MULTIPLICATIVE COMMUTATORS OF OPERATORS

To Professor Paul Halmos on the occasion of his fiftieth birthday

ARLEN BROWN AND CARL PEARCY

1. Introduction. An invertible operator T on a Hilbert space $\mathfrak{H}$ is a *multiplicative commutator* if there exist invertible operators A and B on $\mathfrak{H}$ such that $T = ABA^{-1}B^{-1}$. In this paper we discuss the question of which operators are, and which are not, multiplicative commutators. The analogous question for additive commutators (operators of the form $AB - BA$) has received considerable attention and has, in fact, been completely settled **(2)**. The present results represent the information we have been able to obtain by carrying over to the multiplicative problem the techniques that proved efficacious in the additive situation. While these results remain incomplete, they suffice, for example, to enable us to determine precisely which normal operators are multiplicative commutators.

In what follows we restrict our attention to complex, separable, infinite-dimensional Hilbert space. (It is known **(6)** that a necessary and sufficient condition for an operator on a finite-dimensional Hilbert space to be a multiplicative commutator is that it have determinant 1; as for the hypothesis of separability, it would be easy, but not particularly rewarding, to state all the theorems below so as to make them valid on non-separable spaces.) Let $\mathfrak{H}$ be a separable Hilbert space, and let $\mathfrak{L}(\mathfrak{H})$ denote the algebra of all bounded linear operators on $\mathfrak{H}$. The solution of the additive commutator problem may be described as follows. If $T \in \mathfrak{L}(\mathfrak{H})$ is not congruent to a scalar modulo the ideal (C) of compact operators (these are the operators of *class* (F) of **(2)**), then T is always an additive commutator; if T is congruent to a scalar λ modulo (C), then T is an additive commutator if and only if $\lambda = 0$.

It is plausible to suppose that the multiplicative problem will have a solution that parallels the solution of the additive problem, and this supposition is borne out by the results presented below. A basic distinction appears to subsist between the operators of class (F) and those that are scalar modulo (C). Regarding the former, we conjecture that an arbitrary operator of class (F) (if invertible) is a multiplicative commutator. Our best result in that direction is the theorem that the direct sum of two operators of class (F) is a multiplicative commutator (Theorem 4). As for operators of the form $\lambda + C$ with C compact, it seems most likely that such an operator is a multiplicative commutator provided only that $|\lambda| = 1$. (The converse is certainly valid and

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appears as Theorem 1 below.) In particular, these conjectures are shown to be correct for normal operators (Theorem 5) so that, as noted, our results are definitive in this special case.

It is also tempting to surmise that additive and multiplicative commutators might be somehow related via exponentiation. That such a relation, even supposing one to exist, cannot be a simple one is indicated by the following facts:

(1) e^T may be a multiplicative commutator when T is not an additive commutator.

(2) e^T may fail to be a multiplicative commutator even when T is an additive commutator.

(3) There exist multiplicative commutators that cannot be expressed as e^T for any T.

2. A negative result. The symbols (C), (F), and $\mathfrak{L}(\mathfrak{H})$ introduced above will be used consistently in what follows. Moreover, from now on, we write commutator instead of multiplicative commutator when no confusion can result.

The substance of the present paragraph is the following theorem, which establishes a class of non-commutators.*

THEOREM 1. *If T is a commutator and if T is congruent to λ modulo (C), then* $|\lambda| = 1$.

Proof. The proof rests on the observation, valid in any Banach algebra with unit, that no scalar λ satisfying $|\lambda| \neq 1$ can be a commutator. For if $\lambda = ABA^{-1}B^{-1}$, then $AB = \lambda BA$ and, denoting as usual the spectrum of X by $\Lambda(X)$, one has $\Lambda(AB) = \lambda\Lambda(BA)$. But also $\Lambda(AB) = \Lambda(BA)$ since AB and BA are similar, so that the equation $\Lambda(AB) = \lambda\Lambda(AB)$ is obtained. Since $\Lambda(AB)$ is compact and $\Lambda(AB) \neq \{0\}$, the last relation is clearly possible only when $|\lambda| = 1$.

