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SOME BIANCHI TYPE VI_0 VISCOUS FLUID COSMOLOGICAL MODELS

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Abstract

The Einstein field equations have been solved for Bianchi type VI_0 spacetimes with viscous fluid source. Four cosmological models are derived. They have nonzero expansion and shear. One of them have nonzero constant shear viscosity coefficient.

1. Introduction

Many cosmologists believe that the standard cosmological models are too restrictive because of their insistence on isotropy. Several attempts have been made to study nonstandard (anisotropic) cosmological models (Narlikar [10], Mac Callum [7]). It would therefore be fruitful to carry out detailed studies of gravitational fields which can be described by spacetimes of various Bianchi types.

Viscosity plays an important role in explaining many physical features of the homogeneous world models. Since viscosity counteracts the cosmological collapse, a different picture of the early universe may appear due to dissipative processes caused by viscosity. Homogeneous cosmological models filled with viscous fluid have been widely studied, e.g. by Murphy [8]; Klimek [6]; Banerjee et al. [2]; Dunn and Tupper [4]; Roy and Prakash [13]; Roy and Singh [14]; Ribeiro and Sanyal [12]; Santos et al. [15].

The main purpose of the present work is to derive some new Bianchi type VI_0 cosmological models filled with viscous fluid.

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2. Einstein field equations

The space time is taken to be nontilting and is characterised by the line element

$$dx^2 = A^2(dt^2 - dx^2) - B^2(e^{2mx}\, dy^2 + e^{-2mx}\, dz^2) \qquad (1)$$

where the metric potentials A and B are functions of time t, and m is a nonzero constant.

We assume that the material distribution filling the models is viscous fluid. Therefore, the corresponding energy-momentum tensor is given by

$$T_{ik} = (\overline{p} + \rho)v_i v_k - \overline{p} g_{ik} - \eta\mu_{ik} \qquad (2)$$

where

$$v_i v^i = 1, \qquad \overline{p} = p - (\xi - 2\eta/3)\, v^i_{;i},$$
$$\mu_{ik} = v_{i;k} + v_{k;i} - (v_i v^a v_{k;a} + v_k v^a v_{i;a}). \qquad (3)$$

Here the semicolon denotes the covariant derivative. The variables ξ and η are respectively the bulk and shear viscosity coefficients. In the above, p is pressure, ρ is the density and v_i is the flow vector of the fluid. The variable $\overline{p}$ is called the effective pressure. We shall use the comoving coordinates. Einstein field equations are

$$R_{ik} - \tfrac{1}{2}Rg_{ik} = -8\pi T_{ik} - \Lambda g_{ik} \qquad (4)$$

where Λ is the cosmological constant and T_{ik} and g_{ik} are given by (2) and (1) respectively. The equations (4) give rise to

$$8\pi\rho = -\Lambda + \frac{1}{A^2}\left[\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} - m^2\right], \qquad (5)$$

$$8\pi\overline{p} = \Lambda - \frac{16\pi\eta}{A}\frac{\dot{B}}{B} + \frac{1}{A^2}\left[m^2 + \frac{\dot{A}^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B}\right], \qquad (6)$$

$$\frac{16\pi\eta}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) = \frac{1}{A^2}\left[\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} - 2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + 2m^2\right]. \qquad (7)$$

Here and in what follows, an overhead dot indicates differentiation with respect to time t. The expansion θ and the shear σ of the flow vector v_i are given by

$$\theta = \frac{1}{A}\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right) \qquad (8)$$

and

$$\sigma = \frac{1}{\sqrt{3}A}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right). \qquad (9)$$

Since we are interested in anisotropic models, we take $\sigma \neq 0$, i.e., $\dot{A}/A \neq \dot{B}/B$.

Here it should be mentioned that the coefficient ξ of bulk viscosity does not appear explicitly in (5), (6) and (7). We take $\xi = 0$ for simplicity. If $\xi \neq 0$, it can be determined by assuming an equation of state of the fluid.

We have a system of three equations (5), (6), and (7) for the determination of five unknowns $\overline{p}$, ρ, η, A and B. Therefore we have to assume two appropriate relations among these variables in order to obtain solutions of these equations. In the next section we present some explicit solutions of these equations.

