'(\zeta)||d\zeta|.\end{aligned}$$

Note that $|\zeta'| = d(z'a_n) \leq r$. Meanwhile, $\phi_{a_n}(\zeta) \in \overline{\Delta}_n$ and $(1-|\zeta|^2)|u_n'(\zeta)| = (1-|\phi_{a_n}(\zeta)|^2)|u'(\phi_{a_n}(\zeta))| < \epsilon$ if $|\zeta| \leq r$. Thus, $|u(z') - u(a_n)| \leq \frac{r\epsilon}{1-r^2}$. The lemma is proved, since ϵ may be arbitrarily small. ■

Lemma 2.7 *Let $\alpha \geq 0$, $0 < r < 1$, $u \in H(D)$, and $a_n \to \partial D$ as $n \to \infty$. If*

$$\delta_n = \sup_{d(z,a_n)\leq r} (1-|z|^2)^\alpha |u(z)| \to 0 \quad \textit{as } n \to \infty,$$

then

$$\sup_{d(z,a_n)\leq r'} (1-|z|^2)^{\alpha+1}|u'(z)| \to 0, \quad \textit{as } n \to \infty,$$

for any $r' < r$.

Proof Let $0 < r' < r$. For a fixed n, let $\zeta = \phi_{a_n}(z)$ for $z \in D$, and $u_n = u \circ \phi_{a_n}$. If $|\zeta| \leq r$, then $d(z,a_n) \leq r$ and, by (2.1),

$$|u_n(\zeta)| = |u(z)| \leq \frac{\delta_n}{(1-|z|^2)^\alpha} \leq \frac{\delta_n}{(1-|a_n|^2)^\alpha}\frac{(1+r)^\alpha}{(1-r)^\alpha}.$$

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Thus, by Cauchy's inequality,

$$|u_n'(\zeta)| \leq \frac{\delta_n}{(1-|a_n|^2)^\alpha} \frac{(1+r)^\alpha}{(1-r)^\alpha (r-r')} \quad \text{for } |\zeta| \leq r'.$$

Then, if $d(z, a_n) \leq r'$, we have $|\zeta| \leq r'$ and, by (2.1),

$$(1-|z|^2)^{\alpha+1}|u'(z)| = (1-|z|^2)^\alpha (1-|\zeta|^2)|u_n'(\zeta)| \leq \frac{\delta_n (1+r)^{2\alpha}}{(1-r)^{2\alpha}(r-r')}.$$

This shows the lemma. ∎

Lemma 2.8 *Let $uC_\phi\colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$ be bounded. If there exists a $\delta > 0$ such that $\|uC_\phi f\|_{\mathcal{B}^\beta} \geq \delta \|f\|_\alpha$ holds for $f \in \mathcal{B}^\alpha$, then uC_ϕ is bounded below from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$.*

Proof Suppose on the contrary that there is a sequence $f_n \in \mathcal{B}^\alpha$ such that $\|f_n\|_{\mathcal{B}^\alpha} = 1$ for $n = 1, 2, \ldots$, and $\|uC_\phi f_n\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$. Then, by hypothesis, $\|f_n\|_\alpha \to 0$ and, consequently, $|f_n(0)| \to 1$ as $n \to \infty$. Without loss of generality, assume that $f_n(0) \to 1$ as $n \to \infty$. By Lemma 2.2, we have $f_n \to 1$ and $uC_\phi f_n \to u$ locally uniformly in D as $n \to \infty$. Thus, $\|u\|_\beta \leq \lim_{n\to\infty} \|uC_\phi f_n\|_\beta = 0$ and $u \equiv 0$, which contradicts the assumption of the lemma. The proof is complete. ∎

Lemma 2.9 *Let $\alpha > 0$, $u \in H(D)$, $u \not\equiv 0$, and $f_n \in H(D)$ for $n = 1, 2, \ldots$. If $\|uf_n\|_{\mathcal{B}^\alpha} \to 0$ as $n \to \infty$, then $f_n \to 0$, as $n \to \infty$, locally uniformly in D.*