Suppose now that T is a commutator in $\mathfrak{L}(\mathfrak{H})$ and that $T = \lambda + C$ where C is compact. Then in the quotient algebra $\mathfrak{L}(\mathfrak{H})/(C)$ the scalar λ is a commutator, and the theorem is proved.

3. A basic construction. Let $\mathfrak{K}$ be a Hilbert space, which for present purposes may even be finite-dimensional, and let

$$\mathfrak{K}^* = \sum_{-\infty}^{\infty} \oplus \mathfrak{K} = \ldots \oplus \mathfrak{K} \oplus (\mathfrak{K}) \oplus \mathfrak{K} \oplus \ldots$$

denote the two-way infinite direct sum of copies of $\mathfrak{K}$. In other words, $\mathfrak{K}^*$ is the Hilbert space of square-summable sequences $\{\ldots, x_{-1}, (x_0), x_1, \ldots\}$ of elements of $\mathfrak{K}$. (The parentheses are used here to indicate the 0th term of the

*Theorem 1 and a slightly weaker version of Corollary 4.7 have been known to P. R. Halmos for some time, but were never published.

sequence.) If $\{A_n\}_{n=0,\pm 1,\pm 2,\ldots}$ is a sequence of invertible operators in $\mathfrak{L}(\mathfrak{K})$, then the mapping

$$\{\ldots, x_{-1}, (x_0), x_1, \ldots\} \to \{\ldots, A_{-1}x_{-1}, (A_0 x_0), A_1 x_1, \ldots\}$$

defines an invertible operator on $\mathfrak{K}^*$ if and only if the sequences $\{||A_n||\}$ and $\{||A_n^{-1}||\}$ are both bounded. For such an operator, which is, of course, just the direct sum of the A_n, we employ the notation $\mathrm{diag}(A_n)$, in view of the obvious matricial interpretation. Clearly $[\mathrm{diag}(A_n)]^{-1} = \mathrm{diag}(A_n^{-1})$. We construct commutators on $\mathfrak{K}^*$ by using such diagonal operators and another familiar operator, the *bilateral shift* U defined by

$$U\{\ldots, x_{-1}, (x_0), x_1, \ldots\} = \{\ldots, x_{-2}, (x_{-1}), x_0, \ldots\}.$$

It is well known that U is a unitary operator with inverse given by the shift in the opposite direction:

$$U^{-1}\{\ldots, x_{-1}, (x_0), x_1, \ldots\} = \{\ldots, x_0, (x_1), x_2, \ldots\}.$$

Suppose now that $\mathrm{diag}(A_n)$ is an invertible operator on $\mathfrak{K}^*$. For any x in $\mathfrak{K}$ and fixed index k, we write $x_{(k)}$ for the vector in $\mathfrak{K}^*$ that has x in the kth position and all other entries equal to zero. An easy calculation then shows that

$$U\,\mathrm{diag}(A_n)\,U^{-1}\,\mathrm{diag}(A_n^{-1})x_{(k)} = [A_{k-1}A_k^{-1}x]_{(k)}$$

and hence that

$$(*) \qquad U\,\mathrm{diag}(A_n)\,U^{-1}\,\mathrm{diag}(A_n^{-1}) = \mathrm{diag}(C_n)$$

where

$$C_n = A_{n-1}A_n^{-1}, \qquad n = 0, \pm 1, \ldots.$$

Now this equation may be solved for the A_n in terms of the C_n. Indeed, if $\mathrm{diag}(C_n)$ is a given invertible operator on $\mathfrak{K}^*$, we have but to define

$$(**) \qquad \begin{aligned} A_0 &= 1, \\ A_n &= (C_1 \ldots C_n)^{-1}, \qquad n = 1, 2, \ldots, \\ A_n &= C_{n+1} \ldots C_0, \qquad n = -1, -2, \ldots, \end{aligned}$$

to ensure that, formally at least, $(*)$ is satisfied. Motivated by these equations, we say that a sequence $\{C_n\}_{n=0,\pm 1,\pm 2,\ldots}$ of invertible operators on $\mathfrak{K}$ is *multipliable* if both of the sequences $\{||A_n||\}_{n=0,\pm 1,\pm 2,\ldots}$ and $\{||A_n^{-1}||\}_{n=0,\pm 1,\pm 2,\ldots}$ are bounded where the A_n are the products appearing in $(**)$.

