ENTIRE SOLUTIONS OF THE FUNCTIONAL EQUATION $f(f(z)) = g(z)$

W. J. THRON

In this note it is proved that: *the functional equation*

$$f(f(z)) = g(z), \tag{1}$$

where $g(z)$ is an entire function of finite order, which is not a polynomial, and which takes on a certain value ρ only a finite number of times, does not have a solution $f(z)$ which is an entire function.

The problem solved here arose from the special case $g(z) = e^z - 1$, which was brought to the author's attention by S. Chowla, who, together with A. J. Kempner and T. Rivlin, had proved that even though a formal power series expansion

$$\sum_{n=1}^{\infty} a_n z^n$$

exists for the solution of the functional equation (1) the function cannot be entire. Further investigations of the author seem to indicate that the radius of convergence of this series is zero.

Other aspects of the subject here discussed are considered in recent articles by Hadamard **(1)**, Isaacs **(2)**, and Kneser **(3)**. These authors also give references to earlier work done in this field.

We begin our proof with an application of a theorem of Polya **(4)**. This theorem states that: if $r(z)$ and $s(z)$ are entire functions, and if $r(s(z))$ is of finite order, then either $s(z)$ is a polynomial, and $r(z)$ is of finite order, or $s(z)$ is not a polynomial but of finite order, and $r(z)$ is of order zero. Applying this theorem to $f(f(z))$, which is to be of finite order, we can conclude that $f(z)$ is of order zero. Also, since $f(f(z))$ is assumed not to be a polynomial, $f(z)$ cannot be a polynomial.

Now let $\{f_m\}$ be the set of all values of z for which $f(z) = \rho$. Here the possibility that this set is empty is included. Denote by $z_k^{(m)}$ all those values of z for which $f(z) = f_m$. Then

$$f(f(z_k^{(m)})) = g(z_k^{(m)}) = f(f_m) = \rho.$$

Since the number of values of z for which $g(z) = \rho$ is finite, and since, in view of the above relation, all $z_k^{(m)}$ have this property, there can only be a finite number of $z_k^{(m)}$. Hence $f(z)$ takes on each value f_m only a finite number of times. According to Picard's theorem there exists at most one value which an entire

Received January 17, 1955.

https://doi.org/10.4153/CJM-1956-008-4 Published online by Cambridge University Press

function, not a polynomial, takes on only a finite number of times. It follows either that all f_m are equal to one value, say f^*, or that no f_m, with $f(f_m) = \rho$, exists.

Thus in the first case the only value of z for which $f(z) = \rho$ is f^*, and we can write

$$f(z) = \rho + (z - f^*)^n h(z),$$

where $h(z)$ does not vanish and is an entire function of order zero. It follows from Hadamard's factorization theorem (5, p. 250) that $h(z)$ is a constant. The function $h(z)$ then is a polynomial which we know to be impossible.

In the second case we have immediately, by an analogous argument, that $f(z) = \rho + c$, and thus again a contradiction. Since in the case $g(z) = e^z - 1$ the value -1 is never taken on by $g(z)$ the result mentioned in the beginning of this note is seen to be a special case of our theorem.

References

1. J. Hadamard, *Two works on iteration and related questions*, Bull. Amer. Math. Soc., *50* (1944), 67–75.
2. R. Isaacs, *Iterates of fractional order*, Can. J. Math., *2* (1950), 409–416.
3. H. Kneser, *Reelle analytische Lösungen der Gleichung $\varphi(\varphi(x)) = e^x$ und verwandter Funktionalgleichungen*, J. reine angew. Math., *187* (1950), 56–67.
4. G. Pólya, *On an integral function of an integral function*, J. London Math. Soc., *1* (1926), 12–15.
5. E. C. Titchmarsh, *The Theory of Functions* (2nd ed., Oxford, 1939).

University of Colorado