Proof Since $u \not\equiv 0$, for any positive $r_0 < 1$, there exists an r' such that $r_0 < r' < 1$ and $u(z) \neq 0$ for $|z| = r'$. By Lemma 2.2, $|u(z)f_n(z)| \leq C_{\alpha,r'}\|uf_n\|_{\mathcal{B}^\alpha}$ and, consequently, $|f_n(z)| \leq (C_{\alpha,r'}/\delta)\|uf_n\|_{\mathcal{B}^\alpha}$ for $n = 1, 2, \ldots$, and $|z| = r'$, where $\delta = \min_{|z|=r'} |u(z)| > 0$. By maximum principle, this shows that $f_n \to 0$, as $n \to \infty$, uniformly for $|z| \leq r'$, since $\|uf_n\|_{\mathcal{B}^\alpha} \to 0$, as $n \to \infty$, by hypothesis. The lemma is proved. ∎

3 Theorems and Their Proofs

It is easy to see that if $0 < \beta < \alpha$, $M_u\colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is not bounded unless $u \equiv 0$. Then, M_u is obviously not bounded below. So we only need to consider the case $0 < \alpha \leq \beta$.

Theorem 3.1 *Let $0 < \alpha \leq 1$ and $\alpha < \beta$. If $M_u\colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, then M_u is not bounded below from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$.*

Proof Let $a_n \in D$ be a sequence such that $a_n \to \partial D$ as $n \to \infty$, and let $f_n = f_{\alpha,a_n}$ be functions defined in Lemma 2.3. We have

$$\|uf_n\|_{\mathcal{B}^\beta} \leq |u(0)f_n(0)| + \sup_{z\in D}(h_n(z) + k_n(z)), \tag{3.1}$$

where $h_n(z) = (1-|z|^2)^\beta |u'(z)||f_n(z)|$ and $k_n(z) = (1-|z|^2)^\beta |u(z)||f_n'(z)|$. It is obvious that $u(0)f_n(0) \to 0$ as $n \to \infty$. By (1.3′) and (1.4), $u \in \mathcal{B}_0^\beta$ if $\alpha = 1$, and $u \in \mathcal{B}^\beta$ if $0 < \alpha < 1$. Thus, using Lemmas 2.4 and 2.5 and Equations (2.6) and (2.8), we obtain $\sup_{z\in D}(h_n(z) + k_n(z)) \to 0$ as $n \to \infty$. It is proved that $\|uf_n\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$, which shows that M_u is not bounded below since $\|f_n\|_{\mathcal{B}^\alpha} \geq 1$, by (2.5), for $n = 1, 2, \ldots$. The theorem is proved. ∎

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Theorem 3.2 *Let $0 < \alpha \leq 1$ and $M_u \colon \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ be bounded. Then, the following conditions are equivalent:*

(i) *M_u is bounded below on $\mathcal{B}^\alpha$;*

(ii) *M_u is bounded below on the subset $\{f_{\alpha,a} : a \in D \setminus \{0\}\}$ of $\mathcal{B}^\alpha$, where $f_{\alpha,a}$ denote functions defined in Lemma 2.3;*

(iii) $\liminf_{z\to\partial D} |u(z)| > 0$.

Proof Since M_u is bounded on $\mathcal{B}^\alpha$, we have $u \in \mathcal{B}^\alpha \subset \mathcal{B}_0$ if $\alpha < 1$ by (1.4), and $u \in \mathcal{B}_0$ if $\alpha = 1$ by (1.3′), and

$$\sup_{z\in D} |u(z)| = M < \infty \tag{3.2}$$

for $0 < \alpha \leq 1$ by (1.1′). It is obvious that (i) implies (ii).

Assume that (iii) does not hold, *i.e.,* there exists a sequence $a_n \to \partial D$ such that $u(a_n) \to 0$ as $n \to \infty$. For $n = 1, 2, \dots$, let $f_n = f_{\alpha,a_n}$. We have (3.1) again with the same definition of h_n and k_n as before and $u(0) f_n(0) \to 0$ as $n \to \infty$. By Lemma 2.4, $\sup_{z\in D} h_n(z) \to 0$ as $n \to \infty$.