The following lemma summarizes the above remarks.

LEMMA 3.1. *If the sequence $\{C_n\}_{n=0,\pm 1,\pm 2,\ldots}$ of invertible operators on $\mathfrak{K}$ is multipliable, then* $\mathrm{diag}(C_n)$ *is a commutator on* $\mathfrak{K}^*$.

In order to be able to apply the above construction to a given operator on an abstract Hilbert space, it is necessary that the operator possess a large supply of reducing subspaces. This is a major obstacle, and essentially limits our application of the lemma to operators that have a large normal direct summand. Of course it is also necessary to arrange the reducing subspaces in such a way that, at the appropriate moment, multipliability can be established. In the following theorems we employ, first, the trivial observation that every sequence $\{C_n\}$ of unitary operators is multipliable and, secondly, the easily verified fact that if

$$\sum_{-\infty}^{\infty} ||C_n|| < \infty,$$

then $\{e^{C_n}\}$ is multipliable. As usual, the notation $A|\mathfrak{M}$ is employed for the restriction of an operator A to an invariant subspace $\mathfrak{M}$.

THEOREM 2. *If T is an invertible operator on $\mathfrak{H}$ that admits an infinite-dimensional reducing subspace $\mathfrak{N}$ such that $T|\mathfrak{N}$ is a unitary operator, then T is a commutator.*

Proof. Observe first that any unitary operator W on an infinite-dimensional Hilbert space can be split into the direct sum of two infinite-dimensional operators, both of which are themselves necessarily unitary. (This is an easy exercise in spectral theory. If $\Lambda(W)$ is infinite, we can divide $\Lambda(W)$ into two infinite disjoint Borel sets and use the corresponding spectral projections to split W; if $\Lambda(W)$ is finite, we have but to split some one infinite-dimensional eigenspace of W.) It follows by induction that W can be split into the direct sum of infinitely many unitary operators each acting on an infinite-dimensional space. In particular this is true of $W = T|\mathfrak{N}$.

As a first application of this last remark, we note that we may assume that $\mathfrak{H} \ominus \mathfrak{N}$ is infinite-dimensional also, for if this were not so, we could replace $\mathfrak{N}$ by a smaller reducing subspace and make it so. Under that assumption, let $\mathfrak{M}_0 = \mathfrak{H} \ominus \mathfrak{N}$. Next split $\mathfrak{N}$ as indicated into the direct sum of an infinite family of infinite-dimensional subspaces, all reducing for T, and enumerate these subspaces as $\{\mathfrak{M}_n\}_{n=\pm 1,\pm 2,\ldots}$. Finally, choose a fixed infinite-dimensional Hilbert space $\mathfrak{K}$ and, for each n, a unitary isomorphism of $\mathfrak{M}_n$ onto $\mathfrak{K}$. The direct sum of these isomorphisms is a unitary isomorphism of $\mathfrak{H}$ onto $\mathfrak{K}^*$ that carries T onto a diagonal operator $\operatorname{diag}(C_n)$ on $\mathfrak{K}^*$. Since C_n is unitary for $n \neq 0$, it is clear that $\{C_n\}_{n=0,\pm 1,\pm 2,\ldots}$ is multipliable, and since unitary isomorphisms carry commutators to commutators, an application of Lemma 3.1 completes the argument.

COROLLARY 3.2. *Every unitary operator (on an infinite-dimensional Hilbert space) is a commutator.*

THEOREM 3. *Every invertible normal operator (on an infinite-dimensional Hilbert space) of the form $\lambda + C$, where $|\lambda| = 1$ and C is compact, is a commutator.*

Proof. We prefer to write the operator as $\lambda(1 + C)$, where of course $1 + C$ is invertible and normal, and C is compact. Since the spectrum of $1 + C$

cannot separate the origin from ∞, there exists a compact, normal logarithm: $D = \log(1 + C)$ so that $1 + C = e^D$. Since D is compact and normal, it is an easy matter to split $\mathfrak{H}$ into a direct sum

$$\mathfrak{H} = \sum_{-\infty}^{\infty} \oplus \mathfrak{M}_n$$

of infinite-dimensional reducing subspaces for D in such a way that

$$\lim_n \|D|\mathfrak{M}_n\| = 0.$$

Furthermore this convergence can be accelerated as much as desired by grouping and relabelling the $\mathfrak{M}_n$. In particular, we may and do arrange the direct-sum decomposition so that

$$\sum_{-\infty}^{\infty} \|D|\mathfrak{M}_n\| < \infty,$$

(The reader is referred to **(1**, §3**)** for a similar construction.)

