A METHOD FOR SOLVING PROBLEMS OF ELASTIC PLATES OF ARBITRARY SHAPE

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(Received 20 February, 1968)

Introduction

The methods developed up to now in the theory of plates do not lend themselves easily to the solution of problems in the case of plates of arbitrary shape. Such methods lend themselves only to the solution of problems for plates of certain special simple shapes. Moreover, because of the complicated mathematical tools involved in these methods, the methods are not suitable for engineers. Hence there arises the need for the formulation of an approximate theory which may appear quite suitable for attacking problems on plates of arbitrary shape. In the present paper the author develops a new method which aims at the solution of the problem quite independently of the shape of the plate.

It is well known that the usual methods for the approximate solution of problems of bending of a plate are based upon the rough idea of the shape of the deflected surface of the plate being physically compatible with the type of fastening at the boundary, the nature of the surface loads and the geometrical shape of the plate. In these methods, there arise the principal difficulties of the satisfaction of all boundary conditions which practically remained insurmountable for plates of arbitrary shape. The present method aims at removing these difficulties. In this method, when an elastic plate with clamped or supported boundary is bent under the action of external pressure, the corresponding deflection surface of the plate may be described by a family of curves which may be called 'Lines of Equal Deflection', i.e., lines which are obtained by intersecting the bent plate by planes parallel to the original plane of the plate. In principle, it is always possible to determine the equation of such lines of equal deflection. In some cases, the equation of such lines ot equal deflection can be known by symmetry consideration or by intuition. The outstanding feature of this method is that it is entirely independent of the shape of the plate; its success depends, however, on the determination of the equation of the lines of equal deflection and, consequently, on the type of loading. As illustrations of the procedure, the method has been applied in Section 3 to the calculation of the deflections of an elliptic and a parabolic plate with uniformly distributed normal loads. These examples show that the method of prescribing the lines of equal deflection is quite suitable. All details of the method are explained in Section 1. Several remarks close the paper.

1. An account of the method and derivation of the new equations of equilibrium

The theory which is developed here is based upon the following assumptions:

(i) the plate consists of homogeneous and isotropic material;
(ii) the material obeys Hooke's law;
(iii) the deflection of the plate is small compared with its thickness;
(iv) the thickness of the plate is small compared with its other dimensions.

Taking the xoy-plane as usual to be the middle plane of the plate and directing the z-axis perpendicular to that plane, we shall suppose that the family of lines of equal deflection $u = u(x, y) =$ Const. is known. Hence, intersections between the deflection surface $z = w(x, y)$ and the planes $z =$ Const. yield contours which, after projection on to the xoy plane, are the lines of equal deflection $u(x, y) =$ const. If the boundary of the plate does not move in the direction perpendicular to the plane of the plate (this case corresponds to elastically supported edges), then clearly the contour of the plate belongs to the family of the lines of equal deflection and we may consider this contour as $u = o$ (Fig. 1). In particular, when the

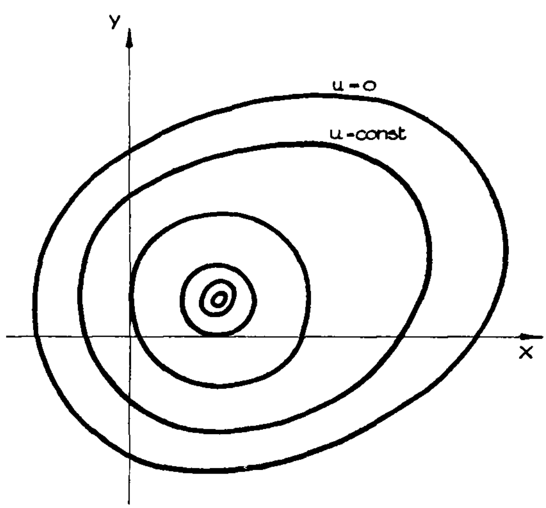

Figure 1

distribution of normal loading is such that it does not change sign anywhere in the region of the plate, then clearly a line of equal deflection cannot intersect the boundary and it cannot end at any point in the plane of the plate. Each line of equal deflection must, therefore, be a closed curve. Moreover, two lines of equal deflection cannot intersect. In general, however, the lines of equal deflection form a system of non-intersecting closed curves, starting from the outer boundary as one of the lines. For the symmetrically loaded circular plate, the lines of equal deflections are concentric circles; for the clamped elliptic plate, they are a set of similar concentric ellipses.

