11

Symplectic invariance of the CCR on Fock spaces

This chapter is a continuation of Chap. 9, where we studied Fock CCR representations. Our goal is to extend the results of Chap. 10 about the symplectic invariance of canonical commutation relations to the case of Fock CCR representations in any dimension.

11.1 Symplectic group on a Kähler space

The basic framework of this section, as well as most other sections of this chapter, is the same as that of Chap. 9.

In particular, throughout the section, $(\mathcal{Y}, \cdot, \omega, \mathrm{j})$ is a complete Kähler space. We recall that for $r \in B(\mathcal{Y})$, $r^{\#}$ denotes the adjoint of r for the Euclidean scalar product of $\mathcal{Y}$.

Recall that the holomorphic space $\mathcal{Z}$ in $\mathbb{C}\mathcal{Y}$ is defined as $\mathrm{Ran}\,\frac{1}{2}(\mathbb{1} - \mathrm{ij})$, so that $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$. $\mathcal{Z}$ is a (complex) Hilbert space.

In this section we study the symplectic group in a complete Kähler space of any dimension. We treat the symplectic form ω as the basic structure of the Kähler space $\mathcal{Y}$. However, the additional structure on $\mathcal{Y}$ plays an important role. In particular, it gives $\mathcal{Y}$ a Hilbertian topology, which is especially useful when we consider the infinite-dimensional case.

11.1.1 Basic properties

Definition 11.1 *The group of linear transformations on $\mathcal{Y}$ that are bounded, symplectic and have a bounded inverse will be denoted by $Sp(\mathcal{Y})$. Similarly, the Lie algebra of bounded infinitesimally symplectic transformations on $\mathcal{Y}$ will be denoted by $sp(\mathcal{Y})$.*

Note that $sp(\mathcal{Y})$ is the set of generators of norm continuous one-parameter groups in $Sp(\mathcal{Y})$.

We can use the anti-involution j instead of the symplectic form ω to describe various properties of symplectic and infinitesimally symplectic transformations.

The following proposition can be compared with Prop. 1.37.

Proposition 11.2 (1) $r \in Sp(\mathcal{Y})$ *iff*

$$a)\ r^{\#}\mathrm{j}r = \mathrm{j},\ \ and\ \ b)\ r\mathrm{j}r^{\#} = \mathrm{j}.$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

(2) *$r \in Sp(\mathcal{Y})$ iff $r^{\#} \in Sp(\mathcal{Y})$.*
(3) *If $r \in Sp(\mathcal{Y})$, then $r^{-1} = -\mathrm{j} r^{\#} \mathrm{j}$.*

Proposition 11.3 (1) *$a \in sp(\mathcal{Y})$ iff $a^{\#}\mathrm{j} + \mathrm{j}a = 0$.*
(2) *$a \in sp(\mathcal{Y})$ iff $a^{\#} \in sp(\mathcal{Y})$.*

11.1.2 Unitary group on a Kähler space

Recall that a complete Kähler space $\mathcal{Y}$ can be viewed as a complex Hilbert space. It is then denoted by $\mathcal{Y}^{\mathbb{C}}$, with the imaginary unit given by j and the scalar product given by $(y_1|y_2) := y_1 \cdot y_2 + \mathrm{i} y_1 \cdot \omega y_2$ (see (1.37)).

Proposition 11.4 *We have the following characterizations of the unitary group and Lie algebra on a Kähler space:*

$$\begin{aligned} U(\mathcal{Y}^{\mathbb{C}}) &= O(\mathcal{Y}) \cap Sp(\mathcal{Y}) = O(\mathcal{Y}) \cap GL(\mathcal{Y}^{\mathbb{C}}) = Sp(\mathcal{Y}) \cap GL(\mathcal{Y}^{\mathbb{C}}), \\ u(\mathcal{Y}^{\mathbb{C}}) &= o(\mathcal{Y}) \cap sp(\mathcal{Y}) = o(\mathcal{Y}) \cap gl(\mathcal{Y}^{\mathbb{C}}) = sp(\mathcal{Y}) \cap gl(\mathcal{Y}^{\mathbb{C}}). \end{aligned}$$

It is easy to characterize elements of $U(\mathcal{Y}^{\mathbb{C}})$ and $u(\mathcal{Y}^{\mathbb{C}})$ by their extensions to $\mathbb{C}\mathcal{Y} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$.

Proposition 11.5 (1) *$r \in U(\mathcal{Y}^{\mathbb{C}})$ iff*

$$r_{\mathbb{C}} = \begin{bmatrix} p & 0 \\ 0 & \overline{p} \end{bmatrix},$$

with $p \in U(\mathcal{Z})$.
(2) *$a \in u(\mathcal{Y}^{\mathbb{C}})$ iff*

$$a_{\mathbb{C}} = \mathrm{i} \begin{bmatrix} -h & 0 \\ 0 & \overline{h} \end{bmatrix},$$

with $h = h^*$.

11.1.3 Symplectic transformations on Kähler spaces

We recall that if $a \in B(\mathcal{Z}_1, \mathcal{Z}_2)$, then $a^{\#} := \overline{a}^* \in B(\overline{\mathcal{Z}}_2, \overline{\mathcal{Z}}_1)$. We recall also that $B_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$ denotes the set of $g \in B(\overline{\mathcal{Z}}, \mathcal{Z})$ such that $g^{\#} = g$, and $B_{\mathrm{h}}(\mathcal{Z})$ denotes the set of $h \in B(\mathcal{Z})$ such that $h^* = h$ (see Subsect. 2.2.3).

Proposition 11.6 *$r \in Sp(\mathcal{Y})$ iff its extension to $\mathbb{C}\mathcal{Y}$ equals*

$$r_{\mathbb{C}} = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix}, \tag{11.1}$$

with $p \in B(\mathcal{Z})$, $q \in B(\overline{\mathcal{Z}}, \mathcal{Z})$, and the following conditions hold:

conditions implied by Prop. 11.2 (1a): $\quad p^* p - q^{\#} \overline{q} = 1\!\!1, \quad p^* q - q^{\#} \overline{p} = 0,$

conditions implied by Prop. 11.2 (1b): $\quad pp^* - qq^* = 1\!\!1, \quad pq^{\#} - qp^{\#} = 0.$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

Proposition 11.7 *$a \in sp(\mathcal{Y})$ iff its extension to $\mathbb{C}\mathcal{Y}$ equals*

$$a_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} -h & g \\ -\overline{g} & \overline{h} \end{bmatrix}, \tag{11.2}$$

with $h \in B_{\mathrm{h}}(\mathcal{Z})$ and $g \in B_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$.

We describe now a convenient factorization of a symplectic map. Let $r \in Sp(\mathcal{Y})$, and let p, q be defined as in (11.1). Note that

$$pp^* \geq \mathbb{1}, \quad p^*p \geq \mathbb{1}.$$

Hence p^{-1} and p^{*-1} are bounded operators, and we can set

$$c := q^{\#}(p^{\#})^{-1}, \tag{11.3}$$

$$d := q\overline{p}^{-1}. \tag{11.4}$$

Recall that, for $a, b \in B_{\mathrm{h}}(\mathcal{Z})$, $a < b$ means $a \leq b$ and $\mathrm{Ker}(b - a) = \{0\}$.

Proposition 11.8 (1) *We have $c, d \in B_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$ and*

$$c^*c < \mathbb{1}, \quad d^*d < \mathbb{1}. \tag{11.5}$$

(2) *The following equivalent characterizations of c, d hold:*

$$c = p^{-1}q, \tag{11.6}$$

$$d = (p^*)^{-1}q^{\#}. \tag{11.7}$$

(3) *One has the following factorization:*

$$r_{\mathbb{C}} = \begin{bmatrix} \mathbb{1} & d \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} (p^*)^{-1} & 0 \\ 0 & \overline{p} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ \overline{c} & \mathbb{1} \end{bmatrix}. \tag{11.8}$$

(4) *We have*

$$\begin{aligned} (r_{\mathbb{C}}^*)^{-1} &= \begin{bmatrix} p & -q \\ -\overline{q} & \overline{p} \end{bmatrix}, \\ (r_{\mathbb{C}}r_{\mathbb{C}}^* - \mathbb{1})(r_{\mathbb{C}}r_{\mathbb{C}}^* + \mathbb{1})^{-1} &= \begin{bmatrix} 0 & d \\ \overline{d} & 0 \end{bmatrix}, \\ (r_{\mathbb{C}}^*r_{\mathbb{C}} - \mathbb{1})(r_{\mathbb{C}}^*r_{\mathbb{C}} + \mathbb{1})^{-1} &= \begin{bmatrix} 0 & c \\ \overline{c} & 0 \end{bmatrix}. \end{aligned} \tag{11.9}$$

(5) *We have the identities*

$$\mathbb{1} - cc^* = (p^*p)^{-1}, \quad \mathbb{1} - d^*d = (\overline{p}\,\overline{p}^*)^{-1}. \tag{11.10}$$

Proof For example, the first inequality of (11.5) follows from the fact that $pp^* > qq^*$, which implies that $(p^*)^{-1}p^{-1} = (pp^*)^{-1} < (q^*)^{-1}q^{-1}$. Now $c^*c = q^*(p^*)^{-1}p^{-1}q < \mathbb{1}$. □

11.1.4 Positive symplectic transformations

Symplectic transformations that are at the same time positive enjoy special properties. We devote this subsection to a discussion of their basic properties.

Let $r \in Sp(\mathcal{Y})$ such that $r = r^{\#}$ and $r > 0$ as an operator on $(\mathcal{Y}, \cdot)$. Recall that the unitary structure on $\mathbb{C}\mathcal{Y}$ is obtained from the Euclidean structure of $\mathcal{Y}$ as in Subsect. 1.3.4. Hence, $r_{\mathbb{C}} = r_{\mathbb{C}}^*$ and $r_{\mathbb{C}} > 0$. We have

$$r_{\mathbb{C}} = \begin{bmatrix} p & q \\ \overline{q} & \overline{p} \end{bmatrix},$$

where $p = p^* > 0$ and $q = q^{\#}$. The conditions in Prop. 11.6 simplify to

$$p^2 - q\overline{q} = 1\!\!1, \quad pq - q\overline{p} = 0.$$

We have

$$r_{\mathbb{C}}^{-1} = \begin{bmatrix} p & -q \\ -\overline{q} & \overline{p} \end{bmatrix}.$$

In the case of positive symplectic transformations some of the identities of Prop. 11.8 simplify:

Proposition 11.9 *Let $r \in Sp(\mathcal{Y})$ such that $r = r^{\#}$ and $r > 0$. Let $c \in B_{\rm s}(\overline{\mathcal{Z}}, \mathcal{Z})$ be defined as in (11.3). Then $c^*c < 1\!\!1$,*

$$\begin{aligned} r_{\mathbb{C}} &= \begin{bmatrix} (1\!\!1 - cc^*)^{-\frac12} & (1\!\!1 - cc^*)^{-\frac12} c \\ c^*(1\!\!1 - cc^*)^{-\frac12} & (1\!\!1 - c^*c)^{-\frac12} \end{bmatrix} \qquad (11.11) \\ &= \begin{bmatrix} 1\!\!1 & c \\ 0 & 1\!\!1 \end{bmatrix} \begin{bmatrix} (1\!\!1 - cc^*)^{\frac12} & 0 \\ 0 & (1\!\!1 - c^*c)^{-\frac12} \end{bmatrix} \begin{bmatrix} 1\!\!1 & 0 \\ c^* & 1\!\!1 \end{bmatrix}, \end{aligned}$$

$$(r_{\mathbb{C}}^2 - 1\!\!1)(r_{\mathbb{C}}^2 + 1\!\!1)^{-1} = \begin{bmatrix} 0 & c \\ \overline{c} & 0 \end{bmatrix}. \qquad (11.12)$$

Conversely, let $c \in B_{\rm s}(\overline{\mathcal{Z}}, \mathcal{Z})$ satisfy $c^*c < 1\!\!1$, and let r be defined by (11.11). Then $r \in Sp(\mathcal{Y})$, $r = r^{\#}$, $r > 0$.

Proof The properties of c follow directly from the properties of p, q given above.