To estimate $k_n(z)$, let $\epsilon > 0$ be given. We have

$$\begin{aligned} (1-|z|^2)^\alpha |f_n'(z)| &= \frac{(1-|z|^2)^\alpha (1-|a_n|^2)}{|1-\overline{a}_n z|^{\alpha+1}} \\ &= \frac{(1-|z|^2)^\alpha (1-|a_n|^2)^\alpha}{|1-\overline{a}_n z|^{2\alpha}} \frac{(1-|a_n|^2)^{1-\alpha}}{|1-\overline{a}_n z|^{1-\alpha}} \\ &= \big(1-|\phi_{a_n}(z)|^2\big)^\alpha \frac{(1-|a_n|^2)^{1-\alpha}}{|1-\overline{a}_n z|^{1-\alpha}}, \end{aligned} \tag{3.3}$$

where the identity (1.6) is used. Let $r' = (1-\epsilon^{1/\alpha})^{1/2}$. By (3.3) and (3.2),

$$k_n(z) \leq 2^{1-\alpha} M\epsilon \quad \text{if } d(z,a_n) = |\phi_{a_n}(z)| \geq r'. \tag{3.4}$$

On the other hand, by Lemma 2.6,

$$\lambda_n = \sup_{d(z,a_n)\leq r'} |u(z)| \leq |u(a_n)| + \sup_{d(z,a_n)\leq r'} |u(z) - u(a_n)| \to 0 \quad \text{as } n \to \infty,$$

since $u \in \mathcal{B}_0$. Thus,

$$\sup_{d(z,a_n)\leq r'} k_n(z) \leq \lambda_n \|f_n\|_\alpha \leq 2^{1-\alpha}\lambda_n \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Combining (3.4) and (3.5), we see that $\sup_{z\in D} k_n(z) \to 0$ as $n \to \infty$, since ϵ may be arbitrarily small. We have proved that the terms at the right side of (3.1) are all convergent to 0 as $n \to \infty$. Therefore, $\|u f_n\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$, which contradicts (ii) for $\|f_n\|_\alpha \geq 1$ by (2.5). The implication (ii)$\Rightarrow$(iii) is proved.

Now assume that (iii) holds. We want to prove (i). Denote

$$\delta = \liminf_{z \to \partial D} |u(z)| > 0.$$

Suppose on the contrary that M_u is not bounded below on $\mathcal{B}^\alpha$. Then, by Lemma 2.8, there exists a sequence $f_n \in \mathcal{B}^a$ such that $\|f_n\|_\alpha = 1$ for $n = 1, 2, \ldots$, and $\|uf_n\|_{\mathcal{B}^\alpha} \to 0$ as $n \to \infty$. By Lemma 2.9, $f_n \to 0$, as $n \to \infty$, locally uniformly in D. Let $z_n \in D$ be a sequence such that $(1-|z_n|^2)^\alpha |f_n'(z_n)| \geq 1/2$ for $n = 1, 2, \ldots$. Then $z_n \to \partial D$ as $n \to \infty$.

Let r' be close to 1 so that $|u(z)| \geq \delta/2$ for $r' < |z| < 1$. By (2.2) and (2.4), for $n = 1, 2, \ldots$, and $r' < |z| < 1$, we have

$$|f_n(z)| \leq \frac{2\|uf_n\|_{\mathcal{B}}}{\delta}\Big(1 + \log \frac{1}{1-|z|^2}\Big) \tag{3.6}$$

or

$$|f_n(z)| \leq \frac{C_\alpha \|uf_n\|_{\mathcal{B}^\alpha}}{\delta}, \tag{3.6$'$}$$

according to $\alpha = 1$ or $\alpha < 1$.

For sufficiently large n, we have $|z_n| > r'$, $|u(z_n)| \geq \delta/2$, and

$$(1-|z_n|^2)^\alpha |u(z_n)||f_n'(z_n)| \geq \frac{\delta}{4}. \tag{3.7}$$

If $\alpha < 1$, then

$$(1-|z_n|^2)^\alpha |u'(z_n)||f_n(z_n)| \to 0 \quad \text{as } n \to \infty, \tag{3.8}$$

since $u \in \mathcal{B}^\alpha$ and $f_n \to 0$, as $n \to \infty$, uniformly on D by (3.6$'$). In the case that $\alpha = 1$, by (1.3$'$),

$$M = \sup_{z \in D} (1-|z|^2)|u'(z)|\Big(1 + \log \frac{1}{1-|z|^2}\Big) < \infty.$$