Now fix a Hilbert space $\mathfrak{K}$, and for each n, choose a unitary isomorphism of $\mathfrak{M}_n$ onto $\mathfrak{K}$. The direct sum of these isomorphisms carries $\mathfrak{H}$ onto $\mathfrak{K}^*$ and D onto a diagonal operator $(\operatorname{diag}(A_n)$ with

$$\sum_{-\infty}^{\infty} \|A_n\| < \infty.$$

Moreover $\lambda(1 + C) = \lambda e^D$ is carried onto $\operatorname{diag}(\lambda e^{A_n})$. Since $\|e^T\| \leqslant e^{\|T\|}$ for any operator T, the summability of the sequence $\{\|A_n\|\}$ implies the multipliability of $\{\lambda e^{A_n}\}$, and the proof is completed by an application of Lemma 3.1.

4. Operators of type (F)**.** In this section we show that the direct sum of any two operators of type (F) is a commutator, and we apply this result to obtain complete information concerning normal commutators. Just as in the study of additive commutators **(2)**, the constructions needed for this purpose involve 2×2 matrices and 3×3 matrices with operator entries. We remind the reader that if A, B, C, and D, are operators on $\mathfrak{H}$, then the matrix

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

defines an operator on $\mathfrak{H} \oplus \mathfrak{H}$ according to the rule

$$T(x, y) = (Ax + By, Cx + Dy).$$

Conversely, any operator $T \in \mathfrak{L}(\mathfrak{H} \oplus \mathfrak{H})$ can be represented as such a 2×2 matrix with entries from $\mathfrak{L}(\mathfrak{H})$. An analogous relationship holds between 3×3 matrices with entries from $\mathfrak{L}(\mathfrak{H})$ and operators in $\mathfrak{L}(\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H})$.

We say that two operators X and X' on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}'$, respectively, are *similar* if there exists a bounded linear transformation $S\colon \mathfrak{H} \to \mathfrak{H}'$ with bounded inverse $S^{-1}\colon \mathfrak{H}' \to \mathfrak{H}$ such that $X' = SXS^{-1}$. Also, if X and Y are operators (on the same or different Hilbert spaces), and if X' and Y' are operators on the same Hilbert space $\mathfrak{H}'$ that are similar to X and Y respectively, then the product $X'Y'$ will be called a *generalized product* of X and Y. In other words, a generalized product of X and Y is any operator of the form

$$(SXS^{-1})(TYT^{-1}).$$

It is easy to see that any operator that is similar to a generalized product of X and Y is another generalized product of X and Y, and that a generalized product of invertible operations is itself invertible.

LEMMA 4.1. *If some generalized product of X and Y is a commutator, then the direct sum $X \oplus Y$ is also a commutator.*

Proof. Note first that an operator that is similar to a commutator is itself a commutator. (This fact will be used frequently below without further notice.) It follows from this observation and the equation

$$SXS^{-1} \oplus TYT^{-1} = (S \oplus T)(X \oplus Y)(S \oplus T)^{-1}$$

that it suffices to treat the case of two operators X and Y on the same Hilbert space with XY a commutator. Let $XY = ADA^{-1}D^{-1}$, and let $C = YDA$, so that $Y = CA^{-1}D^{-1}$ and $X = ADC^{-1}$. Direct calculation shows that

$$\begin{bmatrix} 0 & A \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ A^{-1} & 0 \end{bmatrix}\begin{bmatrix} 0 & D^{-1} \\ C^{-1} & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix},$$

and, since $\begin{bmatrix} 0 & N^{-1} \\ M^{-1} & 0 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}$, the proof is complete.

Remark. This construction can be carried out for generalized products and direct sums of more than two operators, but the 2×2 case suffices for all our purposes.