We end this section by a brief discussion of the Raychaudhuri [11] equation. For our case this equation becomes

$$\theta_{;i}v^i - \theta^2/3 - f^i_{;i} + 2\sigma^2 = R_{ik}v^i v^k \tag{10}$$

where $f_i = v^a v_{i;a}$ is the acceleration vector of the flow vector. Assuming $\Lambda = 0$, the field equations imply $R_{ik}v^i v^k = -4\pi(\rho + 3\overline{p} + 2\eta\theta)$. Therefore the Hawking-Penrose [5] energy conditions are satisfied if $R_{ik}v^k v^k \leq 0$, i.e. if $\rho + 3\overline{p} + 2\eta\theta \geq 0$.

3. Solutions

Let us first assume a linear relation between shear and expansion, i.e. $\sigma/\theta = \text{constant}$. This leads to

$$\dot{A}/A = a\dot{B}/B, \qquad \text{i.e. } A = B^a \tag{11}$$

where a is a constant. Note that $a \neq 1$, otherwise σ becomes zero. Here it should be noted that the condition $\sigma/\theta = \text{constant}$ has been used by Collins et al [3] for the construction of spatially homogeneous cosmological models. Now we must assume one more relation among the variables $\overline{p}$, ρ, η, A and B in order to have an explicit solution of the equations (5), (6) and (7). In the present work, we shall discuss the following three cases:

Case (i). $\overline{p} = \gamma\rho$, $0 \leq \gamma \leq 1$, a barotropic equation of state connecting the density and the effective pressure, and $\Lambda = 0$.

Case (ii). $(8\pi\rho + \Lambda)/\theta^2 = \text{constant}$. This condition has been used by Banerjee et al [1] for deriving a viscous-fluid cosmological model with Bianchi type II space time.

We take $\Lambda = 0$. Then our assumption says that ρ is proportional to θ^2. The effect of viscosity is more prominent at the beginning where θ and ρ

are quite large. Thus, when θ increases, ρ also increases. Therefore it is reasonable to assume ρ to be proportional to θ^2. At later stages the viscosity may play only an insignificant role.

CASE (iii). The metric potential B satisfies the differential equation $\ddot{B}/B + \dot{B}^2/B^2 = 2m^2$. This equation has been used by Narain [9] in connection with the Bianchi type VI_0 perfect fluid model.

The assumption made here is purely mathematical. We are interested to have a viscous fluid generalisation of Narain's solution. This is the only motivation for this assumption.

CASE (i). Using (11) and $\overline{p} = \gamma\rho$, $\Lambda = 0$, we obtain

$$\frac{m^2}{(a-1)}[(a+1)+\gamma(a-1)] + [(a-1)-\gamma(2a+1)]\frac{\dot{B}^2}{B^2} - (a+2)\frac{\ddot{B}}{B} = 0. \quad (12)$$

The above differential equation has a simple solution in the case $\gamma = (a-1)/(2a+1)$. As $\gamma \geq 0$, we must take $a > 1$. Clearly $a > 1$ implies $\gamma < 1$. This simple solution is given by

$$B = e^{bt}, \qquad b^2 = \frac{m^2(3a^2+a+2)}{(a+2)(a-1)(2a+1)}. \quad (13)$$

For this solution, the parameters ρ, $\overline{p}$ and η are determined as

$$8\pi\rho = e^{-2abt}[(2a+1)b^2 - m^2], \quad (14)$$

$$8\pi\overline{p} = e^{-2abt}\frac{(a-1)}{(2a+1)}[(2a+1)b^2 - m^2], \quad (15)$$

and

$$8\pi\eta = \frac{1}{b}e^{-abt}\left[b^2 + \frac{m^2}{1-a}\right]. \quad (16)$$

In view of (13), one can easily verify that the physical requirements $\rho > \overline{p} \geq 0$ are satisfied. The explicit form of the metric of this solution is

$$ds^2 = e^{2abt}(dt^2 - dx^2) - e^{2bt}(e^{2mx}\,dy^2 + e^{-2mx}\,dz^2). \quad (17)$$