For the analysis of small deflections of laterally loaded plates, we shall only consider the external forces perpendicular to the middle surface of the plate and the distribution of moments along its edges. In Fig. 2, the distribu-

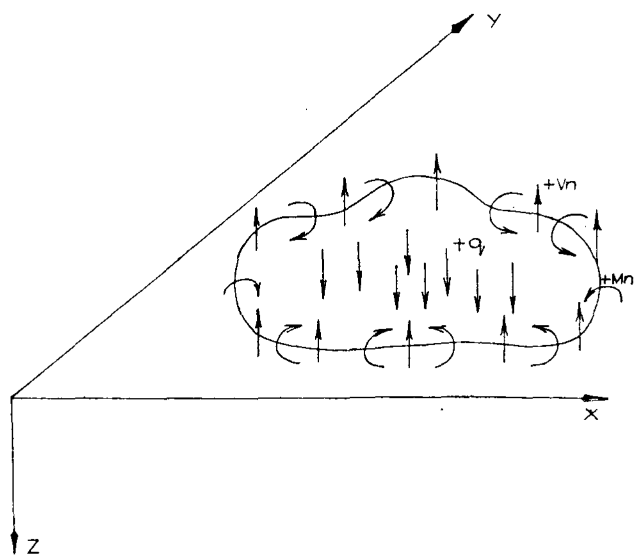

Figure 2

tion of the bending moment M_n along the contour of the plate, the distribution of a continuously transverse load of intensity q over the upper surface of the plate and the distribution of transverse forces V_n which contain the shearing force Q_n and the portion of the edge reaction which is due to the distribution along the edge of the twisting moment M_{nt} on the same contour of the plate are shown. As in the classical theory, the shearing force Q_t is being neglected.

Let us consider the equilibrium of an element of the plate bounded by any line of equal deflection. In Fig. 3, the interior portion of the plate

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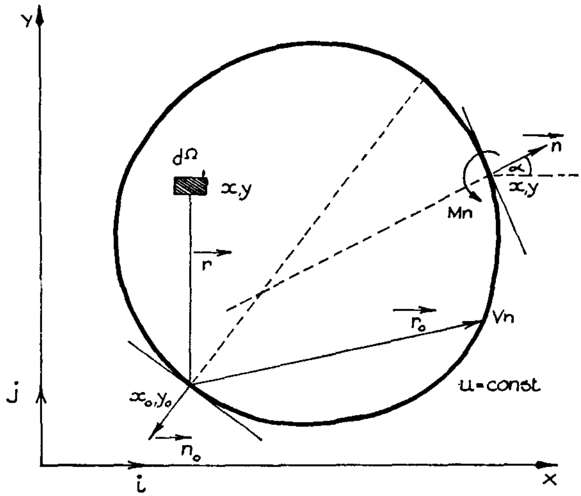

Figure 3

bounded by a line $u(x, y) = \text{const.}$ is shown, where (x_0, y_0) indicates a fixed point on the contour, $\vec{n}$ and $\vec{n}_0$ denote the unit vectors normal to the line $u = \text{const.}$ at any arbitrary point (x, y) and at the fixed point (x_0, y_0), $\vec{r}$ and $\vec{r}_0$ denote the position vectors from the fixed point (x_0, y_0) to any arbitrary point inside the contour and on the contour $u = \text{const.}$, respectively. We thus have the relations

$$
\begin{aligned}
\vec{r} &= (x-x_0)\vec{i}+(y-y_0)\vec{j},\\
\vec{r}_0 &= (x-x_0)\vec{i}+(y-y_0)\vec{j}\Big|_{u(x,y)=\text{Const.}},\\
\vec{n} &= \frac{u_x\vec{i}+u_y\vec{j}}{\sqrt{u_x^2+u_y^2}}\Bigg|_{u(x,y)=\text{Const.}}\\
\vec{n}_0 &= \frac{u_x\vec{i}+u_y\vec{j}}{\sqrt{u_x^2+u_y^2}}\Bigg|_{(x_0 y_0)}
\end{aligned}
\tag{1.1}
$$

The conditions of equilibrium for the element of the plate require that the sum of moments about the tangent line to the curve $u(x, y) = \text{const.}$ at any point (x_0, y_0) of all forces acting on the element and the sum of all forces normal to the plane xoy are zero. We will not demand the vanishing of the sum of moments about the normal to the curve $u(x, y) = \text{const.}$ at that point for the same reason that we will not seek the vanishing of the twisting moment along the free edge of any plate. Therefore we obtain

$$
\sum M = \vec{n}_0 \oint M_n \vec{n}\, ds + \vec{n}_0 \oint V_n \vec{r}_0\, ds - \vec{n}_0 \iint_\Omega q\vec{r}\, d\Omega = 0, \tag{1.2}
$$

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$$\sum Z = \oint V_n ds - \iint_{\Omega} q d\Omega \equiv 0, \tag{1.3}$$

where the contour integrals are taken around a closed path $u =$ const. and the double integrals over the area bounded by the closed contour $u =$ const.