Thus, for sufficiently large n, by (3.6),

$$\begin{aligned}(1-|z_n|^2)|u'(z_n)||f_n(z_n)| &\leq \frac{2\|uf_n\|_{\mathcal{B}}}{\delta}(1-|z_n|^2)^\alpha |u'(z_n)|\Big(1 + \log \frac{1}{1-|z_n|^2}\Big) \\ &\leq \frac{2M\|uf_n\|_{\mathcal{B}}}{\delta}.\end{aligned}$$

This shows that (3.8) holds also for $\alpha = 1$. However,

$$\|uf_n\|_{\mathcal{B}^\alpha} \geq (1-|z_n|^2)^\alpha |u(z_n)||f_n'(z_n)| - (1-|z_n|^2)^\alpha |u'(z_n)||f_n(z_n)|. \tag{3.9}$$

It follows from (3.9), (3.7), and (3.8) that $\|uf_n\|_{\mathcal{B}^\alpha} \geq \delta/8$ for sufficiently large n. We arrive at a contradiction, and the implication (iii)⇒(i) is proved. This completes the proof of the theorem. ∎

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Theorem 3.3 *Let $\beta \geq \alpha > 1$ and $M_u\colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$ be bounded. Then, the following conditions are equivalent:*

(i) *M_u is bounded below from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$;*

(ii) *M_u is bounded below from the subset $\{f_{\alpha,a} : a \in D \setminus \{0\}\}$ of $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$ with $f_{\alpha,a}$ as in Lemma 2.3;*

(iii) *there exist a $\delta > 0$ and a positive $r < 1$ such that for every point $z \in D$ there is a $z' \in D$ with the property that $d(z', z) < r$ and $(1 - |z'|^2)^{\beta-\alpha}|u(z')| \geq \delta$.*

Proof Since M_u is bounded, by (1.2′) and (1.1′), we have

$$\sup_{z\in D}(1 - |z|^2)^{\beta-\alpha+1}|u'(z)| = M_1 < \infty, \tag{3.10}$$

$$\sup_{z\in D}(1 - |z|^2)^{\beta-\alpha}|u(z)| = M_2 < \infty. \tag{3.11}$$

It is obvious that (i) implies (ii).

Assume that (iii) does not hold, *i.e.,* there exist sequences $r_n \to 1$ and $a_n \to \partial D$ such that

$$\delta_n = \sup_{d(z,a_n)\leq r_n} (1 - |z|^2)^{\beta-\alpha}|u(z)| \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Then, using Lemma 2.7, we see that for any $r' < 1$

$$\sup_{d(z,a_n)\leq r'} (1 - |z|^2)^{\beta-\alpha+1}|u'(z)| \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

Assume that $|a_n| > 1/2$ and let $f_n = f_{\alpha,a_n}$ for $n = 1, 2, \dots$. Then, we have (3.1) again with the same definition of h_n and k_n as before and $u(0)f_n(0) \to 0$ as $n \to \infty$.

Let $z \in D$. By (1.6), we have

$$\begin{aligned} h_n(z) &= \left(1 - |z|^2\right)^{\beta-\alpha+1} |u'(z)| \frac{(1 - |z|^2)^{\alpha-1}(1 - |a_n|^2)}{\alpha|a_n||1 - \overline{a}_n z|^\alpha} \\ &\leq \frac{\left(1 - |z|^2\right)^{\beta-\alpha+1} |u'(z)|}{\alpha|a_n|} \frac{(1 - |z|^2)^{\alpha-1-\lambda}(1 - |a_n|^2)^{1-\lambda}}{|1 - \overline{a}_n z|^{\alpha-2\lambda}} \left(1 - |\phi_{a_n}(z)|^2\right)^\lambda \\ &\leq \frac{2^{\alpha+1-2\lambda}}{\alpha} \left(1 - |z|^2\right)^{\beta-\alpha+1} |u'(z)| \left(1 - |\phi_{a_n}(z)|^2\right)^\lambda, \end{aligned}$$

where $\lambda = \min\{\alpha - 1, 1\}$. Consequently, by (3.10),

$$h_n(z) \leq \frac{2^{\alpha+1-2\lambda}}{\alpha} \left(1 - |z|^2\right)^{\beta-\alpha+1} |u'(z)| \text{ and} \tag{3.14}$$

$$h_n(z) \leq \frac{2^{\alpha+1-2\lambda}M_1}{\alpha} \left(1 - |\phi_{a_n}(z)|^2\right)^\lambda. \tag{3.15}$$