Next let $c \in B_{\rm s}(\overline{\mathcal{Z}}, \mathcal{Z})$ with $c^*c < 1\!\!1$. Clearly, $cc^* = \overline{c^*c} < 1\!\!1$, and hence the operators $1\!\!1 - cc^*$ and $1\!\!1 - c^*c$ are invertible. We check that the operator r defined by (11.11) is a positive symplectic transformation. □

Positive symplectic transformations can be obtained as exponentials of self-adjoint infinitesimally symplectic transformations:

Proposition 11.10 *Let $a \in sp(\mathcal{Y})$ such that $a = a^{\#}$. Then $a_{\mathbb{C}} = a_{\mathbb{C}}^*$, and hence there exists $g \in B_{\rm s}(\overline{\mathcal{Z}}, \mathcal{Z})$ such that*

$$a_{\mathbb{C}} = \mathrm{i} \begin{bmatrix} 0 & g \\ -\overline{g} & 0 \end{bmatrix}. \qquad (11.13)$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

Moreover, $r = \mathrm{e}^a$ *belongs to* $Sp(\mathcal{Y})$ *and satisfies* $r = r^\#$, $r > 0$ *and*

$$r_{\mathbb{C}} = \begin{bmatrix} \cosh\sqrt{gg^*} & \mathrm{i}\frac{\sinh\sqrt{gg^*}}{\sqrt{gg^*}}g \\ -\mathrm{i}g^*\frac{\sinh\sqrt{gg^*}}{\sqrt{gg^*}} & \cosh\sqrt{g^*g} \end{bmatrix}, \tag{11.14}$$

$$c = \mathrm{i}\frac{\tanh\sqrt{gg^*}}{\sqrt{gg^*}}g. \tag{11.15}$$

11.1.5 Polar decomposition of symplectic maps

Proposition 11.11 (1) *Let* $r \in Sp(\mathcal{Y})$ *such that* $r > 0$*. Then, for each* $\epsilon \in \mathbb{R}$*,* $r^\epsilon \in Sp(\mathcal{Y})$*.*

(2) *Let* $r \in Sp(\mathcal{Y})$*. Then there exist unique* $k \in Sp(\mathcal{Y})$*,* $u \in U(\mathcal{Y}^{\mathbb{C}})$ *such that* $k = k^\#$*,* $k > 0$ *and* $r = ku$*. The operators* u, k *are given by the polar decomposition of* r *as an operator on the real Hilbert space* $(\mathcal{Y}, \cdot)$*.*

Proof Let $r \in Sp(\mathcal{Y})$ such that $r = r^\#$ and $r > 0$. Then $r\mathrm{j} = \mathrm{j}r^{-1}$, since $r \in Sp(\mathcal{Y})$. This implies that $(z - r)^{-1}\mathrm{j} = \mathrm{j}(z - r^{-1})^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$, and hence

$$f(r)\mathrm{j} = \mathrm{j}f(r^{-1}), \text{ for any measurable function } f.$$

In particular, for $\epsilon \in \mathbb{R}$ we have $r^\epsilon\mathrm{j} = \mathrm{j}r^{-\epsilon}$, and hence $r^\epsilon \in Sp(\mathcal{Y})$. This proves (1).

Now let $r \in Sp(\mathcal{Y})$. Set $k = (rr^\#)^{\frac{1}{2}}$. By (1), $k \in Sp(\mathcal{Y})$. Set $u = k^{-1}r$. Clearly, $u \in Sp(\mathcal{Y})$. By the properties of the polar decomposition in $(\mathcal{Y}, \cdot)$ we have $u \in O(\mathcal{Y})$. Hence $u \in U(\mathcal{Y}^{\mathbb{C}})$, which proves (2). □

11.1.6 Restricted symplectic group

In this subsection we introduce a subgroup of the symplectic group on the Kähler space that plays an important role in Shale's theorem, a basic result of the theory of CCR representations on Fock spaces.

Proposition 11.12 *Let* $r \in Sp(\mathcal{Y})$*. Consider* p, q, c, d *defined by (11.1), (11.3) and (11.4). The following conditions are equivalent:*

(1) $\mathrm{j} - r^{-1}\mathrm{j}r \in B^2(\mathcal{Y})$.

(2) $r\mathrm{j} - \mathrm{j}r \in B^2(\mathcal{Y})$.

(3) $\mathrm{Tr}q^*q < \infty$.

(4) $\mathrm{Tr}(p^*p - \mathbb{1}) < \infty$.

(5) $\mathrm{Tr}(pp^* - \mathbb{1}) < \infty$.

(6) $d \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$.

(7) $c \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$.

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

Proof Clearly, (1)$\Leftrightarrow$ (2).

We have

$$(r\mathrm{j}-\mathrm{j}r)_{\mathbb{C}} = \begin{bmatrix} 0 & -2\mathrm{i}q \\ 2\mathrm{i}\overline{q} & 0 \end{bmatrix}, \qquad (r\mathrm{j}-\mathrm{j}r)_{\mathbb{C}}^*(r\mathrm{j}-\mathrm{j}r)_{\mathbb{C}} = \begin{bmatrix} 4q^{\#}\overline{q} & 0 \\ 0 & 4q^*q \end{bmatrix}.$$

Hence

$$\mathrm{Tr}(r\mathrm{j}-\mathrm{j}r)_{\mathbb{C}}^*(r\mathrm{j}-\mathrm{j}r)_{\mathbb{C}} = 4(\mathrm{Tr}\, q^{\#}\overline{q} + \mathrm{Tr}\, q^*q) = 8\mathrm{Tr}\, q^*q = 8\mathrm{Tr}\,(p^*p - \mathbb{1}),$$

using that $q^{\#}\overline{q} = p^*p - \mathbb{1}$. This implies that (2)$\Leftrightarrow$(3)$\Leftrightarrow$(4). If $v \in U(\mathcal{Z})$ and $p = v|p| = |p^*|v$ is the polar decomposition of p, we have $pp^* = vp^*pv^*$. So (4)$\Leftrightarrow$(5).

The identities $c = p^{-1}q$ and $d = q\overline{p}^{-1}$ and the fact that p is invertible show that (3)$\Rightarrow$(6) and (3)$\Rightarrow$(7). The identities $\mathbb{1} - cc^* = (p^*p)^{-1}$ and $\mathbb{1} - d^*d = (\overline{p}p^{\#})^{-1}$ show that (7)$\Rightarrow$(4) and (6)$\Rightarrow$(5). □

Definition 11.13 *Let $Sp_{\mathrm{j}}(\mathcal{Y})$ be the set of $r \in Sp(\mathcal{Y})$ satisfying the conditions of Prop. 11.12. The set $Sp_{\mathrm{j}}(\mathcal{Y})$ is called the* restricted symplectic group. *We equip it with the metric*

$$d_{\mathrm{j}}(r_1, r_2) := \|p_1 - p_2\| + \|q_1 - q_2\|_2. \tag{11.16}$$

Equivalent metrics are $\|[\mathrm{j}, r_1 - r_2]_+\| + \|[\mathrm{j}, r_1 - r_2]\|_2$ and $\|r_1 - r_2\| + \|[\mathrm{j}, r_1 - r_2]\|_2$.

We say that $a \in sp_{\mathrm{j}}(\mathcal{Y})$ if $a \in sp(\mathcal{Y})$ and $a\mathrm{j} - \mathrm{j}a \in B^2(\mathcal{Y})$, or equivalently $g \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$, where we use the decomposition (11.2).

Proposition 11.14 (1) *$Sp_{\mathrm{j}}(\mathcal{Y})$ is a topological group.*
(2) *$sp_{\mathrm{j}}(\mathcal{Y})$ is a Lie algebra.*
(3) *If $a \in sp_{\mathrm{j}}(\mathcal{Y})$ then $\mathrm{e}^a \in Sp_{\mathrm{j}}(\mathcal{Y})$.*

Proof The fact that $Sp_{\mathrm{j}}(\mathcal{Y})$ is a topological group is clear, since $[r_1r_2, \mathrm{j}] = r_1[r_2, \mathrm{j}] + [r_1, \mathrm{j}]r_2$. To prove (3), we write

$$\mathrm{e}^a\mathrm{j} - \mathrm{j}\mathrm{e}^a = \sum_{n=0}^{\infty} \frac{1}{n!}[a^n, \mathrm{j}],$$

and use that $\|[a^n, \mathrm{j}]\|_2 \leq n\|a\|^{n-1}\|[a, \mathrm{j}]\|_2$, which yields

$$\|\mathrm{e}^a\mathrm{j} - \mathrm{j}\mathrm{e}^a\|_2 \leq \mathrm{e}^{\|a\|}\|a\mathrm{j} - \mathrm{j}a\|_2.$$ □

11.1.7 Anomaly-free symplectic group

In this subsection we introduce another, much smaller subgroup of the symplectic group on the Kähler space. Its name is suggested by the well-known terminology used in quantum field theory.

Definition 11.15 *Let $Sp_{\mathrm{j,af}}(\mathcal{Y})$ be the set of $r \in Sp_{\mathrm{j}}(\mathcal{Y})$ such that $2\mathrm{j} - (\mathrm{j}r + r\mathrm{j}) \in B^1(\mathcal{Y})$, or equivalently $p - \mathbb{1}_{\mathcal{Z}} \in B^1(\mathcal{Z})$, where we use the*

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

decomposition (11.1). $Sp_{\mathrm{j,af}}(\mathcal{Y})$ will be called the anomaly-free symplectic group *and will be equipped with the metric*

$$d_{\mathrm{j,af}}(r_1, r_2) := \|p_1 - p_2\|_1 + \|q_1 - q_2\|_2.$$

An equivalent metric is $\|[\mathrm{j}, r_1 - r_2]_+\|_1 + \|[\mathrm{j}, r_1 - r_2]\|_2$.

We also define $sp_{\mathrm{j,af}}(\mathcal{Y})$ to be the set of $a \in sp_{\mathrm{j}}(\mathcal{Y})$ such that $a\mathrm{j} + \mathrm{j}a \in B^1(\mathcal{Y})$, or equivalently $h \in B^1(\mathcal{Y})$, where we use the decomposition (11.2).

Proposition 11.16 (1) *$Sp_{\mathrm{j,af}}(\mathcal{Y})$ is a topological group.*
(2) *$sp_{\mathrm{j,af}}(\mathcal{Y})$ is a Lie algebra.*
(3) *If $a \in sp_{\mathrm{j,af}}(\mathcal{Y})$ then $\mathrm{e}^a \in Sp_{\mathrm{j,af}}(\mathcal{Y})$.*

Proof If $r \in Sp_{\mathrm{j,af}}(\mathcal{Y})$, then $\mathbb{1} - r \in B^2(\mathcal{Y})$. It follows that if $r_1, r_2 \in Sp_{\mathrm{j,af}}(\mathcal{Y})$, then $r_1 r_2 - (r_1 + r_2) + \mathbb{1} \in B^1(\mathcal{Y})$, which easily implies that $r_1 r_2 \in Sp_{\mathrm{j,af}}(\mathcal{Y})$ and proves (1). To prove (2), note that $sp_{\mathrm{j,af}}(\mathcal{Y}) \subset B^2(\mathcal{Y})$. To prove (3), we use that if $a \in sp_{\mathrm{j,af}}(\mathcal{Y})$, then $\mathrm{e}^a - (\mathbb{1} + a) \in B^1(\mathcal{Y})$. □

Proposition 11.17 (1) *Let $r \in Sp(\mathcal{Y})$ be positive. Then $r \in Sp_{\mathrm{j}}(\mathcal{Y})$ iff $r \in Sp_{\mathrm{j,af}}(\mathcal{Y})$.*
(2) *Let $a \in sp(\mathcal{Y})$ be self-adjoint. Then $a \in sp_{\mathrm{j}}(\mathcal{Y})$ iff $a \in sp_{\mathrm{j,af}}(\mathcal{Y})$.*

Proof (1) We know that $r \in Sp_{\mathrm{j}}(\mathcal{Y})$ iff $c \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$. But (11.11) then implies that $r \in Sp_{\mathrm{j,af}}(\mathcal{Y})$.

(2) By the decomposition (11.2), $a \in sp_{\mathrm{j}}(\mathcal{Y})$ is self-adjoint iff $h = 0$ and $g \in B^2_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$. □

Proposition 11.18 *Let $r \in B(\mathcal{Y})$ and let $r = r_0 u$ be its polar decomposition. Then*

(1) *$r \in Sp(\mathcal{Y})$ iff $r_0 \in Sp(\mathcal{Y})$.*
(2) *$r \in Sp_{\mathrm{j}}(\mathcal{Y})$ iff $r_0 \in Sp_{\mathrm{j}}(\mathcal{Y})$.*
(3) *$r \in Sp_{\mathrm{j,af}}(\mathcal{Y})$ iff $r_0 \in Sp_{\mathrm{j,af}}(\mathcal{Y})$ and $u \in Sp_{\mathrm{j,af}}(\mathcal{Y})$.*

11.1.8 Pairs of Kähler structures on symplectic spaces

In this subsection we study the relationship between two Kähler anti-involutions on a given symplectic space. One of them is denoted j and is treated as the basic one. The other is denoted j_1.