It should be mentioned that if $b^2 + m^2/(1-a) = 0$, η vanishes and we get a perfect fluid model. In this situation $\gamma = \frac{1}{3}$ and $8\pi\rho = 5m^2\exp(-8mt/\sqrt{3})$. Thus the above perfect fluid model is filled with disordered radiation. The differential equation (12) can be made integrable for other values of γ also. Let us define a new time coordinate $\overline{t}$ by $d\overline{t} = \sqrt{B}\,dt$ and assume that $\gamma = (a-4)/[2(2a+1)]$. We must take $a > 4$. Clearly $\gamma < 1$. In this case the solution of (12) can be expressed as

$$B = \frac{1}{2}\alpha\overline{t}^2 + \beta\overline{t} + C, \qquad \alpha = \frac{m^2(5a^2+a+6)}{2(a-1)(a+2)(2a+1)} \quad (18)$$

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where β and C are constants of integration. For the sake of brevity, we shall not give the expressions for $\bar{p}$, ρ and η for the above solution.

CASE (ii). Assuming $(8\pi\rho + \Lambda)/\theta^2 = \text{constant} = l^2$ and using (11) we get $(\dot{B}^2/B^2)[(2a+1) - l^2(a+2)^2] = m^2$. Therefore we must have $(2a+1) - l^2(a+2)^2 = k^{-2} > 0$. In this case we have

$$B = e^{\pm mkt}. \tag{19}$$

The parameters ρ, $\bar{p}$ and η for this solution are given by

$$8\pi\rho = -\Lambda + m^2 e^{\pm 2amkt}[(2a+1)k^2 - 1], \tag{20}$$

$$8\pi\bar{p} = \Lambda + m^2 e^{\pm 2amkt}\left[\frac{a+1}{a-1} - 3k^2\right], \tag{21}$$

$$8\pi\eta = \pm\frac{m}{k(1-a)}e^{\pm amkt}[k^2(1-a)+1]. \tag{22}$$

If $\Lambda = 0$, the physical requirements $\rho > \bar{p} \geq 0$ give the inequality

$$\frac{1}{3}\left(\frac{a+1}{a-1}\right) \geq k^2 > \frac{1}{2a+1}. \tag{23}$$

If $\Lambda = 0$, then clearly $\bar{p}/\rho$ is a constant. This is a note-worthy feature of this solution. For brevity, we shall not write the metric of this solution.

It should be noted that when $k^2 = 1/(a-1)$, $a > 1$, η vanishes and we get a perfect fluid model. The pressure p and the density ρ of this model are given by

$$8\pi\rho = -\Lambda + m^2 \exp\left(\mp\frac{2am}{\sqrt{a-1}}t\right)\left(\frac{a+2}{a-1}\right), \tag{24}$$

$$8\pi p = \Lambda + m^2 \exp\left(\mp\frac{2am}{\sqrt{a-1}}t\right)\left(\frac{a-2}{a-1}\right). \tag{25}$$

If $\Lambda = 0$, the equation of state of the perfect fluid is $\bar{p} = ((a-2)/(a+2))\rho$. This equation of state is physically significant provided $a \geq 2$. Thus in the case $a \geq 2$, we have $p = \gamma\rho$ where $0 \leq \gamma \leq 1$. The perfect fluid satisfies the barotropic equation of state.

CASE (iii). Let us assume that the function B satisfies the differential equation $\ddot{B}/B + \dot{B}^2/B^2 = 2m^2$. The solution of this equation is given by

$$B = B_0 \cosh^{1/2}(2mt). \tag{26}$$

Therefore

$$A = B_0^a \cosh^{a/2}(2mt). \tag{27}$$

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Here B_0 is the constant of integration. The parameters $\overline{p}$, ρ and η are given by

$$8\pi\overline{p} = \Lambda + \frac{m^2}{A^2}\left[\frac{a+1}{a-1} - 3 - (1+2a)\,\mathrm{sech}^2(2mt)\right], \tag{28}$$