Let us now calculate the expressions for the moments and shearing forces on any line of the family of equal deflection, using the well known formulas [1]

(1.4)

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right) = -D\left[\frac{d^2 w}{du^2}(u_x^2+\mu u_y^2)+\frac{dw}{du}(u_{xx}+\mu u_{yy})\right],$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right) = -D\left[\frac{d^2 w}{du^2}(u_y^2+\mu u_x^2)+\frac{dw}{du}(u_{yy}+\mu u_{xx})\right],$$

$$M_{xy} = -M_{yx} = D(1-\mu)\frac{\partial^2 w}{\partial x \partial y} = D(1-\mu)\left[\frac{d^2 w}{du^2}u_x u_y+\frac{dw}{du}u_{xy}\right],$$

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) = -D\left[\frac{d^3 w}{du^3}(u_x^3+u_y^2 u_x)+\frac{d^2 w}{du^2}(3u_{xx}u_x + 2u_{xy}u_y+u_{yy}u_x)+\frac{dw}{du}(u_{xxx}+u_{yyy})\right],$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) = -D\left[\frac{d^3 w}{du^3}(u_y^3+u_x^2 u_y)+\frac{d^2 w}{du^2}(3u_{yy}u_y + 2u_{xy}u_x+u_{xx}u_y)+\frac{dw}{du}(u_{yyy}+u_{xxy})\right],$$

where

w is the deflection of the plate,
$D = Eh^3/(12(1-\mu^2))$ is the flexural rigidity of the plate,
E is Young's modulus,
μ Poisson's ratio,
h the thickness of the plate.

While deriving the above expressions, we make use of the relations

$$\frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = \frac{dw}{du}u_x, \quad \frac{\partial w}{\partial y} = \frac{dw}{du}u_y, \tag{1.5}$$
$$\frac{\partial^2 w}{\partial x \partial y} = \frac{d^2 w}{du^2}u_x u_y + \frac{dw}{du}u_{xy}, \text{ etc.,}$$

substituting the expressions (1.4) into the wellknown expressions for M_n, M_{nt}, etc., and after several transformations, we finally obtain

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$$M_n = M_x \cos^2\alpha + M_y \sin^2\alpha - 2M_{xy}\sin\alpha\cos\alpha = P\frac{d^2w}{du^2} + Q\frac{dw}{du},$$

$$M_{nt} = M_{xy}(\cos^2\alpha - \sin^2\alpha) + (M_x - M_y)\sin\alpha\cos\alpha = H\frac{dw}{du},$$

$$\text{(1.6)}\quad M_t = M_x \sin^2\alpha + M_y\cos^2\alpha + 2M_{xy}\sin\alpha\cos\alpha = P'\frac{d^2w}{du^2} + Q'\frac{dw}{du},$$

$$Q_n = Q_x\cos\alpha + Q_y\sin\alpha,$$

$$V_n = Q_n - \frac{\partial M_{nt}}{\partial s} = R\frac{d^3w}{du^3} + F\frac{d^2w}{du^2} + G\frac{dw}{du},$$

where

$$P = -Dt,$$

$$Q = -\frac{D}{t}[u_{xx}u_x^2 + u_{yy}u_y^2 + \mu u_{yy}u_x^2 + \mu u_{xx}u_y^2 + 2(1-\mu)u_{xy}u_xu_y],$$

$$P' = -D\mu t,$$

$$Q' = -\frac{D}{t}[u_{xx}u_y^2 + u_{yy}u_x^2 + \mu u_{xx}u_x^2 + \mu u_{yy}u_y^2 - 2(1-\mu)u_{xy}u_xu_y],$$

$$R = -Dt^{\frac{3}{2}},$$

$$F = -\frac{D}{t^{\frac{1}{2}}}[3u_{xx}u_x^2 + 3u_{yy}u_y^2 + u_{xx}u_y^2 + u_{yy}u_x^2 + 4u_{xy}u_xu_y],$$

$$\text{(1.7)}\quad \begin{aligned} G = &-\frac{D}{t^{\frac{3}{2}}}[u_{xxx}u_x^3 + u_{yyy}u_y^3 + (2-\mu)(u_{xxx}u_xu_y^2 + u_{yyy}u_x^2u_y \\ &+ u_{xyy}u_x^3 + u_{xxy}u_y^3) + (2\mu-1)(u_{xyy}u_xu_y^2 + u_{xxy}u_x^2u_y) \\ &- 2(1-\mu)u_{xy}(u_xu_yu_{xx} - u_y^2u_{xy} - u_x^2u_{xy} + u_xu_yu_{yy}) \\ &+ (1-\mu)(u_{xx}-u_{yy})(u_{xx}u_y^2 - u_{yy}u_x^2)] \\ &+ \frac{2D(1-\mu)}{t^{\frac{5}{2}}}[u_{xy}(u_x^2-u_y^2) - u_xu_y(u_{xx}-u_{yy})]^2, \end{aligned}$$

$$H = \frac{D(1-\mu)}{t}[u_{xy}u_x^2 - u_{xy}u_y^2 - u_{xx}u_xu_y + u_{yy}u_xu_y],$$

$$t = u_x^2 + u_y^2,$$

$$\cos\alpha = \frac{dy}{ds},\quad \sin\alpha = -\frac{dx}{ds},\quad \frac{dy}{dx} = -\frac{u_x}{u_y}.$$