In the first proposition, j_1 is obtained by conjugating j with an arbitrary symplectic map.

Proposition 11.19 *Let $r \in Sp(\mathcal{Y})$. Set $\mathrm{j}_1 = r^{-1}\mathrm{j}r$.*

(1) *j_1 is a Kähler anti-involution.*
(2) *$r \in U(\mathcal{Y}^{\mathbb{C}})$ iff $\mathrm{j}_1 = \mathrm{j}$.*
(3) *If $r = u|r|$ is the polar decomposition of r, then $\mathrm{j}_1 = |r|^{-1}\mathrm{j}|r|$.*
(4) *$\mathrm{j}_1 = \mathrm{j}r^{\#}r$.*

Proof Since $r \in Sp(\mathcal{Y})$, we have $y \cdot \omega \mathrm{j}_1 y = y \cdot \omega r^{-1} \mathrm{j} r y = (ry) \cdot \omega \mathrm{j} r y$, which shows that (ω, j_1) is Kähler and proves (1). Clearly, $r \in U(\mathcal{Y}^{\mathbb{C}})$ iff $[r, \mathrm{j}] = 0$, which is equivalent to $\mathrm{j}_1 = \mathrm{j}$. This proves (2). If $r = u|r|$, then $[u, \mathrm{j}] = 0$, hence $\mathrm{j}_1 = |r|^{-1} \mathrm{j} |r|$, which proves (3). (4) follows from Prop. 11.2 (1a). □

The next proposition is a partial converse of the previous one. In particular, we compute a positive symplectic map that transforms j into j_1.

Theorem 11.20 (1) *Let* j_1 *be an anti-involution such that* (ω, j_1) *is Kähler. Then* $k := -\mathrm{j}\mathrm{j}_1$ *is a positive symplectic transformation.*
(2) *Let* $k \in Sp(\mathcal{Y})$ *be positive. Then* $\mathrm{j}_1 := \mathrm{j}k$ *is a Kähler anti-involution.*
(3) *Let* k, j_1 *be as in (2). Then* $r = k^{\frac{1}{2}}$ *defined in Subsect. 2.3.2 satisfies*

$$r \in Sp(\mathcal{Y}), \quad r = r^{\#}, \quad r > 0, \quad r^{-1} \mathrm{j} r = \mathrm{j}_1. \tag{11.17}$$

(r is positive symplectic and intertwines j *and* j_1*.)*
(4) *There exists* $c \in B_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$ *such that*

$$\left(\frac{k - \mathbb{1}}{k + \mathbb{1}}\right)_{\mathbb{C}} = \begin{bmatrix} 0 & c \\ \overline{c} & 0 \end{bmatrix}. \tag{11.18}$$

(5) *We have*

$$r_{\mathbb{C}} = \begin{bmatrix} (\mathbb{1} - cc^*)^{-\frac{1}{2}} & (\mathbb{1} - cc^*)^{-\frac{1}{2}} c \\ c^*(\mathbb{1} - cc^*)^{-\frac{1}{2}} & (\mathbb{1} - c^*c)^{-\frac{1}{2}} \end{bmatrix}, \tag{11.19}$$

$$k_{\mathbb{C}} = \begin{bmatrix} (\mathbb{1} + cc^*)(\mathbb{1} - cc^*)^{-1} & 2(\mathbb{1} - cc^*)^{-1} c \\ 2c^*(\mathbb{1} - cc^*)^{-1} & (\mathbb{1} + c^*c)(\mathbb{1} - c^*c)^{-1} \end{bmatrix}, \tag{11.20}$$

$$\mathrm{j}_{1\mathbb{C}} = \mathrm{i} \begin{bmatrix} (\mathbb{1} + cc^*)(\mathbb{1} - cc^*)^{-1} & 2(\mathbb{1} - cc^*)^{-1} c \\ -2c^*(\mathbb{1} - cc^*)^{-1} & -(\mathbb{1} + c^*c)(\mathbb{1} - c^*c)^{-1} \end{bmatrix}. \tag{11.21}$$

Proof Since $\mathrm{j}, \mathrm{j}_1 \in Sp(\mathcal{Y})$, $k = -\mathrm{j}\mathrm{j}_1 \in Sp(\mathcal{Y})$. Since (ω, j_1) is Kähler, we have

$$0 = (\mathrm{j}_1 y_1) \cdot \omega y_2 + y_1 \cdot \omega \mathrm{j}_1 y_2 = -(\mathrm{j}_1 y_1) \cdot \mathrm{j} y_2 - y_1 \cdot \mathrm{j}\mathrm{j}_1 y_2,$$

i.e. $\mathrm{j}_1^{\#} \mathrm{j} = -\mathrm{j}\mathrm{j}_1$. Hence

$$(\mathrm{j}\mathrm{j}_1)^{\#} = \mathrm{j}_1^{\#} \mathrm{j}^{\#} = -\mathrm{j}_1^{\#} \mathrm{j} = \mathrm{j}\mathrm{j}_1,$$

i.e. $k = k^{\#}$. Again using that (ω, j_1) is Kähler, we get $-y \cdot \mathrm{j}\mathrm{j}_1 y = y \cdot \omega \mathrm{j}_1 y > 0$, i.e. $k > 0$. This proves (1).

Let us prove (3). The fact that $r \in Sp(\mathcal{Y})$ follows from Prop. 11.11 (1). Using that $r = r^{\#}$ and $r \in Sp(\mathcal{Y})$, we obtain that $\mathrm{j}_1 = \mathrm{j} r^2 = r^{-1} \mathrm{j} r$.

Set $b := \frac{k - \mathbb{1}}{k + \mathbb{1}}$. We check that $\mathrm{j}b = -b\mathrm{j}$. This implies (4). Then using (11.12) we see that $r_{\mathbb{C}}$ equals (11.11), which is repeated as (11.19). Then we use $k = r^2$ and $\mathrm{j}_1 = \mathrm{j}k$ to obtain (11.20) and (11.21). □

Proposition 11.21 *Let $\mathcal{Z}$ and $\mathcal{Z}_1$ be the holomorphic subspaces of $\mathbb{C}\mathcal{Y} \simeq \mathcal{Z} \oplus \overline{\mathcal{Z}}$ for the anti-involutions* j *and* j_1*. Let c be as above. Then*

$$\mathcal{Z}_1 = \big\{(z, -\overline{c}z) \ : \ z \in \mathcal{Z}\big\},$$

$$\overline{\mathcal{Z}}_1 = \big\{(-c\overline{z}, \overline{z}) \ : \ z \in \mathcal{Z}\big\}.$$

Proof Every vector of $\mathcal{Z}_1$ is of the form $(\mathbb{1} - \mathrm{i}\mathrm{j}_1)y_1$ for some $y_1 \in \mathcal{Y}$. Since $k > 0$, every vector of $\mathcal{Y}$ is of the form $y_1 = (\mathbb{1} + k)^{-1}y$ for some $y \in \mathcal{Y}$. Now

$$\begin{aligned}(\mathbb{1} - \mathrm{i}\mathrm{j}_1)(\mathbb{1} + k)^{-1}y &= (\mathbb{1} - \mathrm{i}\mathrm{j}k)(\mathbb{1} + k)^{-1}y \\ &= \mathbb{1}_{\mathcal{Z}}y - \mathbb{1}_{\overline{\mathcal{Z}}}(\tfrac{k-\mathbb{1}}{k+\mathbb{1}})y \\ &= z - \overline{c}z,\end{aligned}$$

where $z = \mathbb{1}_{\mathcal{Z}}y \in \mathcal{Z}$. Hence every vector of $\mathcal{Z}_1$ is of the form $z - \overline{c}z$ for $z \in \mathcal{Z}$. Applying the canonical conjugation on $\mathbb{C}\mathcal{Y}$ we obtain the corresponding result for $\overline{\mathcal{Z}}_1$. □

The following proposition will be used to describe the unitary equivalence of two Fock CCR representations (one of the versions of Shale's theorem).

Proposition 11.22 *Let* $\mathrm{j}, \mathrm{j}_1, k, c$ *be as above. The following conditions are equivalent:*

(1) $\mathrm{j} - \mathrm{j}_1 \in B^2(\mathcal{Y})$.
(2) $\mathbb{1} - k \in B^2(\mathcal{Y})$.
(3) $c \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$.
(4) *There exists a positive $r \in Sp_{\mathrm{j,af}}(\mathcal{Y})$ such that* $\mathrm{j}_1 = r\mathrm{j}r^{-1}$.
(5) *There exists $r \in Sp_{\mathrm{j}}(\mathcal{Y})$ such that* $\mathrm{j}_1 = r\mathrm{j}r^{-1}$.

Proof The identity $-\mathrm{j}(\mathrm{j} - \mathrm{j}_1) = \mathbb{1} - k$ and $\mathrm{j} \in Sp(\mathcal{Y})$ imply the equivalence of (1) and (2).

(11.18) and the boundedness of $(\mathbb{1} + k)^{-1}$ show that (2) implies (3).

Since $c^*c < \mathbb{1}_{\overline{\mathcal{Z}}}$ and $c = c^{\#}$, we have $cc^* < \mathbb{1}_{\overline{\mathcal{Z}}}$, and hence $(\mathbb{1}_{\overline{\mathcal{Z}}} - c^*c)^{-1}$ and $(\mathbb{1}_{\mathcal{Z}} - cc^*)^{-1}$ are bounded. From (11.20) we obtain that (3) implies (2).

(4) ⇒ (5) is obvious. (5) ⇒ (1) is obvious. (3)⇒(4) follows from (11.19). □

Remark 11.23 *The Hilbert–Schmidt property in conditions (1) and (2) uses the real scalar product on $\mathcal{Y}$ that belongs to the Kähler structure $(\cdot, \omega, \mathrm{j})$. Therefore, conditions (1) and (2) may not seem symmetric w.r.t. the anti-involutions* j *and* j_1*. Nevertheless, if they are satisfied, then the scalar products $\cdot$ and $\cdot k$ are related with the operator k, which is bounded with a bounded inverse, hence the set of Hilbert–Schmidt operators w.r.t. the scalar products $\cdot$ and $\cdot k$ coincide.*

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

11.1.9 Conjugation adapted to a pair of Kähler involutions

Generically, a pair of Kähler anti-involutions determines a conjugation for both Kähler structures, as expressed in the following proposition:

Proposition 11.24 *Suppose that* j_1 *is an anti-involution such that* (ω, j_1) *is Kähler. Then the following is true:*

(1) *There exists* $\tau \in B(\mathcal{Y})$ *such that*

$$\tau^2 = \mathbb{1}, \quad \tau \mathrm{j} \tau = -\mathrm{j}, \quad \tau \mathrm{j}_1 \tau = -\mathrm{j}_1.$$

(τ is a conjugation for both j *and* j_1*.)*

(2) *Let* $k = -\mathrm{j}\mathrm{j}_1$*. Then* $\tau k \tau = k$*.*

(3) *If* $\mathrm{Ker}(\mathrm{j} - \mathrm{j}_1) = \{0\}$*, or equivalently* $\mathrm{Ker}(\mathbb{1} - k) = \{0\}$*, then we can take*

$$\tau := \mathbb{1}_{]1,+\infty[}(k) - \mathbb{1}_{]0,1[}(k).$$

Proof Recall that $k^{\#} = k$, $k > 0$.

Assume first that $\mathrm{Ker}(\mathrm{j} - \mathrm{j}_1) = \{0\}$, and hence $\mathrm{Ker}(\mathbb{1} - k) = \{0\}$. Set $\tau = \mathbb{1}_{]1,+\infty[}(k) - \mathbb{1}_{]0,1[}(k)$. We have $\tau^2 = \mathbb{1}$ and $\tau \in O(\mathcal{Y})$. Using that $k\mathrm{j} = \mathrm{j}k^{-1}$, we see that $\tau\mathrm{j} = -\mathrm{j}\tau$. Since $\tau k = k\tau$, we have also $\tau \mathrm{j}_1 = -\mathrm{j}_1 \tau$.