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Similarly, for $k_n(z)$, we have

$$
\begin{aligned}
(3.16) \qquad k_n(z) &= (1-|z|^2)^{\beta-\alpha}|u(z)| \cdot \frac{(1-|z|^2)^{\alpha}(1-|a_n|^2)}{|1-\overline{a}_n z|^{\alpha+1}} \\
&\le (1-|z|^2)^{\beta-\alpha}|u(z)| \cdot \frac{(1-|z|^2)^{\alpha-1}}{|1-\overline{a}_n z|^{\alpha-1}} \left(1-|\phi_{a_n}(z)|^2\right) \\
&\le 2^{\alpha-1}(1-|z|^2)^{\beta-\alpha}|u(z)| \left(1-|\phi_{a_n}(z)|^2\right), \\
&\le 2^{\alpha-1}(1-|z|^2)^{\beta-\alpha}|u(z)|,
\end{aligned}
$$

and, by (3.11),

$$(3.17) \qquad k_n(z) \le 2^{\alpha-1}M_2\left(1-|\phi_{a_n}(z)|^2\right).$$

For $\epsilon > 0$, let $r' = (1-\epsilon)^{1/2}$. If $d(z,a_n) = |\phi_{a_n}(z)| > r'$, by (3.15) and (3.17), we have

$$h_n(z) < 2^{\alpha+1-2\lambda}M_1\epsilon^{\lambda}/\alpha \quad \text{and} \quad k_n(z) < 2^{\alpha-1}M_2\epsilon.$$

On the other hand, by (3.12), (3.13), (3.14), and (3.16),

$$\sup_{d(z,a_n)\le r'} (h_n(z)+k_n(z)) \to 0 \quad \text{as} \quad n\to\infty.$$

Now, it is proved that

$$\sup_{z\in D}(h_n(z)+k_n(z)) \to 0 \quad \text{as} \quad n\to\infty,$$

since ϵ may be arbitrarily small. We have proved that all of the terms in the right side of the inequality (3.1) tend to 0 as $n\to\infty$. So, $\|uf_n\|_{\mathcal{B}^\beta} \to 0$ as $n\to\infty$, which means that (ii) does not hold. This shows the implication (ii)⇒(iii).

Now, we will proceed to prove (iii)⇒(i). Assume that (iii) holds. We want to prove (i). Suppose on the contrary that M_u is not bounded below from $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$. Then, by Lemma 2.8, there exists a sequence $f_n \in \mathcal{B}^\alpha$ such that $\|f_n\|_\alpha = 1$ for $n = 1,2,\dots,$ and $\|uf_n\|_{\mathcal{B}^\beta} \to 0$ as $n\to\infty$. By Lemma 2.9, $f_n \to 0$, as $n\to\infty$, locally uniformly in D. Let $z_n \in D$ be a sequence such that $(1-|z_n|^2)^\alpha|f_n'(z_n)| \ge 1/2$ for $n = 1,2,\cdots$. Then $z_n \to \partial D$ as $n\to\infty$.

Let $\delta > 0$ and $r < 1$ be the number in (iii). For $n = 1,2,\cdots$, let $z_n \in \Delta_n$ be a point such that $d(z_n, z_n') < r$ and

$$(3.18) \qquad (1-|z_n'|^2)^{\beta-\alpha}|u(z_n')| \ge \delta,$$

and let

$$\zeta_n' = \phi_{z_n}(z_n'), \quad u_n = (1-|z_n|^2)^{\beta-\alpha}u\circ\phi_{z_n}, \quad g_n = (1-|z_n|^2)^{\alpha-1}f_n\circ\phi_{z_n}.$$

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Since $|\zeta_n'| = d(z_n', z_n) < r$, without loss of generality, assume that $\zeta_n' \to \zeta_0' \in D$. By (2.3) and (2.1), we have

$$|g_n(0)| = (1-|z_n|^2)^{\alpha-1}|f_n(z_n)| \le C_\alpha \|f_n\|_{\mathcal{B}^\alpha} \le C_\alpha(1+|f_n(0)|), \tag{3.19}$$