Substituting the expressions for M_n and V_n from (1.6) and taking into account that w and its derivatives with respect to u are constant on the line $u = \text{const.}$, we finally represent the equilibrium equations (1.2) and (1.3) in the form

$$(1.8)\quad \begin{aligned} &\vec{n}_0 \frac{d^2w}{du^2} \oint P\vec{n}ds + \vec{n}_0 \frac{dw}{du} \oint Q\vec{n}ds + \vec{n}_0 \frac{d^3w}{du^3} \oint R\vec{r}_0 ds \\ &\qquad + \vec{n}_0 \frac{d^2w}{du^2} \oint F\vec{r}_0 ds + \vec{n}_0 \frac{dw}{du} \oint G\vec{r}_0 ds - \vec{n}_0 \iint_\Omega q\vec{r}\, d\Omega = 0, \end{aligned}$$

$$(1.9)\quad \frac{d^3w}{du^3} \oint R ds + \frac{d^2w}{du^2} \oint F ds + \frac{dw}{du} \oint G ds - \iint_\Omega q\, d\Omega = 0.$$

For the analysis of the deflection of a plate bent by a lateral load q, it is sufficient to consider only one of the above two equilibrium equations. For the sake of simplicity, we shall henceforth consider only the second equation. It is to be noted here that when the equation $u = u(x, y)$ appears to be an exact equation for a line of equal deflection then the first of the above two equations is satisfied identically with the help of the second equation. Consequently, our problem reduces to that of finding out the solution of the second equation giving the exact value for the function $u(x, y)$.

2. Boundary conditions

Typical boundary conditions for a plate of arbitrary shape are here expressed in terms of the deflection w and its derivatives with respect to u. However, the boundary conditions depend on the nature of fastening of the edge of the plate which, in general, will be a curved boundary with normal $\vec{n}$.

(a) *Clamped Edge.* Along a clamped edge, the deflection and slope normal to the boundary are zero, so that

$$(2.1)\quad \begin{aligned} w\big|_{u=0} &= 0, \\ \frac{\partial w}{\partial n}\bigg|_{u=0} &= 0, \end{aligned}$$

where the derivative $\partial w/\partial n$ is expressed in the form

$$(2.2)\quad \frac{\partial w}{\partial n} = \frac{dw}{du}(u_x \cos\alpha + u_y \sin\alpha) = \sqrt{u_x^2 + u_y^2}\,\frac{dw}{du} = \sqrt{t}\,\frac{dw}{du}.$$

(b) *Simply Supported Edge.* Along a boundary which is simply supported the deflection and the moment per unit length, M_n, are zero, so that

$$(2.3)\quad \begin{aligned} w\big|_{u=0} &= 0, \\ P\frac{d^2w}{du^2} + Q\frac{dw}{du}\bigg|_{u=0} &= 0. \end{aligned}$$

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Still one more condition for both of these cases is obtained at the centre, i.e., at the point of maximum deflection of the plate. The deflection of the plate at the centre must be a finite quantity. For an axially symmetrical, external load, the deflection surface of the plate will also have axial symmetry and, therefore, the tangent plane at the centre must be horizontal.

Thus we obtain three conditions (two at the boundary and one at the centre) for transversely bent thin elastic plates. We see that the essence of the method lies in the fact that it reduces to the integration of a third order ordinary differential equation with three boundary conditions. Consequently, with the exact equation of the lines of equal deflection, the above method gives us an exact solution which can be verified in the case of circular and clamped elliptic plates under symmetrical loading. However, for cases where the preassigned equation of the lines of equal deflection is not correct, we get an approximate solution and obviously this approximation will be as close to the exact solution as the assigned equation of the lines of equal deflection has been selected close to the exact equation. So the crux of the problem is how to obtain the best selection of the equation of the lines of equal deflection close to their exact equation. This problem will be discussed in the next section.

Further, we note from (1.6) and (2.2) that along clamped edges the twisting moment M_{nt} is always zero and, in conjunction with specified boundary conditions, the bending moments M_n and M_t are given by

$$\begin{aligned} M_n &= -D(u_x^2+u_y^2)\,\frac{d^2w}{du^2} \\ M_t &= -D\mu(u_x^2+u_y^2)\,\frac{d^2w}{du^2} \end{aligned} \tag{2.4}$$

Since $d^2w/du^2 = \text{const.}$ on the contour which belongs to the family of lines of equal deflection we have

$$\begin{aligned} M_n &= K(u_x^2+u_y^2), \\ M_t &= K\mu(u_x^2+u_y^2); \end{aligned} \tag{2.5}$$

consequently, knowing the function $u(x, y)$, we may readily detect the critical point or points of bending moments on the contour, for which we are required to determine the extremum of the function $t = u_x^2+u_y^2$ along the boundary $u = 0$.