If $\mathrm{Ker}(\mathrm{j} - \mathrm{j}_1) \neq \{0\}$, we set $\mathcal{Y}_1 := \mathbb{1}_{\{1\}}(k)\mathcal{Y}$. The spaces $\mathcal{Y}_1$ and $\mathcal{Y}_0 := \mathcal{Y}_1^{\perp}$ are invariant under j since $k\mathrm{j} = \mathrm{j}k^{-1}$. We first construct the conjugation τ_0 on $\mathcal{Y}_0$ as above. On $\mathcal{Y}_1$ we have $\mathrm{j} = \mathrm{j}_1$. We can choose an arbitrary anti-unitary involution τ_1 of $\mathcal{Y}_1^{\mathbb{C}}$. Then we set $\tau = \tau_1 \oplus \tau_0$ on $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_0$. □

Recall that $\mathcal{Y}^{\pm\tau} := \{y \in \mathcal{Y} \ : \ \tau y = \pm y\}$. Set $\mathcal{X} = \mathcal{Y}^{-\tau}$. As in Subsect. 1.3.10, we can identify the Kähler space with conjugation $\mathcal{Y}$ with $\mathcal{X} \oplus \mathcal{X}$ by the map

$$\mathcal{Y} \ni y \mapsto \Big(\mathrm{j}\frac{1}{2\sqrt{2}}(\mathbb{1} + \tau)y, \frac{1}{\sqrt{2}}(\mathbb{1} - \tau)y\Big) \in \mathcal{X} \oplus \mathcal{X},$$

which corresponds to the choice $c = \mathbb{1}$ in (1.38). We set $m := k^{-1}\big|_{\mathcal{X}}$, which is a positive self-adjoint operator on the real Hilbert space $\mathcal{X}$. The symplectic form on $\mathcal{Y} \simeq \mathcal{X} \oplus \mathcal{X}$ is

$$(x_1^+, x_1^-)\cdot\omega(x_2^+, x_2^-) = x_1^+\cdot x_2^- - x_1^-\cdot x_2^+.$$

Proposition 11.25 *We have*

$$\tau = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \quad \mathrm{j} = \begin{bmatrix} 0 & -\frac{1}{2}\mathbb{1} \\ 2\mathbb{1} & 0 \end{bmatrix}, \quad \mathrm{j}_1 = \begin{bmatrix} 0 & -(2m)^{-1} \\ 2m & 0 \end{bmatrix}, \quad k = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}.$$

Proof The matrix representation of τ and j on $\mathcal{X} \oplus \mathcal{X}$ was shown in Subsect. 1.3.10. To compute k, we note that if $y \in \mathcal{Y}$ is identified with $(\mathrm{j}x_1, x_2) \in \mathcal{X} \oplus \mathcal{X}$, then ky is identified with $(k\mathrm{j}x_1, kx_2) = (\mathrm{j}mx_1, m^{-1}x_2)$ since $\mathrm{j}k = k^{-1}\mathrm{j}$. The formula for j_1 follows from $\mathrm{j}_1 = \mathrm{j}k$. □

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

11.2 Bosonic quadratic Hamiltonians on Fock spaces

The basic framework of this section is the same as that of the previous one. Recall, in particular, that $\mathcal{Z}$ is a Hilbert space and $\mathcal{Y} := \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ is the corresponding Kähler space. We consider the Fock CCR representation over $\mathcal{Y}$ in $\Gamma_{\mathrm{s}}(\mathcal{Z})$.

11.2.1 Wick and anti-Wick quantizations of quadratic polynomials

Let us consider various kinds of complex quadratic polynomials on $\mathcal{Y}$ and their quantizations. We recall that $B^{\mathrm{fd}}(\mathcal{Z})$ denotes the set of finite-dimensional operators on $\mathcal{Z}$.

Let $h \in B^{\mathrm{fd}}(\mathcal{Z})$. Consider the polynomial

$$\mathcal{Y}^{\#} \ni (\overline{z}, z) \mapsto \overline{z}{\cdot}hz. \tag{11.22}$$

The Wick, Weyl–Wigner and anti-Wick quantizations of (11.22) are, respectively,

$$\mathrm{d}\Gamma(h), \qquad \mathrm{d}\Gamma(h) + \frac{\mathrm{Tr}\, h}{2}\mathbb{1}, \qquad \mathrm{d}\Gamma(h) + (\mathrm{Tr}\, h)\mathbb{1}.$$

Note that the anti-Wick and Weyl–Wigner quantizations of (11.22) can be extended to the case $h \in B^1(\mathcal{Z})$, that is, to trace class h. The Wick quantization of (11.22) can be defined, e.g. for $h \in B(\mathcal{Z})$, or even for much more general h.

Suppose that $g \in B_{\mathrm{s}}^{\mathrm{fd}}(\overline{\mathcal{Z}}, \mathcal{Z}) \simeq \overset{\mathrm{al}}{\Gamma}{}^2(\mathcal{Z})$. Consider the polynomial

$$\mathcal{Y}^{\#} \ni (\overline{z}, z) \mapsto \overline{z}{\cdot}g\overline{z}. \tag{11.23}$$

The Wick, anti-Wick and Weyl–Wigner quantizations of (11.23) coincide with the two-particle creation operator $a^*(g)$, according to the notation of Subsect. 3.4.4. Following the notation of Def. 9.46, this can be written as $\mathrm{Op}^{a^*,a}\big(|g)\big)$. It can be defined as a closable operator also if $g \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z}) \simeq \Gamma_{\mathrm{s}}^2(\mathcal{Z})$. It will act on $\Psi_n \in \Gamma_{\mathrm{s}}^n(\mathcal{Z})$ as

$$a^*(g)\Psi_n := \sqrt{(n+2)(n+1)}\, g \otimes_{\mathrm{s}} \Psi_n. \tag{11.24}$$

(Note that on the right of (11.24) g is treated as an element of $\Gamma_{\mathrm{s}}^2(\mathcal{Z})$.)

The complex conjugate of (11.23) equals

$$\mathcal{Y}^{\#} \ni (z, \overline{z}) \mapsto z{\cdot}g^* z = z{\cdot}\overline{g}z. \tag{11.25}$$

Its Wick, anti-Wick and Weyl–Wigner quantizations coincide with the two-particle annihilation operator $a(g)$; see again Subsect. 3.4.4. Following the notation of Def. 9.46, this can be written as $\mathrm{Op}^{a^*,a}\big((g|\big)$. It is clear that $a(g)$ extends to a closable operator iff $g \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z}) \simeq \Gamma_{\mathrm{s}}(\mathcal{Z})$, and $a(g)^* = a^*(g)$.

A general element of $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}^2(\mathcal{Y}^{\#})$ is

$$\mathcal{Y}^{\#} \ni (\overline{z}, z) \mapsto 2\overline{z}{\cdot}hz + \overline{z}{\cdot}g_1\overline{z} + z{\cdot}\overline{g}_2 z, \tag{11.26}$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

where $h \in B^{\mathrm{fd}}(\mathcal{Z})$, $g_1, g_2 \in B_{\mathrm{s}}^{\mathrm{fd}}(\overline{\mathcal{Z}}, \mathcal{Z})$. We can write (11.26) as

$$(\overline{z}, z)\cdot\zeta(\overline{z}, z), \quad \text{where} \quad \zeta_{\mathbb{C}} = \begin{bmatrix} g_1 & h \\ h^{\#} & \overline{g}_2 \end{bmatrix} \in B_{\mathrm{s}}(\overline{\mathcal{Z}} \oplus \mathcal{Z}, \mathcal{Z} \oplus \overline{\mathcal{Z}}). \tag{11.27}$$

(Recall that $\mathbb{C}\mathcal{Y} \simeq \mathcal{Z} \oplus \overline{\mathcal{Z}}$, $\mathbb{C}\mathcal{Y}^{\#} \simeq \overline{\mathcal{Z}} \oplus \mathcal{Z}$ and we use elements of $B_{\mathrm{s}}^{\mathrm{fd}}(\mathcal{Y}^{\#}, \mathcal{Y})$ for symbols of bosonic quadratic Hamiltonians, as in Def. 10.15.)

The quantizations of ζ are

$$\mathrm{Op}^{a^*,a}(\zeta) = 2\mathrm{d}\Gamma(h) + a^*(g_1) + a(g_2), \tag{11.28}$$

$$\mathrm{Op}(\zeta) = 2\mathrm{d}\Gamma(h) + (\mathrm{Tr}\, h)\mathbb{1} + a^*(g_1) + a(g_2), \tag{11.29}$$

$$\mathrm{Op}^{a,a^*}(\zeta) = 2\mathrm{d}\Gamma(h) + (2\mathrm{Tr}\, h)\mathbb{1} + a^*(g_1) + a(g_2).$$

Clearly,

$$\mathrm{Op}(\zeta) = \frac{1}{2}\left(\mathrm{Op}^{a^*,a}(\zeta) + \mathrm{Op}^{a,a^*}(\zeta)\right).$$

We can extend the definition of $\mathrm{Op}(\zeta)$ and $\mathrm{Op}^{a,a^*}(\zeta)$ to the case of $g_1, g_2 \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ and $h \in B^1(\mathcal{Z})$. $\mathrm{Op}^{a^*,a}(\zeta)$ is defined under much more general conditions. All these quantizations are Hermitian operators iff h is Hermitian and $g_1 = g_2$.

11.2.2 Bosonic Schwinger term

For simplicity, in this subsection we assume that $\mathcal{Z}$ is finite-dimensional. Recall from Thm. 10.13 that the Weyl–Wigner quantization restricted to quadratic symbols yields an isomorphism of the Lie algebra $sp(\mathcal{Y})$ into quadratic Hamiltonians in $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$. This is no longer true in the case of the Wick quantization, where the so-called Schwinger term appears. This is described in the following proposition:

Proposition 11.26 *Let $\zeta, \zeta_i \in B_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$, $i = 1, 2$. Then*

$$\mathrm{Op}(\zeta) = \mathrm{Op}^{a^*,a}(\zeta) - \frac{1}{2}(\mathrm{Tr}\, \zeta\omega\mathrm{j})\, \mathbb{1}, \tag{11.30}$$

$$\left[\mathrm{Op}^{a^*,a}(\zeta_1), \mathrm{Op}^{a^*a}(\zeta_2)\right] = 2\mathrm{i}\mathrm{Op}^{a^*,a}(\zeta_2\omega\zeta_1 + \zeta_1\omega\zeta_2) - \mathrm{i}(\mathrm{Tr}\,[\zeta_2\omega, \zeta_1\omega]\mathrm{j})\, \mathbb{1}.$$

Proof Let ζ be as in (11.27). We have $\omega_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix} \in B(\mathcal{Z} \oplus \overline{\mathcal{Z}}, \overline{\mathcal{Z}} \oplus \mathcal{Z})$. Therefore,

$$\zeta_{\mathbb{C}}\omega_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} -h & g_1 \\ -\overline{g}_2 & h^{\#} \end{bmatrix}, \quad \zeta_{\mathbb{C}}\omega_{\mathbb{C}}\mathrm{j}_{\mathbb{C}} = \begin{bmatrix} h & g_1 \\ \overline{g}_2 & h^{\#} \end{bmatrix}.$$

Therefore, (11.30) follows from (11.29).

Now to compute the Schwinger term we apply Prop. 10.16 (1) and (11.30). □

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

11.2.3 Infimum of bosonic quadratic Hamiltonians

For simplicity, in this subsection we again assume that $\mathcal{Z}$ is finite-dimensional. The Wick quantization of a positive quadratic symbol is then bounded from below. In the following theorem we compute the infimum of the spectrum of such Hamiltonians.