LEMMA 4.2. *If T is an operator on $\mathfrak{H} \oplus \mathfrak{H}$ of the form*

$$T = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

and if the spectra of A and C are disjoint, then T is similar to

$$\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}.$$

Proof. Since the spectra of A and C are disjoint, it follows from a theorem of Lumer and Rosenblum (**4**, Theorem 10) that there exists an operator $X \in \mathfrak{L}(\mathfrak{H})$ satisfying $CX - XA = -B$. The following computation completes the proof:

$$\begin{bmatrix} 1 & 0 \\ -X & 1 \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ X & 1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}.$$

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LEMMA 4.3. *Let F, G, and S be invertible operators on $\mathfrak{H}$, and suppose F and G are of class (F). Then for all sufficiently large positive numbers t, there exists a generalized product of F and G of the form*

$$\begin{bmatrix} C(t) & 0 \\ 0 & tS \end{bmatrix}$$

on the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}$.

Proof. Since G is of class (F), it is similar (**2**, Theorem 2) to an operator G_1 on $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H}$ of the form

$$G_1 = \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 1 \\ G_{31} & G_{32} & 0 \end{bmatrix}.$$

Applying the same theorem to F and then interchanging the second and third copies of $\mathfrak{H}$, we obtain an operator F_1 similar to F of the form

$$F_1 = \begin{bmatrix} F_{11} & 0 & F_{31} \\ F_{21} & 0 & F_{32} \\ F_{31} & 1 & F_{33} \end{bmatrix}.$$

Since F_1 and G_1 are similar to F and G, it suffices to find a generalized product of F_1 and G_1 of the specified form. Let S_1 and $T(t)$ denote the diagonal operators

$$S_1 = \operatorname{diag}(1, 1, S), \qquad T(t) = \operatorname{diag}(t, t, 1), \quad t > 0.$$

on $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H}$, and write $F_2 = S_1 F_1 S_1^{-1}$ and $G_2(t) = T(t) G_1 T(t)^{-1}$. Direct calculation shows that

$$F_2 = \begin{bmatrix} F_{11} & 0 & F_{13} S^{-1} \\ F_{21} & 0 & F_{23} S^{-1} \\ SF_{31} & S & SF_{33} S^{-1} \end{bmatrix},$$

$$G_2(t) = \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & t \\ \frac{1}{t} G_{31} & \frac{1}{t} G_{32} & 0 \end{bmatrix},$$

and

$$F_2 G_2(t) = \begin{bmatrix} M + \frac{1}{t} \mathrm{N} & 0 \\ * & tS \end{bmatrix},$$

where

$$M = \begin{bmatrix} F_{11} G_{11} & F_{11} G_{12} \\ F_{21} G_{11} & F_{21} G_{12} \end{bmatrix}, \qquad N = \begin{bmatrix} F_{13} S^{-1} G_{31} & F_{13} S^{-1} G_{32} \\ F_{23} S^{-1} G_{31} & F_{23} S^{-1} G_{32} \end{bmatrix}.$$

(We have here taken the liberty of writing the upper left-hand 2×2 block of $F_2 G_2(t)$ as the sum of two 2×2 blocks.) The operator $F_2 G_2(t)$ is, of course, a generalized product of F_1 and G_1, and we complete the proof by showing that, for sufficiently large t, $F_2 G_2(t)$ is similar to an operator of the desired form.

Let ϕ be a unitary isomorphism of $\mathfrak{H} \oplus \mathfrak{H}$ onto $\mathfrak{H}$, and let A and B be the operators on $\mathfrak{H}$ that correspond to M and N under this isomorphism. Then the direct sum of ϕ acting on the first two copies of $\mathfrak{H}$ with the identity operator on the third copy of $\mathfrak{H}$ is a unitary isomorphism of $\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H}$ onto $\mathfrak{H} \oplus \mathfrak{H}$ that carries $F_2 G_2(t)$ onto an operator of the form

$$Z(t) = \begin{bmatrix} A + \frac{1}{t} B & 0 \\ * & tS \end{bmatrix}.$$

Write $C(t) = A + (1/t)B$, and observe that for $t > 1$, $||C(t)|| \leqslant ||A|| + ||B||$. Also note that since S is invertible, $\Lambda(tS) = t\Lambda(S)$ lies entirely outside the circle $|z| = ||A|| + ||B||$ for sufficiently large t. Thus the spectra of $C(t)$ and tS are disjoint for sufficiently large t, and an application of Lemma 4.2 completes the argument.