3. Illustration of the method

As illustrations, this method will be applied to a number of problems of the determination of deflections in thin elastic plates. We shall see that the above method enables us to arrive at an approximate solution of such problem, where the exact solution is unknown or is very inconvenient for numerical calculations.

In cases where the exact equation of the line of equal deflection is not known, we can find quite good approximations with the aid of an equation. If the boundary of the plate is given by the equation $F(x, y) = 0$ and the function F is different from zero within the region of the plate, the way to use this equation is to assume a reasonable expression for u of the form

$$(3.1) \qquad u(x, y) = F(x, y) \sum_{\substack{m=0 \\ n=0}}^{m, n} A_{mn} x^m y^n,$$

where m and n are natural numbers and the coefficients A_{mn} can be interpreted as coordinates, which determine the form of the deflection surface. We shall now see how to obtain the solution when the boundary of the plate has one or the other special form.

(a) *Elliptic Plate with Simply Supported End.*

As a first example of the above method, consider the case of a simply supported, thin, elliptic plate subject to uniform normal pressure on the upper surface. The exact solution of this classical problem in elliptic coordinates was given by Galerkin [2]. Approximate solutions of this problem have been given by several authors.

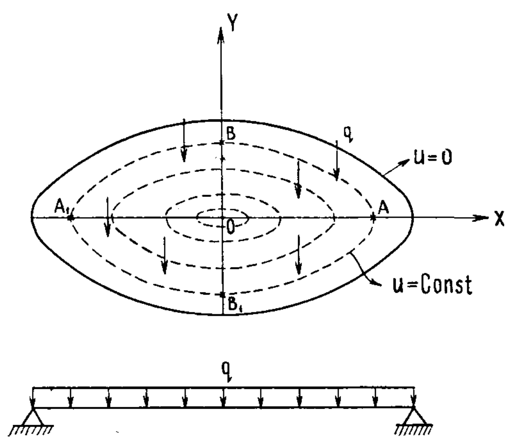

Figure 4

Taking the coordinates as shown in Fig. 4, the equation of the rim of the plate is

$$(3.2) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

At first, limiting ourselves to the first term of the series in (3.1) consider the equation of the lines of equal deflection in the form

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$$(3.3) \qquad u(x, y) = A_{00}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right);$$

without loss of generality, we can, for the sake of simplicity, set $A_{00} = 1$. Thus the equation of the lines of equal deflection reduces to the form

$$(3.4) \qquad u(x, y) = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2};$$

of course, we know from symmetry that (3.4) is an exact equation of the lines of equal deflection for the clamped elliptic plate under uniform loading.

It is noted that $u = 1$ at the centre and $u = 0$ on the boundary of the plate. Calculating the values of the expressions in (1.7), we obtain

$$(3.5) \qquad \begin{aligned} P &= -\frac{4D}{p^2}, \\ Q &= 2Dp^2\left[\frac{\mu(1-u)}{a^2b^2}+\frac{x^2}{a^6}+\frac{y^2}{b^6}\right], \\ R &= -\frac{8D}{p^3}, \\ F &= 4Dp\left[\frac{1-u}{a^2b^2}+3\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}\right)\right], \\ G &= \frac{2D(1-\mu)}{a^2b^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)p^5\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)(1-u), \\ t &= \frac{4}{p^2}, \end{aligned}$$

where

$$(3.6) \qquad p^2 = \frac{1}{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}}.$$

Substituting the above expression into (1.9), we obtain the equation

$$(3.7) \qquad \begin{aligned} &-8D\frac{d^3w}{du^3}\oint\frac{1}{p^3}\,ds+4D\frac{d^2w}{du^2}\oint p\left[\frac{1-u}{a^2b^2}+3\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}\right)\right]ds \\ &+\frac{2D(1-\mu)}{a^2b^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\frac{dw}{du}\oint p^5\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)(1-u)ds-\iint_\Omega q\,dxdy = 0, \end{aligned}$$

where the contour integrations are taken around the closed contour

$$u = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2} = \text{Const.}$$

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and the double integration extends over ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-u.$$

The values of these integrals are found to be

$$\oint \frac{1}{p^3}\,ds=\frac{\pi}{4}\,\frac{(1-u)^2}{a^3b^3}\,(3a^4+2a^2b^2+3b^4),$$

$$\oint p\,ds=2\pi ab,$$

$$\oint p\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}\right)ds=\pi ab(1-u)\left(\frac{1}{a^4}+\frac{1}{b^4}\right), \tag{3.8}$$

$$\oint p^5\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)ds=0,$$

$$\iint_\Omega dxdy=\pi ab(1-u).$$

With the help of (3.8), the differential equation (3.7) reduces finally to the form

$$\frac{d^3w}{du^3}-\frac{2}{1-u}\,\frac{d^2w}{du^2}=-\frac{q_1}{1-u}, \tag{3.9}$$

where

$$q_1=\frac{a^4b^4}{3a^4+2a^2b^2+3b^4}\cdot\frac{q}{2D}. \tag{3.10}$$

Thus the problem of bending of uniformly loaded elliptic plates reduces to the solution of the ordinary third order differential equation (3.9) with the solution

$$w=\frac{q_1}{4}\,(1-u)^2+A\log(1-u)+B(1-u)+C, \tag{3.11}$$

where A, B and C are constants of integration. The constants A, B and C are now to be determined from the conditions at the edges as well as at the centre of the plate. These conditions are given by