Theorem 11.27 *Let $h \in B_{\mathrm{h}}(\mathcal{Z})$, $g \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Suppose that for $z \in \mathcal{Z}$*

$$(\overline{z}, z)\cdot\zeta(\overline{z}, z) = 2\overline{z}\cdot hz + z\cdot g^* z + \overline{z}\cdot g\overline{z} \geq 0. \tag{11.31}$$

Set

$$\zeta_{\mathbb{C}} = \begin{bmatrix} g & h \\ h^{\#} & g^* \end{bmatrix} \in B_{\mathrm{s}}(\overline{\mathcal{Z}} \oplus \mathcal{Z}, \mathcal{Z} \oplus \overline{\mathcal{Z}}). \tag{11.32}$$

Then

$$\inf \mathrm{Op}^{a^*,a}(\zeta) = \frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} h^2 - gg^* & -hg + gh^{\#} \\ g^*h - h^{\#}g^* & h^{\#\,2} - g^*g \end{bmatrix}^{\frac{1}{2}} - \begin{bmatrix} h & 0 \\ 0 & h^{\#} \end{bmatrix}\right).$$

Proof We have $(\zeta_{\mathbb{C}}\omega_{\mathbb{C}})^2 = -\begin{bmatrix} h^2 - gg^* & -hg + gh^{\#} \\ g^*h - h^{\#}g^* & h^{\#\,2} - g^*g \end{bmatrix}$. Thus, by Thm. 10.17,

$$\begin{aligned} \inf \mathrm{Op}^{a^*,a}(\zeta) + \mathrm{Tr}\, h &= \inf \mathrm{Op}(\zeta) \\ &= \frac{1}{2}\mathrm{Tr}\,|\zeta\omega| = \frac{1}{2}\mathrm{Tr}\begin{bmatrix} h^2 - gg^* & -hg + gh^{\#} \\ g^*h - h^{\#}g^* & h^{\#\,2} - g^*g \end{bmatrix}^{\frac{1}{2}}. \end{aligned}$$

□

11.2.4 Gaussian vectors

Let $c \in \Gamma_{\mathrm{s}}^2(\mathcal{Z})$. Recall that we can define the two-particle creation operator $a^*(c)$ acting on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ as in Subsect. 3.4.4, and that we can identify c with an operator $c \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ (see Subsect. 3.3.4). Since $c \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$, c^*c is trace-class, so $\det(\mathbb{1} - c^*c)$ is well defined. If we assume also that $c^*c < \mathbb{1}$, then $\det(\mathbb{1} - c^*c) > 0$. So we can define

$$\Omega_c := \det(\mathbb{1} - c^*c)^{\frac{1}{4}}\mathrm{e}^{\frac{1}{2}a^*(c)}\Omega. \tag{11.33}$$

Theorem 11.28 (1) *If $c \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ and $c^*c < \mathbb{1}$, then Ω_c is a normalized vector in $\Gamma_{\mathrm{s}}(\mathcal{Z})$.*

(2) *Let k be a positive number with $k^2|c| < \mathbb{1}$. Then Ω_c belongs to* $\mathrm{Dom}\, k^N$ *and $k^N\Omega_c = \Omega_{k^2c}$, where N is the number operator.*

(3) *Suppose that c is a densely defined operator from $\overline{\mathcal{Z}}$ to $\mathcal{Z}$ such that $(z_1|c\overline{z}_2) = (z_2|c\overline{z}_1)$, i.e. $c \subset c^{\#}$. Suppose that there exists $\Psi \in \Gamma_{\mathrm{s}}(\mathcal{Z})$ satisfying*

$$\big(a(z) - a^*(c\overline{z})\big)\Psi = 0, \quad \overline{z} \in \mathrm{Dom}\ c,$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

in the weak sense. Then $c \in B_{\rm s}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ *and* $c^*c < \mathbb{1}$*. Moreover,* Ψ *is proportional to* Ω_c*.*

(4) *Let* $c_1, c_2 \in B_{\rm s}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ *and* $c_i^* c_i < \mathbb{1}$*. Then*

$$(\Omega_{c_1}|\Omega_{c_2}) = \det(\mathbb{1} - c_1^* c_1)^{\frac{1}{4}} \det(\mathbb{1} - c_2^* c_2)^{\frac{1}{4}} \det(\mathbb{1} - c_1^* c_2)^{-\frac{1}{2}}.$$

Proof First let Ψ be as in (3), and let $z_1, z_2, \dots$ be a sequence of vectors in $\operatorname{Dom} \overline{c}$. For $I \subset \{1, 2, \dots\}$ finite, set

$$M(I) = \Big(\prod_{i \in I} a^*(z_i)\Omega \Big| \Psi \Big).$$

From

$$0 = \Big(\prod_{i \in \{1, \cdots n\}} a^*(z_i)\Omega \Big| (a^*(z_{n+1}) - a(c\overline{z}_{n+1}))\Psi \Big)$$

we obtain

$$M(\{1, \cdots, n+1\}) = \sum_{i=1}^{n} (z_i | c\overline{z}_{n+1}) M(\{1, \cdots, n\} \backslash \{i\}).$$

This yields

$$\begin{aligned} (a^*(z_1) \cdots a^*(z_{2m+1})\Omega|\Psi) &= 0, \\ (a^*(z_1) \cdots a^*(z_{2m})\Omega|\Psi) &= \lambda \sum_{\sigma \in \mathrm{Pair}_{2m}} \prod_{i=0}^{m-1} (z_{\sigma(2i+1)} | c\overline{z}_{\sigma(2i+2)}), \end{aligned}$$

where $\lambda := (\Omega|\Psi)$ and Pair_{2m} is the set of pairings in $\{1, \dots, 2m\}$ (see Subsect. 3.6.10).

In particular, for $\overline{z}_1, \overline{z}_2 \in \operatorname{Dom} c$ this gives the following formula for the two-particle component of Ψ:

$$\sqrt{2}(z_1 \otimes_{\rm s} z_2 | \Psi) = \lambda(z_1 | c\overline{z}_2). \tag{11.34}$$

Since $\Psi \in \Gamma_{\rm s}(\mathcal{Z})$, the l.h.s. of (11.34) can be extended to a bounded functional on $\Gamma_{\rm s}^2(\mathcal{Z})$. This implies that either $\lambda = 0$ or $c \in \Gamma_{\rm s}^2(\mathcal{Z})$, and then the l.h.s. of (11.34) equals $\lambda(z_1 \otimes_{\rm s} z_2 | c)$.

We have

$$\begin{aligned} (z_1 \otimes_{\rm s} \cdots \otimes_{\rm s} z_{2m} | c^{\otimes_{\rm s} m}) &= \frac{1}{2m!} \sum_{\sigma \in S_{2m}} \prod_{i=0}^{m-1} (z_{\sigma(2i+1)} | c\overline{z}_{\sigma(2i+2)}) \\ &= \frac{m! 2^m}{2m!} \sum_{\sigma \in \mathrm{Pair}_{2m}} \prod_{i=0}^{m-1} (z_{\sigma(2i+1)} | c\overline{z}_{\sigma(2i+2)}), \end{aligned}$$

which implies that

$$\Psi_{2m} = \lambda \frac{\sqrt{(2m)!}}{2^m m!} c^{\otimes_{\rm s} m} = \lambda \frac{1}{2^m m!} (a^*(c))^m \Omega, \qquad \Psi_{2m+1} = 0,$$

i.e. $\Psi = \lambda \mathrm{e}^{\frac{1}{2} a^*(c)} \Omega$.

Let us compute $\|\mathrm{e}^{\frac{1}{2}a^*(c)}\Omega\|^2$. Since c^*c is trace-class, we can by Corollary 2.88 find an o.n. basis $\{e_i, : i \in I\}$ of $\mathcal{Z}$ such that $c\overline{e_i} = \lambda_i e_i$, $\lambda_i \geq 0$. Using this basis, we can unitarily identify $\mathcal{Z}$ with $\underset{i\in I}{\oplus}\mathbb{C}$. By the exponential law of Subsect. 3.5.4, we unitarily identify $\Gamma_{\mathrm{s}}(\mathcal{Z})$ with $\underset{i\in I}{\otimes}(\Gamma_{\mathrm{s}}(\mathbb{C}),\Omega)$. Under this identification,

$$\mathrm{e}^{\frac{1}{2}a^*(c)}\Omega \simeq \underset{i\in I}{\otimes}\mathrm{e}^{\frac{1}{2}\lambda_i a^{*2}}\Omega,$$

$$\begin{aligned}\left\|\mathrm{e}^{\frac{1}{2}a^*(c)}\Omega\right\|^2_{\Gamma_{\mathrm{s}}(\mathcal{Z})} &= \prod_{i\in I}\left\|\mathrm{e}^{\frac{1}{2}\lambda_i a^{*2}}\Omega\right\|^2_{\Gamma_{\mathrm{s}}(\mathbb{C})}\\ &= \prod_{i\in I}\sum_{m=0}^{\infty}\frac{(2m)!\lambda_i^{2m}}{2^{2m}(m!)^2}\\ &= \prod_{i\in I}\left(1-\lambda_i^2\right)^{-\frac{1}{2}} = \det\left(\mathbb{1}-c^*c\right)^{-\frac{1}{2}}.\end{aligned} \tag{11.35}$$

This shows that the vector Ω_c in (1) is normalized. Moreover, if $\lambda_i \geq 1$ for some $i \in I$, then one of the series on the r.h.s. of (11.35) is divergent, which contradicts the fact that the vector Ψ is normalizable. This shows the necessity of the condition $c^*c < \mathbb{1}$ and completes the proof of (3).

Let us now prove (2). Since $\mathrm{e}^{\mathrm{i}tN}a^*(c)\mathrm{e}^{-\mathrm{i}tN} = \mathrm{e}^{2\mathrm{i}t}a^*(c)$, we obtain that

$$\mathrm{e}^{\mathrm{i}tN}\Omega_c = \det(\mathbb{1}-c^*c)^{\frac{1}{4}}\mathrm{e}^{\frac{1}{2}a^*(c_t)}\Omega,$$

for $c_t = \mathrm{e}^{2\mathrm{i}t}c$. It follows that if $k^4c^*c < \mathbb{1}$, $\mathrm{e}^{\mathrm{i}tN}\Omega_c$ extends holomorphically in $\{z \in \mathbb{C} : \ \mathrm{Im}\, z > -\log k\}$ and is uniformly bounded on this set. Therefore, $\Omega_c \in \mathrm{Dom}\, k^N$ and $k^N\Omega_c = \det(\mathbb{1}-c^*c)^{1/4}\mathrm{e}^{\frac{1}{2}a^*(k^2c)}\Omega$, which proves (2).

It remains to prove (4). Let us first assume that $\mathcal{Z}$ is finite-dimensional. In the complex-wave representation, $\mathrm{e}^{\frac{1}{2}a^*(c)}\Omega$ equals $\mathrm{e}^{\frac{1}{2}\overline{z}\cdot c\overline{z}}$ and

$$\left(\mathrm{e}^{\frac{1}{2}a^*(c_1)}\Omega|\mathrm{e}^{\frac{1}{2}a^*(c_2)}\Omega\right) = (2\pi\mathrm{i})^{-d}\int_{\mathrm{Re}(\overline{\mathcal{Z}}\oplus\mathcal{Z})}\mathrm{e}^{-|z|^2}\mathrm{e}^{\frac{1}{2}z\cdot\overline{c_1}z}\mathrm{e}^{\frac{1}{2}\overline{z}\cdot c_2\overline{z}}\mathrm{d}z\mathrm{d}\overline{z}. \tag{11.36}$$

To compute this integral, we use the arguments in Subsect. 4.1.9. We are led to compute $\det\nu$, where ν is an operator on $\mathrm{Re}(\overline{\mathcal{Z}}\oplus\mathcal{Z})$ given by

$$\nu_{\mathbb{C}} = \det\begin{bmatrix}\mathbb{1} & -\overline{c}_1\\ -c_2 & \mathbb{1}\end{bmatrix}.$$

But $\det\nu = \det\nu_{\mathbb{C}} = \det(\mathbb{1}-\overline{c}_1c_2)$. From (4.10), we obtain that (11.36) equals

$$(\mathrm{e}^{\frac{1}{2}a^*(c_1)}\Omega|\mathrm{e}^{\frac{1}{2}a^*(c_2)}\Omega) = \det(\mathbb{1}-\overline{c}_1c_2)^{-\frac{1}{2}}.$$

Let us now prove (4) in the general case. For simplicity, we will assume that $\mathcal{Z}$ is separable (the non-separable case can be treated by the same argument, replacing sequences by nets). Let us fix an increasing sequence of finite rank projections $\pi_1, \pi_2, \ldots$ such that $\mathrm{s}-\lim_{n\to\infty}\pi_n = \mathbb{1}$. For $c \in B^2_{\mathrm{s}}(\overline{\mathcal{Z}},\mathcal{Z})$ we set $c_n = \pi_n c\overline{\pi}_n$, so

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

that $c_n \to c$ in the Hilbert–Schmidt norm. We claim that

$$\lim_{n\to\infty} \Omega_{c_n} = \Omega_c. \tag{11.37}$$

By approximating c_1, c_2 by $c_{1,n}, c_{2,n}$, this implies (4) in the general case.