$$|g_n'(0)| = (1-|z_n|^2)^{\alpha}|f_n'(z_n)| \ge 1/2, \tag{3.20}$$

and

$$\begin{aligned} |g_n'(\zeta)| &= \frac{1}{1-|\zeta|^2}(1-|z_n|^2)^{\alpha-1}(1-|\phi_{z_n}(\zeta)|^2)|f_n'(\phi_{z_n}(\zeta))| \\ &\le \frac{(1+|\zeta|)^{\alpha-1}}{(1-|\zeta|)^{\alpha-1}(1-|\zeta|^2)}(1-|\phi_{z_n}(\zeta)|^2)^{\alpha}|f_n'(\phi_{z_n}(\zeta))| \\ &\le \frac{(1+|\zeta|)^{\alpha-1}}{(1-|\zeta|)^{\alpha-1}(1-|\zeta|^2)} \quad \text{for } \zeta \in D. \end{aligned} \tag{3.21}$$

For u_n, by (2.1), (3.11), and (3.18), we have

$$\begin{aligned} |u_n(\zeta_n')| &= (1-|z_n|^2)^{\beta-\alpha}|u(z_n')| \\ &\ge \frac{(1-r)^{\beta-\alpha}}{(1+r)^{\beta-\alpha}}(1-|z_n'|^2)^{\beta-\alpha}|u(z_n')| \ge \frac{\delta(1-r)^{\beta-\alpha}}{(1+r)^{\beta-\alpha}} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} |u_n(\zeta)| &\le \frac{(1+|\zeta|)^{\beta-\alpha}}{(1-|\zeta|)^{\beta-\alpha}}(1-|\phi_{z_n}(\zeta)|^2)^{\beta-\alpha}|u(\phi_{z_n}(\zeta))| \\ &\le \frac{M_2(1+|\zeta|)^{\beta-\alpha}}{(1-|\zeta|)^{\beta-\alpha}} \quad \text{for } \zeta \in D. \end{aligned} \tag{3.23}$$

It follows from (2.3) that

$$|u_n(0)g_n(0)| = (1-|z_n|^2)^{\beta-1}|u_n(z_n)g_n(z_n)| \le C_\beta\|uf_n\|_{\mathcal{B}^\beta}. \tag{3.24}$$

By (3.19), (3.21), and (3.23), g_n and u_n are bounded locally uniformly in D. Thus, by Montel's theorem, g_n and u_n contain locally uniformly convergent subsequences. Without loss of generality, we may assume that $g_n \to g_0$ and $u_n \to u_0$, as $n \to \infty$, locally uniformly in D. For a fixed n, letting $z = \phi_{z_n}(\zeta)$, by (2.1), we have

$$\begin{aligned} \|uf_n\|_{\mathcal{B}^\beta} &\ge (1-|z|^2)^\beta|(uf_n)'(z)| \\ &= (1-|\phi_{z_n}(\zeta)|^2)^\beta|(uf_n)'(\phi_{z_n}(\zeta))| \\ &= (1-|\phi_{z_n}(\zeta)|^2)^{\beta-1}(1-|\zeta|^2)|\big((u\circ\phi_{z_n})(f_n\circ\phi_{z_n})\big)'(\zeta)| \\ &\ge \frac{(1-|\zeta|^2)(1-|\zeta|)^{\beta-1}}{(1+|\zeta|)^{\beta-1}}(1-|z_n|^2)^{\beta-1}|\big((u\circ\phi_{z_n})(f_n\circ\phi_{z_n})\big)'(\zeta)| \\ &= \frac{(1-|\zeta|^2)(1-|\zeta|)^{\beta-1}}{(1+|\zeta|)^{\beta-1}}|\big(u_ng_n\big)'(\zeta)| \quad \text{for } \zeta \in D. \end{aligned}$$

Letting $n \to \infty$ in the above estimate, we see that $u_0 g_0$ is a constant. Note that $u_0(0)g_0(0) = 0$ by (3.24). Thus, $u_0 g_0 \equiv 0$. However, both u_0 and g_0 are not equal to 0 identically, since $|g_0'(0)| > 0$ and $|u_0(\zeta_0')| > 0$ by (3.20) and (3.22), respectively. We arrive at a contradiction, and this shows (iii)⇒(i). ■

Remark. We indicate that condition (iii) in Theorem 3.2 can be replaced by an apparently weaker one:

(iii′) there exist a $\delta > 0$ and a positive $r < 1$ such that for every point $z \in D$ there is a $z' \in D$ with the property that $d(z', z) < r$ and $|u(z')| \geq \delta$.