The emphasis in Lemma 4.3 is on the form of the matrix diag$(*, tS)$; the operator S is entirely at our disposal. The next pair of lemmas show, somewhat surprisingly, that it is possible to construct an invertible S such that diag$(*, tS)$ is almost always a commutator.

LEMMA 4.4. *Let λ and μ be scalars and let $M = M(\lambda, \mu)$ denote the 2×2 diagonal matrix*

$$M(\lambda, \mu) = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.$$

If $\lambda \neq \mu$ and $\lambda^2\mu^2 \neq 1$, then the diagonal matrix $M(\lambda^2\mu^2, 1)$ is a generalized product of M with itself.

Proof. Consider the matrix

$$T_\beta = \begin{bmatrix} \beta + 1 & \beta \\ 1 & 1 \end{bmatrix}$$

which has inverse

$$T_\beta^{-1} = \begin{bmatrix} 1 & -\beta \\ -1 & \beta + 1 \end{bmatrix}$$

for all scalars β. For every β, $T_\beta M T_\beta^{-1} M = Q(\beta)$ is a generalized product of M with itself, and consideration of the characteristic equation of $Q(\beta)$ yields the fact that if β_0 is defined by

$$\beta_0 = \frac{(1 - \mu^2)(1 - \lambda^2)}{(\mu - \lambda)^2},$$

then $Q(\beta_0)$ has eigenvalues 1 and $\lambda^2\mu^2$. Since by assumption $\lambda^2\mu^2 \neq 1$, $Q(\beta_0)$ is similar to the diagonal matrix $M(\lambda^2\mu^2, 1)$, and the proof is complete.

LEMMA 4.5. *There exists an invertible operator S on $\mathfrak{H}$ with the property that* $\mathrm{diag}(C, tS)$ *is a commutator for every invertible $C \in \mathfrak{L}(\mathfrak{H})$ and all positive $t \neq 1$.*

Proof. We begin by defining an invertible operator S such that (a) S is spatially isomorphic to the direct sum of two (or more) copies of itself, and also such that (b) for the stated values of t, there exists a generalized product of tS with itself that is unitary on a large reducing subspace. The balance of the argument shows that such an S satisfies the conditions of the lemma.

On the basis of the preceding construction it is easy to write down such an S. Fix a positive number $\lambda_0 \neq 1$, let $\mu_0 = \lambda_0^{-1}$, and write

$$S = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{bmatrix}$$

where the matrix entries denote scalar operators on an infinite-dimensional space. Then $S = M(\lambda_0, \mu_0)$ in the notation of Lemma 4.4, and since

$$tS = M(t\lambda_0, t\mu_0),$$

a direct application of that lemma shows that

$$N = \begin{bmatrix} t^4 & 0 \\ 0 & 1 \end{bmatrix}$$

is a generalized product of tS with itself whenever $t > 0$, $t \neq 1$.

In particular, this generalized product N acts as the identity operator on an infinite-dimensional reducing subspace so S satisfies (b). That S satisfies (a) is clear; in fact, S is indistinguishable from the direct sum of infinitely many copies of itself.

It remains to show that S satisfies the conditions of the lemma. Let C be an invertible element of $\mathfrak{L}(\mathfrak{H})$, and let t be a fixed positive number different from 1. We first split S into the direct sum of three copies of itself and write (up to spatial isomorphism)

$$\mathrm{diag}(C, tS) = \mathrm{diag}(C \oplus tS, tS \oplus tS).$$

According to Lemma 4.1, the lemma will be proved if we exhibit a generalized product of $C \oplus tS$ and $tS \oplus tS$ that is a commutator. Let N be as above; i.e., a generalized product of tS with itself that is the identity operator on an infinite-dimensional reducing subspace. Clearly $tCS \oplus N$ is a generalized product of $C \oplus tS$ and $tS \oplus tS$, and, since $tCS \oplus N$ is a commutator by Theorem 2, the proof is complete.