$$8\pi\rho = -\Lambda + \frac{m^2}{A^2}[(2a+1)\tanh^2(2mt) - 1], \tag{29}$$

$$8\pi\eta = \frac{m(a-2)}{A(a-1)}\coth(2mt) \tag{30}$$

where A is given by (27). The reality conditions $\overline{p} \geq 0$ and $\rho > \overline{p}$ imply that

$$\frac{m^2}{2A^2}\left[2a+3-\frac{a+1}{a-1}\right] > \Lambda > \frac{m^2}{A^2}\left[\frac{2(a-2)}{a-1} + (2a+1)\,\mathrm{sech}^2(2mt)\right] \tag{31}$$

where A is given by (27).
The geometry of this solution is described by the line element

$$ds^2 = B_0^{2a}\cosh^a(2mt)(dt^2 - dx^2) - B_0^2\cosh(2mt)(e^{2mx}\,dy^2 + e^{-2ms}\,dz^2). \tag{32}$$

Here it should be noted that when $a = 2$, the coefficient η of shear viscosity vanishes. In this case our solution reduces to the perfect fluid solution given by Narain [9]. Thus, our solution is a viscous-fluid generalisation of Narain's solution.

We have verified that if $a > 1$, then the Hawking-Penrose energy condition $\rho + 3\overline{p} + 2\eta\theta \geq 0$ is satisfied for all three solutions discussed above.

The present upper limit for the ratio σ/θ is 10^{-3}, obtained from indirect arguments concerning the isotropy of primordial black body radiation (Collins et al. [9]). For the above three solutions $a > 1$ and $\sigma/\theta = (a-1)/(a+2)\sqrt{3}$. This ratio can be made considerably greater than 10^{-3}. For example, taking $a = 1.5$ we have $\sigma/\theta = 1/(7\sqrt{3}) = 0.0824$ which is larger than 10^{-3}. This shows that our solutions describe the early stages of the evolution of the universe.

In viscous hydrodynamics, the shear viscosity coefficient η is often assumed to be constant. Let us now assume that η is a constant. In this case, (7) becomes

$$16\pi\eta A\dot{Z} = \ddot{Z} + 2\frac{\dot{B}}{B}\dot{Z} + 2m^2 \tag{33}$$

where $e^Z = B/A$. We shall integrate (33) for the case in which A and B^2 are linear functions of prime t.

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CASE (iv). Under the above assumptions, (33) admits the solution

$$B^2 = qt + l, \qquad A = -\frac{4m^2(qt+l)}{16\pi\eta q} \tag{34}$$

where q and l are arbitrary constants. The parameters $\overline{p}$ and ρ are given by

$$8\pi\overline{p} = \Lambda + \frac{3}{A^2}\left[m^2 + \frac{q^2}{4(qt+l)^2}\right] \tag{35}$$

$$8\pi\rho = -\Lambda + \frac{1}{A^2}\left[\frac{5q^2}{4(qt+l)^2} - m^2\right] \tag{36}$$

where the function A is given by (34). The physical requirements $\overline{p} \geq 0$ and $\rho > \overline{p}$ imply that

$$\frac{-3}{A^2}\left[m^2 + \frac{q^2}{4(qt+l)^2}\right] \leq \Lambda < \frac{1}{A^2}\left[\frac{q^2}{2(qt+l)^2} - 4m^2\right] \tag{37}$$

where A is given by (34).

In this case, $\dot{A}/A = 2\dot{B}/B$ and hence the ratio $\sigma/\theta = 1/4\sqrt{3} = 0.147$. This value is considerably greater than the present value 10^{-3}. Therefore the above solution represents an early stage of the evolution of the universe.

When $\eta = 0$, the metric potential A becomes singular. Therefore we cannot put $\eta = 0$ and consequently there is no perfect-fluid counterpart of our above solution.

Roy and Singh [14] have considered two equations

$$\ddot{Z} + 2m^2 = 0, \qquad \dot{B}/B = 8\pi\eta A$$

instead of (33) and discussed a viscous-fluid cosmological model. Here it should be noted that the differential equation (33) can be made integrable for other choices of A and B also. But we shall not go into these mathematical details here.

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