It remains to prove (11.37). Using (4) in the finite-dimensional case, we get that

$$\|\Omega_{c_n} - \Omega_{c_m}\|^2 = 2 - 2\mathrm{Re}(\Omega_{c_n}|\Omega_{c_m}) \to 0 \text{ when } n, m \to \infty,$$

hence the sequence $\Omega_{c_1}, \Omega_{c_2}, \dots$ converges to a normalized vector Ψ. There exists $k > 1$ and $k_0 < 1$ such that, for all n, $k^4 c_n^* c_n < k_0^4 \mathbb{1}$. Using (2), we obtain that $\Psi \in \mathrm{Dom}\, N$ and that Ω_{c_n} converges to Ψ in $\mathrm{Dom}\, N$. Therefore, we can let $n \to \infty$ in the identity

$$\big(a(z) - a^*(c_n \overline{z})\big)\Omega_{c_n} = 0$$

to get

$$\big(a(z) - a^*(c\overline{z})\big)\Psi = 0.$$

Since $(\Psi|\Omega) = \lim_{n\to\infty} (\Omega_{c_n}|\Omega) \geq 0$, (3) implies that $\Psi = \Omega_c$. $\square$

11.2.5 Gaussian vectors in the real-wave representation

Let $c \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ such that $c^* c < \mathbb{1}$. Let k be the positive symplectic transformation defined in terms of c by formula (11.20), that is,

$$k_{\mathbb{C}} = \begin{bmatrix} (\mathbb{1} + cc^*)(\mathbb{1} - cc^*)^{-1} & 2(\mathbb{1} - cc^*)^{-1} c \\ 2c^*(\mathbb{1} - cc^*)^{-1} & (\mathbb{1} + c^*c)(\mathbb{1} - c^*c)^{-1} \end{bmatrix}.$$

As discussed in Subsect. 11.1.9, we can identify $\mathcal{Y}$ with $\mathcal{X} \oplus \mathcal{X}$, where $\mathcal{X}$ is a real Hilbert space, the symplectic form on $\mathcal{Y}$ has the standard form and

$$\mathrm{j} = \begin{bmatrix} 0 & -\frac{1}{2}\mathbb{1} \\ 2\mathbb{1} & 0 \end{bmatrix}, \quad k = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix},$$

where $m \in B_{\mathrm{s}}(\mathcal{X})$, $m > 0$ and $\mathbb{1} - m \in B^2(\mathcal{X})$.

In Sect. 9.3 we described the unitary map T^{rw} between the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$ and the Gaussian $\mathbf{L}^2$ space $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$ intertwining the Fock and the real-wave representations such that $T^{\mathrm{rw}}\Omega = 1$.

Proposition 11.29 *In the real-wave representation, we have*

$$T^{\mathrm{rw}}\Omega_c(x) = C\mathrm{e}^{\frac{1}{4}x\cdot(\mathbb{1} - m^{-1})x},$$

where C is a "normalizing constant" (see Prop. 5.79). If $\mathbb{1} - m \in B^1(\mathcal{X})$ then

$$T^{\mathrm{rw}}\Omega_c(x) = (\det m)^{-\frac{1}{4}}\mathrm{e}^{\frac{1}{4}x\cdot(\mathbb{1} - m^{-1})x}.$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

Proof Assume first that $\mathcal{Y}$ is finite-dimensional. The proposition can then be proved by a direct computation, using that

$$\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x) = L^2(\mathcal{X}, (2\pi)^{-\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x).$$

In the general case, we use the same approximation argument as in the proofs of Thm. 11.28 and of Thm. 5.78, given in Subsect. 11.4.6. $\square$

11.2.6 Two-particle creation and annihilation operators

In this subsection we discuss certain properties of two-particle creation and annihilation operators.

Proposition 11.30 *Let $c \in \Gamma_{\mathrm{s}}^2(\mathcal{Z}) \simeq B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Then*

(1) *$a^*(c)$ and $a(c)$ with domain $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ are densely defined closable operators.*
(2) *$a^*(c) + a(c)$ is essentially self-adjoint on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$.*
(3) *$\mathrm{e}^{\frac{1}{2}a^*(c)}$ and $\mathrm{e}^{\frac{1}{2}a(c)}$ are closable on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$ iff $c^*c < 1\!\!1$, and we have*

$$\mathrm{e}^{-\frac{1}{2}a^*(c)}a(z) = \big(a(z) + a^*(c\overline{z})\big)\mathrm{e}^{-\frac{1}{2}a^*(c)}, \qquad z \in \mathcal{Z}; \tag{11.38}$$

$$\mathrm{e}^{\frac{1}{2}a(c)}a^*(z) = \big(a^*(z) + a(c\overline{z})\big)\mathrm{e}^{\frac{1}{2}a(c)}, \qquad z \in \mathcal{Z}. \tag{11.39}$$

Proof We have $a(c) \subset a^*(c)^*$, $a^*(c) \subset a(c)^*$, which proves (1).

To prove (2) we will use Nelson's commutator theorem, Thm. 2.74 (1), with the comparison operator $N + 1\!\!1$. By Prop. 9.50 we get that $(N + 1\!\!1)^{-1}a^*(c)$ and $a(c)(N + 1\!\!1)^{-1}$ are bounded. Since

$$Na^*(c) = a^*(c)(N + 21\!\!1), \quad Na(c) = a(c)(N - 21\!\!1),$$

we obtain that $a^*(c)(N + 1\!\!1)^{-1}$ is bounded, so $a^*(c) + a(c)$ is bounded on $\mathrm{Dom}\, N$. Next, since

$$[N, a^*(c) + a(c)] = 2(a^*(c) - a(c)),$$

we get that $\pm[N, a^*(c) + a(c)] \leq C(N + 1\!\!1)$. Applying Nelson's commutator theorem, we see that $a^*(c) + a(c)$ is essentially self-adjoint on $\mathrm{Dom}\, N$, hence on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$.

It remains to prove (3). Clearly, $\mathrm{e}^{\frac{1}{2}a(c)}$ is defined on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$. We note also that $\mathrm{e}^{\frac{1}{2}a^*(c)}\Omega \in \Gamma_{\mathrm{s}}(\mathcal{Z})$ iff $c^*c < 1\!\!1$, by Thm. 11.28. It remains to prove that if $c^*c < 1\!\!1$, then $\mathrm{e}^{\frac{1}{2}a^*(c)}$ is defined on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$. This is equivalent to showing that $\mathrm{e}^{\frac{1}{2}a^*(c)}\Omega \in \mathrm{Dom} \prod_{i=1}^{n} a^*(z_i)$ for all $z_1, \dots, z_n \in \mathcal{Z}$. But this follows from Thm. 11.28 (2). Since $\mathrm{e}^{\frac{1}{2}a(c)} \subset (\mathrm{e}^{\frac{1}{2}a^*(c)})^*$, $\mathrm{e}^{\frac{1}{2}a^*(c)} \subset (\mathrm{e}^{\frac{1}{2}a(c)})^*$, we see that $\mathrm{e}^{\frac{1}{2}a(c)}$ and $\mathrm{e}^{\frac{1}{2}a^*(c)}$ are closable. Identities (11.38) and (11.39) are direct computations. $\square$

11.3 Bosonic Bogoliubov transformations on Fock spaces

We use the same framework as in the previous section. $(\mathcal{Y}, \cdot, \omega, \mathrm{j})$ is a complete Kähler space. $\mathcal{Z}$ is the holomorphic subspace of $\mathbb{C}\mathcal{Y}$, so that we can identify $\mathcal{Y}$ with $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$.

$$\mathcal{Y} \ni y \mapsto W(y) \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big) \tag{11.40}$$

is the Fock CCR representation.

The central result of this section is the version of Shale's theorem which says that a symplectic transformation is implementable in the Fock CCR representation iff it belongs to the *restricted symplectic group*. The corresponding automorphism of the algebra of operators will be called the *Bogoliubov automorphism*. Unitary operators implementing Bogoliubov automorphisms (the so-called *Bogoliubov implementers*) form a group $Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$, which can be viewed as the natural generalization of the group $Mp^{\mathrm{c}}(\mathcal{Y})$ to the case of an infinite dimension.

We will also describe the group $Mp_{\mathrm{j,af}}(\mathcal{Y})$, which is a generalization of the group $Mp(\mathcal{Y})$ to infinite dimensions. Note that both $Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ and $Mp_{\mathrm{j,af}}(\mathcal{Y})$ depend on the Kähler structure on $\mathcal{Y}$ (which is expressed by putting j as a subscript).

11.3.1 Symplectic transformations in the Fock representation

Definition 11.31 *We define $Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ to be the set of $U \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ such that*

$$\big\{UW(y)U^* \ : \ y \in \mathcal{Y}\big\} = \big\{W(y) \ : \ y \in \mathcal{Y}\big\}.$$

We equip $Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ with the strong operator topology.

Definition 11.32 *Let $r \in Sp(\mathcal{Y})$.*

(1) *We say that $U \in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$* intertwines *$r$ if*

$$UW(y)U^* = W(ry), \quad y \in \mathcal{Y}. \tag{11.41}$$

(2) *If in addition U is unitary, then we say that U* implements *r.*
(3) *If there exists $U \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ that implements r, then we say that r is* implementable on $\Gamma_{\mathrm{s}}(\mathcal{Z})$.

We will prove:

Theorem 11.33 (Shale's theorem about Bogoliubov transformations) (1) *Let $r \in Sp(\mathcal{Y})$. Then r is implementable iff $r \in Sp_{\mathrm{j}}(\mathcal{Y})$.*
(2) *Let $U \in Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$. Then there exists a unique $r \in Sp_{\mathrm{j}}(\mathcal{Y})$ such that r is implemented by U. $Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y})$ is a group and the map $Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y}) \to Sp_{\mathrm{j}}(\mathcal{Y})$ obtained this way is a group homomorphism.*

(3) *Let $r \in Sp_{\mathrm{j}}(\mathcal{Y})$. Let p, d, c be defined as in Subsect. 11.1.3. Define*

$$U_r^{\mathrm{j}} = |\det pp^*|^{-\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}a^*(d)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac{1}{2}a(c)}. \tag{11.42}$$

Then U_r^{j} is the unique unitary operator implementing r in the Fock representation such that

$$(\Omega|U_r^{\mathrm{j}}\Omega) > 0. \tag{11.43}$$

All operators implementing r in the Fock representation are of the form μU_r, where $|\mu| = 1$.

(4) *We have an exact sequence*

$$1 \to U(1) \to Mp_{\mathrm{j}}^{\mathrm{c}}(\mathcal{Y}) \to Sp_{\mathrm{j}}(\mathcal{Y}) \to 1. \tag{11.44}$$

Proof of Thm. 11.33. Let us prove (3). By Prop. 3.53, we have

$$\Gamma(p^{*-1})a^*(z) = a^*(p^{*-1}z)\Gamma(p^{*-1}), \tag{11.45}$$

$$\Gamma(p^{*-1})a(z) = a(pz)\Gamma(p^{*-1}). \tag{11.46}$$

Set $V := \mathrm{e}^{-\frac{1}{2}a^*(d)} \Gamma\big((p^*)^{-1}\big) \mathrm{e}^{\frac{1}{2}a(c)}$. Using (11.45), (11.46), (11.38) and (11.39), we see that

$$\begin{aligned} Va^*(z) &= \big(a^*(p^{*-1}z + d\overline{pc}z) + a(pc\overline{z})\big)V = \big(a^*(pz) + a(q\overline{z})\big)V, \\ Va(z) &= \big(a(pz) + a^*(d\overline{pz})\big)V = \big(a^*(q\overline{z}) + a(pz)\big)V. \end{aligned}$$

Therefore,

$$V\phi(y) = \phi(ry)V, \quad y \in \mathcal{Y}.$$

Thus V intertwines the representations (11.40) and (11.41). These representations are irreducible. Hence, by Prop. 8.13, V is proportional to a unitary operator. Clearly,

$$V\Omega = \mathrm{e}^{-\frac{1}{2}a^*(d)}\Omega.$$

By Thm. 11.28,

$$\|V\Omega\|^2 = \det(1\!\!1 - d^*d)^{-\frac{1}{2}} = (\det pp^*)^{\frac{1}{2}}.$$

Hence, $U_r^{\mathrm{j}} = (\det pp^*)^{-\frac{1}{4}}V$ is unitary. $(\Omega|U_r^{\mathrm{j}}\Omega) = (\det pp^*)^{-\frac{1}{4}} > 0$, hence U_r^{j} also satisfies the condition (11.43). The uniqueness is obvious.