This last lemma essentially completes the proof of the following main result.

THEOREM 4. *If F and G are both invertible operators of class (F), then $F \oplus G$ is a commutator.*

Proof. Lemmas 4.3 and 4.5 together say that there exists a generalized product of F and G that is a commutator, and the theorem then follows from Lemma 4.1.

COROLLARY 4.6. *Every invertible normal operator of type* (F) *is a commutator.*

Proof. A normal operator is of type (F) if and only if its spectrum contains at least two limit points. (A point λ is a limit point of the spectrum of a normal operator if every Borel neighbourhood of λ has infinite spectral measure.) Using this observation and the spectral theorem one shows easily that every invertible normal operator of class (F) is a direct sum of two (necessarily invertible) infinite-dimensional operators of class (F), and the proof is completed by applying Theorem 4.

Corollary 4.6, together with Theorems 1 and 3, yields definitive information about normal commutators.

THEOREM 5. *An invertible normal operator is a commutator if and only if it is not of the form* $\lambda + C$ *where* C *is compact and* $|\lambda| \neq 1$.

This theorem enables us to show that the commutator subgroup of the group G of invertible operators (on a separable infinite-dimensional Hilbert space) is G itself. In fact, we obtain the following better result.

COROLLARY 4.7. *Every invertible operator is the product of two commutators.*

Proof. Any invertible operator A has a polar decomposition $A = UP$ where U is unitary and P is positive definite. If P is of type (F), the proof is completed by applying Corollary 3.2 and Theorem 5 to U and P respectively. If P is of the form $P = \lambda + C$ for C compact, then C is necessarily Hermitian, and one can easily construct a unitary operator W of class (F) such that W commutes with P. Then $A = (UW^*)(WP)$, where UW^* is unitary, and WP is a normal operator of class F. (The fact that WP is normal follows from $WP = PW$; the fact that WP is of class (F) follows from the observation that the inverse of any operator of the form $\lambda + C$ is again of that form.) The proof is completed by applying Corollary 3.2 and Theorem 5 to the operators UW^* and WP respectively.

5. Two further results. A central role in the theory of additive commutators is played by the theorem (**5**, Theorem 1) that every operator of the form

$$\begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix}$$

(on the direct sum of an infinite-dimensional Hilbert space with itself) is a commutator. An analogous result in the present setting would be that every invertible operator of the form

$$\begin{bmatrix} * & 0 \\ * & 1 \end{bmatrix}$$

is a multiplicative commutator, and it seems plausible that a proof of this would likewise lead to further progress in the structure theory of multiplicative commutators. (It is easily seen that this result must hold if the conjectures stated in the introduction are valid.) While we have been unable to establish this result in general, we present two theorems in this section that bear on the problem and serve to establish special cases.

First we note the following lemma.

LEMMA 5.1. *Let T denote the triangular operator*

$$T = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}$$

on $\mathfrak{H} \oplus \mathfrak{H}$. If X and Z are both invertible, then so is T and

$$T^{-1} = \begin{bmatrix} X^{-1} & 0 \\ -Z^{-1}YX^{-1} & Z^{-1} \end{bmatrix}.$$

On the other hand, if T is invertible on $\mathfrak{H} \oplus \mathfrak{H}$, and if at least one diagonal entry has range $\mathfrak{H}$, then both X and Z must be invertible.

Proof. The first half of the lemma is proved by multiplying T on the left and right by the given candidate for T^{-1}. Suppose now that T is invertible and one diagonal entry, say Z, has range $\mathfrak{H}$. Let T^{-1} be the matrix

$$\begin{bmatrix} K & M \\ L & N \end{bmatrix}.$$

Since $T^{-1}T = 1$, we obtain $MZ = 0$ and thus $M = 0$. Hence $KY = NZ = 1$. Similarly, using the fact that $TT^{-1} = 1$, we have $XK = ZN = 1$, and the result follows.

THEOREM 6. *If X and Z are both multiplicative commutators on $\mathfrak{H}$, and Y is any operator on $\mathfrak{H}$, then*

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}$$

is a commutator on $\mathfrak{H} \oplus \mathfrak{H}$.