(3.12)
(i) $w=0$ for $u=0$,

(ii) $P\dfrac{d^2w}{du^2}+Q\dfrac{dw}{du}=0$ for $u=0$,

(iii) $\sqrt{1-u}\,\dfrac{dw}{du}=0$ for $u=1$;

further, clearly, dw/du is finite for $u=1$.

The condition (iii) is here obtained from the consideration that the slope of the deflected surface in the x-direction or in the y-direction must be zero at the centre. As we see, the boundary condition $M_n = 0$ may be satisfied in this particular case only approximately, because the functions P and Q in this equation are not functions of u alone. For example,

$$(3.13) \qquad P = -\frac{4D}{p^2} = 4D\left[\frac{1}{a^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)x^2 - \frac{1-u}{b^2}\right].$$

The dependence of P on x around the line of equal deflection $u =$ const. is shown in the Fig. 5. It is clear that the maximum value of P, as it varies

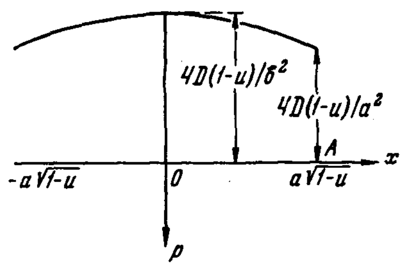

Figure 5

along the line $u =$ const., is attained at the points $A(a\sqrt{1-u}, 0)$ and $A_1(-a\sqrt{1-u}, 0)$ and the minimum value at the points $B(0, b\sqrt{1-u})$ and $B_1(0, -b\sqrt{1-u})$ (Fig. 4). Therefore the mean value of P on the line $u =$ const. is given by

$$(3.14) \qquad P = -2D(1-u)\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$$

Similarly, the mean value of Q on the line $u =$ const. is given by

$$(3.15) \qquad Q = D(1+\mu)\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$$

Let us satisfy the condition $M_n = 0$ on the boundary using the mean values of P and Q. Consequently, we have

$$(3.16) \qquad (1-u)\frac{d^2w}{du^2} - \frac{1+\mu}{2}\frac{dw}{du} = 0, \text{ for } u = 0.$$

With the help of the conditions (3.12) and (3.16), the equation (3.11) gives the following expression for the deflection:

$$(3.17) \quad w = \frac{q_1}{4}u\left(\frac{4}{1+\mu}+u\right) = \frac{q_1}{4}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(\frac{5+\mu}{1+\mu}-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right).$$

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The maximum deflection is at the centre of the plate and, by (3.17),

$$w_{\max} = \frac{q}{D\left(\frac{24}{a^4}+\frac{16}{a^2b^2}+\frac{24}{b^4}\right)}\cdot\frac{5+\mu}{1+\mu}. \tag{3.18}$$

It is interesting to note that for $a = b$, the equation (3.17) reduces to

$$w = \frac{q}{64D}(a^2-x^2-y^2)\left(\frac{5+\mu}{1+\mu}a^2-x^2-y^2\right), \tag{3.19}$$

which is the exact expression for the deflection of a uniformly loaded circular plate with simply supported edge. It is further interesting to note that (3.17) exactly satisfies the biharmonic equation of Sophie Germain, unlike other existing approximate solutions.

In order to estimate the accuracy of this approximate solution, let us calculate the deflection at the centre of the plate ($x = 0$, $y = 0$) for various values of the ratio a/b and $\mu = 0.3$. Assuming $a/b > 1$, we represent the deflection at the centre by the formula

$$(w)_{x=y=0} = \alpha\frac{qb^4}{Eh^3}. \tag{3.20}$$

The numerical values of the constant factor α obtained from (3.18) as well as those given by Galerkin are shown in the following table:

a/b	1	1.1	1.2	1.3	1.4	1.5	2	∞
by (3.18)	0.70	0.83	0.95	1.07	1.16	1.24	1.51	1.86
following Galerkin	0.70	0.83	0.96	1.07	1.17	1.26	1.58	2.28

On the basis of the numerical results in this table, it can be concluded that the expression (3.1) for the lines of equal deflection gives a fairly good solution even in first approximation. However, when necessary, we can always increase the accuracy of the solution by increasing the number of terms in the general expression (3.1). We may therefore, expect that for other cases (at least in the case of uniform pressure) where the equation of the lines of equal deflection is not predetermined, the method outlined above for prescribing the equation of the lines of equal deflection may give us a fairly good result. As we see, the practical importance of the above method stems not so much from the fact that it constitutes an experimental verification for the determination of the lines of equal deflection, but from the fact that it furnishes the basis for an intuitive, qualitative discussion of the lines of equal deflection in cases where its exact determination is cumbersome.