Let $r \in Sp(\mathcal{Y})$. Suppose that $UW(y) = W(ry)U$, $y \in \mathcal{Y}$. Define the operator c as in (11.3). Then the vector $U\Omega$ satisfies the conditions of Thm. 11.28 (3). Hence, $c \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$. By Prop. 11.12, this is equivalent to $r \in Sp_{\mathrm{j}}(\mathcal{Y})$. □

11.3.2 One-parameter groups of Bogoliubov transformations

Let $h \in B_{\mathrm{h}}(\mathcal{Z})$, $g \in B^2_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$. Let $\zeta \in B_{\mathrm{s}}(\mathcal{Y}^{\#}, \mathcal{Y})$ be defined by $\zeta_{\mathbb{C}} = \begin{bmatrix} g & h \\ \overline{h} & \overline{g} \end{bmatrix}$. Recall that

$$\mathrm{Op}^{a^*,a}(\zeta) = 2\mathrm{d}\Gamma(h) + a^*(g) + a(g)$$

is a self-adjoint operator. If in addition $h \in B^1_{\mathrm{h}}(\mathcal{Z})$, then we can use the Weyl–Wigner quantization to quantize ζ, obtaining

$$\mathrm{Op}(\zeta) = 2\mathrm{d}\Gamma(h) + a^*(g) + a(g) + (\mathrm{Tr}\, h)\mathbb{1}. \tag{11.47}$$

Let $a \in sp(\mathcal{Y})$ be given by

$$a_{\mathbb{C}} = \zeta_{\mathbb{C}}\omega_{\mathbb{C}} = \mathrm{i}\begin{bmatrix} -h & g \\ -\overline{g} & \overline{h} \end{bmatrix}$$

(see (11.2)). Let $r_t = \mathrm{e}^{ta}$ and

$$r_{t\mathbb{C}} = \begin{bmatrix} p_t & q_t \\ \overline{q}_t & \overline{p}_t \end{bmatrix}$$

For $t \in \mathbb{R}$ we set

$$d_t := q_t \overline{p}_t^{-1}, \quad c_t := q_t^{\#}(p_t^{\#})^{-1}.$$

The following theorem gives the unitary group generated by the Wick and Weyl–Wigner quantizations of ζ:

Theorem 11.34 (1) *For any* $t \in \mathbb{R}$, $p_t\mathrm{e}^{-\mathrm{i}th} - \mathbb{1} \in B^1(\mathcal{Z})$, $d_t, c_t \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$ *and*

$$\mathrm{e}^{\mathrm{i}t\mathrm{Op}^{a^*,a}(\zeta)} = \left(\det p_t \mathrm{e}^{-\mathrm{i}th}\right)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}a^*(d_t)}\Gamma(p_t^{*-1})\mathrm{e}^{\frac{1}{2}a(c_t)}. \tag{11.48}$$

Besides, (11.48) implements r_t.

(2) *If in addition* $h \in B^1_{\mathrm{h}}(\mathcal{Z})$, *then* $p_t - \mathbb{1} \in B^1(\mathcal{Z})$ *and*

$$\mathrm{e}^{\mathrm{i}t\mathrm{Op}(\zeta)} = (\det p_t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}a^*(d_t)}\Gamma(p_t^{*-1})\mathrm{e}^{\frac{1}{2}a(c_t)}. \tag{11.49}$$

(In both (11.48) and (11.49) the branch of the square root is determined by continuity.)

11.3.3 Implementation of positive symplectic transformations

Let us consider a positive $r \in Sp_{\mathrm{j}}(\mathcal{Y})$. From formula (11.11) we know that there exists $c \in B^2_{\mathrm{s}}(\overline{\mathcal{Z}}, \mathcal{Z})$ such that

$$r_{\mathbb{C}} = \begin{bmatrix} \mathbb{1} & c \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} (\mathbb{1} - cc^*)^{\frac{1}{2}} & 0 \\ 0 & (\mathbb{1} - c^*c)^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ c^* & \mathbb{1} \end{bmatrix}.$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

By Thm. 11.33, r is then implemented by

$$U_r^{\mathrm{j}} = \det(\mathbb{1} - cc^*)^{\frac{1}{4}} \mathrm{e}^{-\frac{1}{2}a^*(c)} \Gamma(\mathbb{1} - cc^*)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{2}a(c)}. \tag{11.50}$$

We recall also that $a \in sp_{\mathrm{j}}(\mathcal{Y})$ is self-adjoint iff

$$a_{\mathbb{C}} = \mathrm{i} \begin{bmatrix} 0 & -g \\ g^* & 0 \end{bmatrix}, \tag{11.51}$$

for $g \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$.

The following formula is essentially a special case of (11.48) for $r_t = \mathrm{e}^{ta}$:

$$\begin{aligned} U_{r_t}^{\mathrm{j}} &= \mathrm{e}^{\frac{\mathrm{i}t}{2}\left(a^*(g)+a(g)\right)} \qquad (11.52) \\ &= \left(\det \cosh\left(t\sqrt{gg^*}\right)\right)^{-\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}t}{2}a^*\left(\frac{\tanh\sqrt{gg^*}}{\sqrt{gg^*}}g\right)} \Gamma\left(\cosh(t\sqrt{gg^*})\right)^{-1} \mathrm{e}^{-\frac{\mathrm{i}t}{2}a\left(\frac{\tanh\sqrt{gg^*}}{\sqrt{gg^*}}g\right)}. \end{aligned}$$

Suppose now that τ is a conjugation as in Thm. 11.24. By Prop. 11.25 we can identify $\mathcal{Y}$ with $\mathcal{X} \oplus \mathcal{X}$, so that for $m \in L_{\mathrm{s}}(\mathcal{X})$, $m > 0$,

$$r = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}.$$

Recall also that we defined the unitary map T^{rw} between the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$ and Gaussian $\mathbf{L}^2$ space $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$ intertwining the Fock and real-wave representations such that $T^{\mathrm{rw}}\Omega = 1$.

Proposition 11.35 *In the real-wave representation on $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$ the operator U_r^{j} takes the form*

$$T^{\mathrm{rw}} U_r^{\mathrm{j}} T^{\mathrm{rw}*} F(x) = (\det m)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{4}x\cdot(\mathbb{1}-m^2)x} F(mx).$$

11.3.4 Metaplectic group in the Fock representation

$Sp_{\mathrm{j}}(\mathcal{Y}) \ni r \mapsto U_r^j$ is not a representation – it is only a projective representation. To construct a true representation we need to restrict ourselves to $Sp_{\mathrm{j,af}}(\mathcal{Y})$. Thus we will obtain a generalization of the metaplectic representation to infinite dimensions.

Definition 11.36 *For $r \in Sp_{\mathrm{j,af}}(\mathcal{Y})$ we define*

$$\pm U_r := \pm(\det p^*)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}a^*(d)} \Gamma\left((p^*)^{-1}\right) \mathrm{e}^{\frac{1}{2}a(c)}. \tag{11.53}$$

(We take both signs of the square root, thus $\pm U_r$ denotes a pair of operators differing by the sign.)

Definition 11.37 *We denote by $Mp_{\mathrm{j,af}}(\mathcal{Y})$ the set of operators of the form $\pm U_r$ for some $r \in Sp_{\mathrm{j,af}}(\mathcal{Y})$. We equip it with the strong operator topology.*

Theorem 11.38 *$Mp_{\mathrm{j,af}}(\mathcal{Y})$ is a topological group. We have the exact sequence*

$$1 \to \mathbb{Z}_2 \to Mp_{\mathrm{j,af}}(\mathcal{Y}) \to Sp_{\mathrm{j,af}}(\mathcal{Y}) \to 1. \tag{11.54}$$

If $\mathcal{Y}$ is finite-dimensional, then $Mp_{\mathrm{j,af}}(\mathcal{Y})$ coincides with $Mp(\mathcal{Y})$ introduced in Def. 10.28.

The proof of this theorem is based on the following lemma:

Lemma 11.39 (1) *$\pm U_r$ are unitary.*
(2) *$U_r W(y) U_r^* = W(ry)$.*
(3) *$U_{r_1} U_{r_2} = \pm U_{r_1 r_2}$.*
(4) *If $\mathcal{Y}$ is finite-dimensional, then $\pm U_r$ coincides with $\pm U_r$ introduced in Def. 10.27.*

Proof The operators U_r differ by a phase factor from U_r^{j} from Thm. 11.33. Therefore, they are unitary and implement r. This proves (1) and (2).

Let us prove (3). We know that

$$(\Omega|U_{r_1 r_2}\Omega) = \pm(\det p^*)^{-\frac{1}{2}} = \pm\big(\det(p_1 p_2 + q_1 \overline{q}_2)^*\big)^{-\frac{1}{2}}. \tag{11.55}$$

Moreover,

$$\begin{aligned}(\Omega|U_{r_1} U_{r_2}\Omega) &= \pm\big(\mathrm{e}^{\frac{1}{2}a^*(c_1)}\Omega|\mathrm{e}^{-\frac{1}{2}a^*(d_2)}\Omega\big)(\det p_1^*)^{-\frac{1}{2}}(\det p_2^*)^{-\frac{1}{2}} \\ &= \pm\det(\mathbb{1} + d_2 c_1^*)^{-\frac{1}{2}}(\det p_1^*)^{-\frac{1}{2}}(\det p_2^*)^{-\frac{1}{2}},\end{aligned}$$

using Thm. 11.28 (4) and the fact that c_1, d_2 are symmetric. Using the formulas in Subsect. 11.1.3, we see that

$$(p_1 p_2 + q_1 \overline{q}_2)^* = p_2^*(\mathbb{1} + d_2 c_1^*) p_1^*,$$

which implies that

$$(\Omega|U_{r_1} U_{r_2}\Omega) = \pm(\Omega|U_{r_1 r_2}\Omega). \tag{11.56}$$

We know that $U_{r_1 r_2}$ and $U_{r_1} U_{r_2}$ differ by a phase factor. This phase factor must be equal to ± 1 by (11.56), which completes the proof of (3). □

The following theorem gives an alternative definition of the group $Mp_{\mathrm{j,af}}(\mathcal{Y})$:

Theorem 11.40 *$Mp_{\mathrm{j,af}}(\mathcal{Y})$ is the subgroup of $U\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$ generated by $\mathrm{e}^{\mathrm{i}\mathrm{Op}(\zeta)}$, where $\mathrm{Op}(\zeta)$ are defined as in (11.47) with $g \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}, \mathcal{Z})$ and $h \in B_{\mathrm{h}}^1(\mathcal{Z})$.*

11.4 Fock sector of a CCR representation

The main result of this section is a necessary and sufficient criterion for two Fock CCR representations to be unitarily equivalent. This result goes under the name *Shale's theorem* and is closely related to Thm. 11.33 about the implementability of bosonic Bogoliubov transformations, which we also call Shale's theorem.

Another, closely related, subject of this chapter can be described as follows. Consider a symplectic space $\mathcal{Y}$ and a CCR representation in a Hilbert space $\mathcal{H}$. Suppose that we are given a Kähler anti-involution j. We will describe how to

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

find a subspace of $\mathcal{H}$ on which this representation is unitarily equivalent to a multiple of the Fock representation associated with j.

The basic framework of this section is slightly different from that of the preceding sections of this chapter. As previously, in this section $(\mathcal{Y},\omega)$ is a symplectic space and $\mathrm{j}\in L(\mathcal{Y})$ is a Kähler anti-involution. We do not, however, assume that $\mathcal{Y}$ is complete.

The notation and terminology of this section is based on Subsects. 1.3.6 and 1.3.8, 1.3.9, and was also recalled at the beginning of Chap. 9. Recall, in particular, that $y_1\cdot y_2 := -y_1\cdot\omega\mathrm{j}y_2$ is the corresponding symmetric form, so that $(\mathcal{Y},\cdot,\omega,\mathrm{j})$ is a Kähler space (not necessarily complete). $\mathcal{Z} := \frac{\mathbb{1}-\mathrm{i}\mathrm{j}_{\mathbb{C}}}{2}\mathbb{C}\mathcal{Y}$ is the corresponding holomorphic space. We have identifications $\mathbb{C}\mathcal{Y}\simeq\mathcal{Z}\oplus\overline{\mathcal{Z}}$ and

$$\mathcal{Y}\ni y\mapsto\left(\frac{1}{2}(y-\mathrm{i}\mathrm{j}y),\frac{1}{2}(y+\mathrm{i}\mathrm{j}y)\right)\in\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}}).\tag{11.57}$$

We recall also that $\mathcal{Z}$ inherits a unitary structure. If $(z_i,\overline{z_i})=y_i$, then

$$\begin{aligned}y_1\cdot y_2&=2\mathrm{Re}(z_1|z_2),\\ y_1\cdot\omega y_2&=2\mathrm{Im}(z_1|z_2).\end{aligned}\tag{11.58}$$

Recall that the completion of $\mathcal{Y}$ is denoted $\mathcal{Y}^{\mathrm{cpl}}$. Clearly, $\mathcal{Z}^{\mathrm{cpl}}$ is the holomorphic subspace of the complete Kähler space $\mathcal{Y}^{\mathrm{cpl}}$.