This condition is the same as in Theorem 3.3 for $\beta = \alpha > 1$. In fact, (iii′) and (iii) are equivalent if M_u is bounded on $\mathcal{B}^\alpha$ for some $\alpha \leq 1$. Let u be such a function. Then $u \in \mathcal{B}_0$ by (1.3′) or (1.4). If (iii) does not hold, *i.e.,* there exists a sequence $z_n \to \partial D$ with $u(z_n) \to 0$, then for any $\delta > 0$ and $0 < r < 1$, $|u(z)| < \delta$ for $d(z, z_n) < r$ and sufficiently n, since $\sup_{d(z,z_n)<r} |u(z) - u(z_n)| \to 0$ by Lemma 2.6. This means that (iii′) is not true. This shows that (iii′) ⇒ (iii) and they are equivalent. However, the following example shows that in the case $\alpha = \beta > 1$, the condition (iii) in Theorem 3.3 cannot be replaced by the stronger one: $\liminf_{z\to\partial D} |u(z)| > 0$.

Example. Let $r = 1/4$, $r_1 = 1/2$, $\Delta_1 = \{z : d(z, r_1) < r\}$, and r_1', r_1'' be the left and right intersection points of $\partial\Delta_1$ and the positive real axis. Generally, when Δ_n, r_n, r_n', and r_n'' have been defined, we let $r_{n+1} > r_n$ be the point with $d(r_n'', r_{n+1}) = 2^{-2^{-n}}$, $\Delta_{n+1} = \{z : d(z, r_{n+1}) < r\}$, and r_{n+1}', r_{n+1}'' be the intersection points of $\partial\Delta_{n+1}$ and the positive real axis. Then $\overline{\Delta}_n$, $n = 1, 2, \dots$, are disjoint from one another. We define the function u by the Blaschke product $u(z) = \prod_{n=1}^{\infty} \frac{r_n - z}{1 - r_n z}$. If $z \in \partial\Delta_n$ for some n, then

$$\begin{aligned}
|u(z)| &= \prod_{k=1}^{\infty} d(z, r_k) = \frac{1}{4} \prod_{k=1}^{n-1} d(z, r_k) \prod_{k=n+1}^{\infty} d(z, r_k) \\
&\geq \frac{1}{4} \prod_{k=1}^{n-1} d(r_n', r_k) \prod_{k=n+1}^{\infty} d(r_n'', r_k) \geq \frac{1}{4} \prod_{k=1}^{n-1} d(r_k'', r_n) \prod_{k=n+1}^{\infty} d(r_{k-1}'', r_k) \\
&\geq \frac{1}{4} \prod_{k=1}^{n-1} d(r_k'', r_{k+1}) \prod_{k=n+1}^{\infty} d(r_{k-1}'', r_k) = \frac{1}{4} \prod_{k=1}^{\infty} d(r_k'', r_{k+1}) = \frac{1}{8}.
\end{aligned}$$

This shows that $|u(z)| \geq 1/8$ for $z \in \bigcup_{n=1}^{\infty} \partial\Delta_n$. Let u_n be the partial product of the Blaschke product, $U_n = \bigcup_{k=1}^{n} \Delta_k$ and $U = \bigcup_{k=1}^{\infty} \Delta_k$. Then, for $n = 1, 2, \dots$, by using the maximum principle to the function $1/u_n$, we see that $|u_n(z)| \geq 1/8$ for $z \in D \setminus U_n$, since $|u_n(z)| \geq |u(z)| \geq 1/8$ for $z \in \partial U_n$ and $|u_n(z)| = 1$ for $z \in \partial D$. Thus, $|u(z)| \geq 1/8$ for $z \in D \setminus U$ and, consequently, u satisfies condition (iii) in Theorem 3.3 with $\alpha = \beta > 1$, $r = 1/4$, and $\delta = 1/8$. Meanwhile, M_u is bounded on $\mathcal{B}^\alpha$ for $\alpha > 1$ since u satisfies (1.1′) and (1.2′) with $\beta = \alpha > 1$. Therefore, M_u is bounded below by Theorem 3.3. However, $\liminf_{z\to\partial D} |u(z)| = 0$. This shows that for $\alpha = \beta > 1$, condition (iii) in Theorem 3.3 cannot be replaced by the stronger one: $\liminf_{z\to\partial D} |u(z)| > 0$.

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