(b) *Uniformly Loaded Parabolic Plate with a Clamped Edge.*

As a second example of the above method, let us consider the case of bending of a parabolic plate which is clamped at the rim and subject to uniformly distributed vertical load. According to the Author, this technically important problem has so far not been solved.

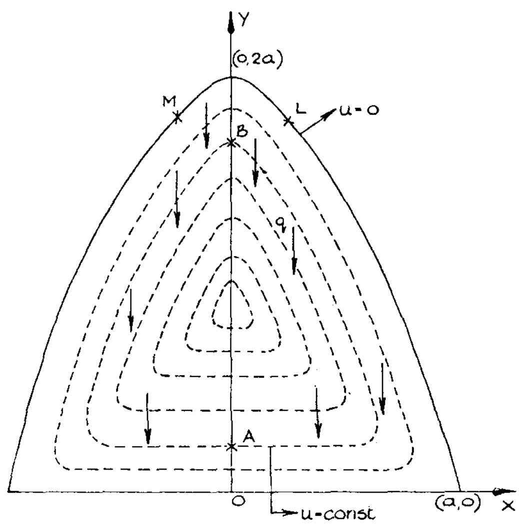

Figure 6

Let the contour of the plate be bounded by the parabola (Fig. 6).

$$x^2 = \frac{a}{2}(2a-y) \tag{3.21}$$

and the line

$$y = 0. \tag{3.22}$$

In agreement with the method outlined above, we will select for the equation of the lines of equal deflection

$$u(x, y) = y\left[\frac{a}{2}(2a-y)-x^2\right] \tag{3.23}$$

Evidently this expression vanishes along the boundary of the plate. Calculating the values of the expressions in (1.7) we obtain

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$$
\begin{aligned}
t &= 4x^2y^2+x^4+2ax^2y+a^2y^2-2a^2x^2-2a^3y+a^4,\\
R &= -Dt^{\frac{3}{2}},\\
F &= \frac{D}{t^{\frac{1}{2}}}\left[24x^2y^3+(a^2-ay-x^2)^2(3a+2y)-16x^2y(a^2-ay-x^2)+4ax^2y^2\right],\\
G &= \frac{D}{t^{\frac{3}{2}}}\big[2(2-\mu)(a^2-ay-x^2)^3+8(2\mu-1)x^2y^2(a^2-ay-x^2)-4(1-\mu)x\\
&\quad\times\{4xy^2(a^2-ay-x^2)+2x(a^2-ay-x^2)^2+8x^3y^2+2axy(a^2-ay-x^2)\}\\
&\quad-(1-\mu)(2y-a)\{2y(a^2-ay-x^2)^2-4ax^2y^2\}\big]\\
&\quad+\frac{2D(1-\mu)}{t^{\frac{5}{2}}}\left[2x\{4x^2y^2-(a^2-ay-x^2)^2\}+2xy(a^2-ay-x^2)(2y-a)\right]^2.
\end{aligned}
$$

In the present case, the formula (1.9) for the deflection of a thin plate turns out to be complicated, because the evaluation of the integrals $\oint Rds$, $\oint Fds$, $\oint Gds$ and $\iint_\Omega qd\Omega$ appearing in the equation is not simple. As a consequence the following procedure may be adopted. Using Green's formula, the double integral $\iint_\Omega qd\Omega$ is transformed into a contour integral of the form $\oint Tds$ and then the line integrals $\oint Rds, \cdots, \oint Tds$ are evaluated approximately by the mean value theorem

$$
\oint (R, F, G, T)ds = (\bar{R}, \bar{F}, \bar{G}, \bar{T})S, \tag{3.25}
$$

where, as usual, $\bar{R}, \bar{F}, \bar{G}, \bar{T}$ denote the mean values of R, F, G, T on the contour $u =$ Const. with the perimeter S. However, in this particular case, for the sake of simplicity, we shall take $\bar{R}, \bar{F}, \bar{G}, \bar{T}$ as the arithmetic mean values of R, F, G, T evaluated at the points of intersection of the lines $u =$ Const. and $x = 0$, i.e., at

$$
A\left[0, a\left(1-\sqrt{1-\frac{2u}{a^3}}\right)\right] \quad \text{and} \quad B\left[0, a\left(1+\sqrt{1-\frac{2u}{a^3}}\right)\right] \quad \text{(Fig. 6).}
$$

