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A NOTE ON THE HOPFICITY OF FINITELY GENERATED FREE-BY-NILPOTENT GROUPS

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Our purpose is to deduce from a theorem of P. Hall the following observation.

THEOREM 1. *Let G be a finitely generated group and F a free normal subgroup of G with G/F nilpotent. Then G is hopfian.*

Here a group G is hopfian if every epiendomorphism $G \to G$ is an automorphism.

Proof of Theorem 1. Because G/F is nilpotent and subgroups of free groups are free we may assume $F = G_n$, some term of the lower central series of G. Thus F and its commutator subgroup F' are fully invariant in G, giving rise to an induced epiendomorphism $\theta_1 : (G/F') \to (G/F')$.

Now G/F' is finitely generated abelian-by-nilpotent. P. Hall [4] showed that such groups are residually finite and hence by a theorem of A. I. Mal'cev [5] are hopfian. This means that θ_1 is an automorphism so that ker $\theta \subseteq F'$. If we let θ_2 be θ restricted to F and $\theta_3 : (F/F') \to (F/F')$ the induced map, we must show ker $\theta_2 = 1$. But θ_3 is 1–1 since ker $\theta \subset F'$, and any endomorphism θ_2 of a free group whose induced map θ_3 on the commutator quotient is a monomorphism must itself be one-one. Thus ker $\theta = 1$ and Theorem 1 is proved.

Theorem 1 provides a corollary to the recent result of R. Bieri [3] that if G is finitely generated with a non-trivial centre and the cohomological dimension cdG of G is ≤ 2, G' is free. In this case G is of course free-by-abelian so that by Theorem 1 we have

COROLLARY 1. *If G is finitely generated, $\zeta G \neq 1$ and $cdG \leq 2$, G is hopfian.*

Thus by a result of B. B. Newman [6], each finitely generated torsion-free one-relator group with centre is hopfian. G. Baumslag and D. Solitar [2] have provided examples of non-hopfian torsion-free one-relator groups.

G. Baumslag [1] has shown by an ingenious argument that finitely generated free-by-cyclic groups are residually finite. The problem of strengthening Theorem 1 by replacing hopficity by residual finiteness seems difficult however. It is not even known whether finitely generated free-by-abelian of rank 2 groups are residually finite.

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