Proof. Choose A, C, K, and M so that $AKA^{-1}K^{-1} = X$ and $CMC^{-1}M^{-1} = Z$. Choose $\lambda > 0$ large enough so that the spectra of $\lambda^{-1}A^{-1}$ and $C^{-1}Z$ are disjoint; let $A_\lambda = \lambda A$, and note that $A_\lambda K(A_\lambda)^{-1}K^{-1} = X$. Apply **(4,** Theorem **10)** once again to obtain an operator L such that

$$(C^{-1}Z)L - L(A_\lambda)^{-1} = -C^{-1}YK,$$

and observe that

$$CLA_\lambda^{-1}K^{-1} - CMC^{-1}M^{-1}LK^{-1} = CLA_\lambda^{-1}K^{-1} - ZLK^{-1} = Y.$$

An easy calculation now shows that

$$\begin{bmatrix} A_\lambda & 0 \\ 0 & C \end{bmatrix} \cdot \begin{bmatrix} K & 0 \\ L & M \end{bmatrix} \cdot \begin{bmatrix} A_\lambda^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} \cdot \begin{bmatrix} K^{-1} & 0 \\ -M^{-1}LK^{-1} & M^{-1} \end{bmatrix} = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}.$$

THEOREM 7. *Every invertible operator on $\mathfrak{H} \oplus \mathfrak{H}$ of the form*

$$T = \begin{bmatrix} N & 0 \\ * & 1 \end{bmatrix}$$

is a commutator provided N is normal.

Proof. The normal operator N is invertible by Lemma 5.1. If N is a commutator, then so is T by Theorem 6. Hence, according to Theorem 5, we may assume that $N = \lambda + C$, where $|\lambda| \neq 1$ and C is normal and compact. Moreover, if the scalar 1 does not lie in the spectrum of N, then, by Lemma 4.2, T is similar to $N \oplus 1$ and so is a commutator by Theorem 2. On the other hand, if 1 belongs to the spectrum of N, it must appear as an isolated eigenvalue of finite multiplicity, say of multiplicity k. Hence there exists an orthonormal basis $\{e_n\}_{n=1,2,\ldots}$ in $\mathfrak{H}$ with respect to which the matrix of N is diagonal, with the first k diagonal entries equal to 1 and all other entries bounded away from 1. By removing the finite-dimensional subspace $\bigvee\{e_1, \ldots, e_k\}$ from the first copy of $\mathfrak{H}$ in $\mathfrak{H} \oplus \mathfrak{H}$ and adding it (directly) to the second copy, it is possible to construct a unitary equivalence between T and an operator T_1 on $\mathfrak{H} \oplus \mathfrak{H}$ of the form

$$T_1 = \begin{bmatrix} M & 0 \\ X & 1 + F \end{bmatrix},$$

where M is now a normal operator that does not have 1 in its spectrum and F is a nilpotent operator of finite rank. Since the spectra of M and $1 + F$ are disjoint, we may apply Lemma 4.2 to show that T_1 is similar to

$$M \oplus (1 + F).$$

Finally, since F has finite rank, it is clear that $1 + F$ has a large reducing subspace on which it acts as the identity operator, whence it follows from Theorem 2 that $M \oplus 1 + F$ is a commutator. Thus T is a commutator and the proof is complete.

6. Concluding remarks.

I. Examples of the phenomena labelled (1), (2), and (3) in the introduction can be given as follows:

(1) Let T by any non-zero pure imaginery scalar operator on an infinite-dimensional space.

(2) Let T be the operator

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 + 2\pi i \end{bmatrix}$$

where the entries are scalar operators on an infinite-dimensional space.

(3) In **(3)**, some invertible operators without square roots were constructed, each of which is an infinite direct sum of finite-dimensional operators. Examination shows that each such operator A is the direct sum of two operators of

class (F), and is thus a multiplicative commutator by Theorem 4. But A cannot be written as $A = e^T$ for any T, since e^T clearly has square roots.

II. It follows easily from Theorem 2 that every invertible operator (on an infinite-dimensional space) of the form $e^{i\theta} + F$ where F is a finite rank operator is a commutator.

III. If A is an additive commutator, then so is $A + A$. But there are multiplicative commutators A such that A^2 is not a multiplicative commutator. Example:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

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University of Michigan