We thus obtain

$$
\begin{aligned}
\bar{R} &= -Da^6\left(1-\frac{2u}{a^3}\right)^{\frac{3}{2}},\\
\bar{F} &= 5Da^3\left(1-\frac{2u}{a^3}\right)^{\frac{1}{2}},\\
\bar{G} &= \frac{-4D(1-\mu)}{\left(1-\frac{2u}{a^3}\right)^{\frac{1}{2}}}\left(\frac{3}{2}-\frac{2u}{a^3}\right),\\
\bar{T} &= \frac{aq}{2}.
\end{aligned} \tag{3.26}
$$

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The differential equation (1.9) finally reduces to

$$a^6\left(1-\frac{2u}{a^3}\right)^2\frac{d^3w}{du^3}-5a^3\left(1-\frac{2u}{a^3}\right)\frac{d^2w}{du^2}+4(1-\mu)\left(\frac{3}{2}-\frac{2u}{a^3}\right)\frac{dw}{du} = -\frac{aq}{2D}\left(1-\frac{2u}{a^3}\right)^{\frac{1}{2}} \tag{3.27}$$

Let us now consider the boundary conditions at the edge of the plate. If the plate is assumed to be clamped, the corresponding boundary conditions are

$$w\big]_{u=0}=0,\quad \frac{dw}{du}\bigg]_{u=0}=0. \tag{3.28}$$

From the symmetry of the loading, we conclude that the inclination of the deflected surface at the centre on the direction of axis Oy must be zero. The centre is obviously a point on the y-axis which is obtained by considering the extremum value of the function $u(x, y)$. The value of $u(x, y)$ at the centre is found to be $a^3/2$. Consequently, we have at the centre

$$\sqrt{1-\frac{2u}{a^3}}\,\frac{dw}{du}\bigg]_{u=a^3/2}=0, \tag{3.29}$$

where, clearly,

$$\frac{dw}{du}\bigg|_{u=a^3/2}$$

is finite. In order to solve the differential equation (3.29), let us substitute

$$1-\frac{2u}{a^3}=v^2;\quad \sqrt{1-\frac{2u}{a^3}}\,\frac{dw}{du}=Y. \tag{3.30}$$

In terms of the new variables (3.27) takes the form ($\mu=\frac{1}{3}$)

$$v^2\frac{d^2Y}{dv^2}+2v\frac{dY}{dv}-\frac{2}{3}(1-4v^2)Y=-\frac{aq}{2D}v^2 \tag{3.31}$$

and the conditions given by (3.28) and (3.29) become

$$w\big|_{v=1}=0,\quad Y\big|_{v=1}=0,\quad Y\big|_{v=0}=0. \tag{3.32}$$

Any conventional method may be used to solve the differential equation (3.31). Let us solve this equation by the method of Galerkin, assuming the function Y, in conjunction with the specified boundary conditions, to be of the form

$$Y=v(1-v^2)(a_0+a_1v^2+a_2v^4+\cdots). \tag{3.33}$$

The coefficient a_0, in the first approximation

$$Y = a_0 v(1-v^2), \tag{3.34}$$

is determined by the usual orthogonality condition, from which we find

$$a_0 = 0 \cdot 10362 \frac{aq}{D}; \tag{3.35}$$

we thus, finally, obtain

$$w = a_0 \frac{u^2}{a^3} = 0 \cdot 10362 \frac{q}{a^2 D} y^2 \left[\frac{a}{2}(2a-y)-x^2\right]^2. \tag{3.36}$$

The maximum deflection occurs at the centre and is given by

$$(w)_{\max} = 0 \cdot 0259 \frac{qa^4}{D}. \tag{3.37}$$

We may, however, improve this result by considering more number of terms in the equation (**3.33**).

Let us now examine the distribution of the bending moments M_n and M_t along the edge of the plate. One can easily see that the maximum value of

$$t = 4x^2y^2 + (a^2-ay-x^2)^2 \tag{3.38}$$

at the edge of the plate occurs at $(\pm\sqrt{\frac{7}{24}}a, \frac{17}{12}a)$ and the minimum value at $(\pm a, 0)$ and, therefore, in the light of what has been discussed at the end of Section 2, one can infer that the strongest points with respect to bending moments appear to be at the end of the horizontal axis and the weakest at

$$L(\sqrt{\tfrac{7}{24}}a, \tfrac{17}{12}a) \quad \text{and} \quad M(-\sqrt{\tfrac{7}{24}}a, \tfrac{17}{12}a) \quad \text{(Fig. 6)}.$$

Clearly, great care must be taken in such situations.

4. Additional remarks

An estimate can be made of the error involved in the use of the foregoing method. In fact, as has been shown by the Author [**3**], the error involved in prescribing the above type of equation of lines of deflection is found to be only 14 %. It should be noted that the method described in the present paper can be extended without difficulty to the study of the stability of elastic plates of arbitrary shape (see [**4**]).

It is believed that the concepts given here may be developed further to give a general theory of the equation of lines of equal deflection. The author hopes to pursue this and other related questions in a later paper.

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