11.4.1 Vacua of CCR representations

Suppose that

$$\mathcal{Y}\ni y\mapsto W^{\pi}(y)=\mathrm{e}^{\mathrm{i}\phi^{\pi}(y)}\in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)\tag{11.59}$$

is a regular CCR representation. As in Subsect. 8.2.4, we introduce the creation, resp. annihilation operators $a^{\pi*}(z)$, resp. $a^{\pi}(z)$ by

$$a^{\pi*}(z):=\phi^{\pi}(z),\quad a^{\pi}(z):=\phi^{\pi}(\overline{z}),\quad z\in\mathcal{Z}.\tag{11.60}$$

Note that these operators depend not only on the representation π, but also on the Kähler anti-involution j, so in some situations it is natural to call them j-*creation*, resp. j-*annihilation operators.*

Definition 11.41 *The* space of j-vacua *is defined as*

$$\mathcal{K}^{\pi}:=\big\{\Psi\in\mathcal{H}\ :\ a^{\pi}(z)\Psi=0,\ z\in\mathcal{Z}\big\}.$$

Proposition 11.42 (1) *$\mathcal{K}^{\pi}$ is a closed subspace of $\mathcal{H}$.*
(2) *Let $\Psi\in\mathcal{H}$. Then*

$$\Psi\in\mathcal{K}^{\pi}\quad\Leftrightarrow\quad\big(\Psi|W^{\pi}(y)\Psi\big)=\|\Psi\|^2\mathrm{e}^{-\frac{1}{4}\|y\|^2},\ y\in\mathcal{Y}.$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

(3) *Elements of* $\mathcal{K}^\pi$ *are analytic vectors of* $\phi^\pi(y)$, $y \in \mathcal{Y}$.
(4) *If* $\Phi, \Psi \in \mathcal{K}^\pi$, *then*

$$\big(\Phi|W^\pi(y)\Psi\big) = (\Phi|\Psi)\mathrm{e}^{-\frac{1}{4}\|y\|^2}, \qquad y \in \mathcal{Y};$$
$$\big(\Phi|\phi^\pi(y_1)\phi^\pi(y_2)\Psi\big) = \frac{1}{2}(\Phi|\Psi)(y_1 \cdot y_2 + \mathrm{i}y_1{\cdot}\omega y_2), \quad y_1, y_2 \in \mathcal{Y}.$$

Proof Let us suppress the superscript π to simplify notation.

(1). The space of vacua $\mathcal{K}$ is closed as an intersection of kernels of closed operators.

Let us prove (2) $\Leftarrow$. Let $\Psi \in \mathcal{H}$ such that $\big(\Psi|W(y)\Psi\big) = \|\Psi\|^2\mathrm{e}^{-\frac{1}{4}\|y\|^2}$. Without loss of generality we can assume that $\|\Psi\| = 1$. Taking the first two terms of the Taylor expansion of

$$\mathbb{R} \ni t \mapsto \big(\Psi|W(ty)\Psi\big) = \|\Psi\|^2\mathrm{e}^{-\frac{1}{4}t^2\|y\|^2},$$

we obtain

$$\big(\Psi|\phi(y)\Psi\big) = 0, \quad \big(\Psi|\phi(y)^2\Psi\big) = \frac{1}{2}\|y\|^2. \tag{11.61}$$

In particular, $\Psi \in \operatorname{Dom}\phi(y)$, $y \in \mathcal{Y}$. Let $z = y - \mathrm{i}\mathrm{j}y \in \mathcal{Z}$. Then

$$\begin{aligned} a^*(z)a(z) &= \phi(z)\phi(\overline{z}) \\ &= \big(\phi(y) - \mathrm{i}\phi(y))(\phi(y) + \mathrm{i}\phi(\mathrm{j}y)\big) = \phi(y)^2 + \phi(\mathrm{j}y)^2 - \|y\|^2. \end{aligned}$$

Hence,

$$\|a(z)\Psi\|^2 = \big(\Psi|\phi(y)^2\Psi\big) + \big(\Psi|\phi(\mathrm{j}y)^2\Psi\big) - \|y\|^2 = 0.$$

To prove (2) $\Rightarrow$, note that if $\Psi \in \mathcal{K}$, then $\Psi \in \operatorname{Dom}\phi(y)$, $y \in \mathcal{Y}$. In particular, the function

$$\mathbb{R} \ni t \mapsto F(t) = (\Psi|W(ty)\Psi)$$

is C^1. Let $y \in \mathcal{Y}$, $z = \frac{1}{2}(y - \mathrm{i}\mathrm{j}y) \in \mathcal{Z}$. Using Thm. 8.25 (1), we get

$$\phi(y)W(ty) = a^*(z)W(ty) + W(ty)a(z) + \frac{\mathrm{i}t}{2}\|y\|^2W(ty). \tag{11.62}$$

This yields

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}F(t) &= \mathrm{i}\big(\Psi|\phi(y)W(ty)\Psi\big) \\ &= \tfrac{\mathrm{i}}{2}\big(a(z)\Psi|W(ty)\Psi\big) + \tfrac{\mathrm{i}}{2}\big(\Psi|W(ty)a(z)\Psi\big) - \tfrac{t}{2}\|y\|^2\big(\Psi, W(ty)\Psi\big) \\ &= -\tfrac{t}{2}\|y\|^2F(t). \end{aligned}$$

Since $F(0) = \|\Psi\|^2$, we get that $F(t) = \|\Psi\|^2\mathrm{e}^{-\frac{1}{4}\|y\|^2}$.

From (2) we know that $F(t)$ is analytic, hence by the spectral theorem $\Psi \in \mathcal{K}$ is an analytic vector for $\phi(y)$, $y \in \mathcal{Y}$, which proves (3). Finally, (4) follows from (11.61) by polarization. $\square$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

11.4.2 Fock CCR representations

As in Sect. 3.4, for $z \in \mathcal{Z}^{\mathrm{cpl}}$ we introduce creation, resp. annihilation operators $a^*(z)$, resp. $a(z)$ acting on the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$.

Definition 11.43 *The regular CCR representation*

$$\mathcal{Y} \ni y \mapsto W(y) = \mathrm{e}^{\mathrm{i}a^*(z)+\mathrm{i}a(z)} \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})\big),\ y = (z, \overline{z}), \tag{11.63}$$

is called the Fock representation over the Kähler space $\mathcal{Y}$.

This is a slight generalization of the definition used in Subsect. 9.1.1, since we allow for a non-complete space $\mathcal{Y}$. Clearly, the representation (11.63) can be extended to a representation over $\mathcal{Y}^{\mathrm{cpl}}$ in an obvious way.

Note that j-creation, resp. j-annihilation operators defined for the CCR representation (11.63) coincide with the usual creation, resp. annihilation operators $a^*(z)$, resp. $a(z)$. Likewise, a vector $\Psi \in \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$ is a j-vacuum for (11.63) iff it is proportional to Ω.

We can also consider another Kähler anti-involution j_1, not necessarily equal to j. The following theorem describes the vacua inside $\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$ corresponding to j_1. It is essentially a restatement of parts of Thm. 11.28.

Theorem 11.44 (1) *Let $c \in B_{\mathrm{s}}^2(\overline{\mathcal{Z}}^{\mathrm{cpl}}, \mathcal{Z}^{\mathrm{cpl}})$, $cc^* < \mathbb{1}$, and let j_1 be the Kähler anti-involution determined by c as in Subsect. 11.1.8. Then Ω_c is the unique vector in $\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$ satisfying the following conditions:*

(i) $\|\Omega_c\| = 1$,

(ii) $(\Omega_c|\Omega) > 0$,

(iii) Ω_c *is a* j_1*-vacuum.*

(2) *The statement (1)(iii) is equivalent to*

$$\big(a(z) - a^*(c\overline{z})\big)\Omega_c = 0, \quad z \in \mathcal{Z}. \tag{11.64}$$

11.4.3 Unitary equivalence of Fock CCR representations

Suppose that we are given a symplectic space $(\mathcal{Y}, \omega)$ endowed with two Kähler structures, defined e.g. by two Kähler anti-involutions. Each Kähler structure determines a Fock representation. In this subsection we will prove a necessary and sufficient condition for the equivalence of these two representations.

Theorem 11.45 (Shale's theorem about Fock representations) *Let $\mathcal{Z}$, $\mathcal{Z}_1$ be the holomorphic subspaces of $\mathbb{C}\mathcal{Y}$ corresponding to Kähler anti-involutions* j *and* j_1*. Let*

$$\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi(y)} \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big), \tag{11.65}$$

$$\mathcal{Y} \ni y \mapsto \mathrm{e}^{\mathrm{i}\phi_1(y)} \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z}_1)\big) \tag{11.66}$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

be the corresponding Fock CCR representations. Then the following statements are equivalent:

(1) *There exists a unitary operator* $W : \Gamma_s(\mathcal{Z}) \to \Gamma_s(\mathcal{Z}_1)$ *such that*

$$W\phi(y) = \phi_1(y)W. \tag{11.67}$$

(2) $\mathrm{j} - \mathrm{j}_1$ *is Hilbert–Schmidt (or any of the equivalent conditions of Prop. 11.22 hold).*

Proof Let a_1^*, a_1, Ω_1 denote the creation and annihilation operators and the vacuum for the representation (11.66).

(2)⇒(1). Assume that $\mathrm{j} - \mathrm{j}_1 \in B^2(\mathcal{Y})$. We know by Prop. 11.22 that there exists $r \in Sp_{\mathrm{j}}(\mathcal{Y})$ such that $\mathrm{j}_1 = r\mathrm{j}r^{\#}$. Thus, by Thm. 11.33 there exists $U_r \in U\big(\Gamma_s(\mathcal{Z})\big)$ such that $U_r\phi(y)U_r^* = \phi(ry)$.

Note that $r_{\mathbb{C}}$ is a unitary operator on $\mathbb{C}\mathcal{Y}$ and $r_{\mathbb{C}}\mathcal{Z} = \mathcal{Z}_1$. Set $u := r_{\mathbb{C}}\big|_{\mathcal{Z}}$. Then $u \in U(\mathcal{Z}, \mathcal{Z}_1)$, hence $\Gamma(u) \in U\big(\Gamma_s(\mathcal{Z}), \Gamma_s(\mathcal{Z}_1)\big)$ and

$$\Gamma(u)a^*(z)\Gamma(u)^* = a_1^*(uz), \quad \Gamma(u)a(z)\Gamma(u)^* = a_1(uz), \quad z \in \mathcal{Z}.$$

Consequently, $\Gamma(u)\phi(y)\Gamma(u)^* = \phi_1(ry)$. Therefore, $W := \Gamma(u)U_r^*$ satisfies (11.67).

(1)⇒(2). Suppose that the representations (11.65) and (11.66) are equivalent with the help of the operator $W \in U\big(\Gamma_s(\mathcal{Z}_1), \Gamma_s(\mathcal{Z})\big)$. Then $\Psi := W\Omega_1$ satisfies

$$\big(a(z) - a^*(c\overline{z})\big)\Psi = 0, \quad z \in \mathcal{Z}.$$

By Thm. 11.28, this implies that $c \in B^2(\overline{\mathcal{Z}}, \mathcal{Z})$. Hence, $\mathrm{j} - \mathrm{j}_1 \in B^2(\mathcal{Y})$. □

11.4.4 Fock sector of CCR representations

Let us go back to an arbitrary regular CCR representation (11.59) over a Kähler space $\mathcal{Y}$. We will describe how to determine the largest sub-representation of (11.59) unitarily equivalent to a multiple of the j-Fock representation over $\mathcal{Y}$ in $\Gamma_s(\mathcal{Z}^{\mathrm{cpl}})$.

Theorem 11.46 *Set*

$$\mathcal{H}^\pi := \mathrm{Span}^{\mathrm{cl}}\big\{W^\pi(y)\Psi \ : \ \Psi \in \mathcal{K}^\pi, \quad y \in \mathcal{Y}\big\}. \tag{11.68}$$

(1) $\mathcal{H}^\pi$ *is invariant under* $W^\pi(y)$, $y \in \mathcal{Y}$.
(2) *There exists a unique unitary operator*

$$U^\pi : \mathcal{K}^\pi \otimes \Gamma_s(\mathcal{Z}^{\mathrm{cpl}}) \to \mathcal{H}^\pi$$

satisfying

$$U^\pi \ \Psi \otimes W(y)\Omega = W^\pi(y)\Psi, \quad \Psi \in \mathcal{K}^\pi, \quad y \in \mathcal{Y},$$

where $W(y)$ *denote Weyl operators in the Fock representation on* $\Gamma_s(\mathcal{Z}^{\mathrm{cpl}})$.

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

(3)

$$U^\pi\ 1\!\!1\otimes W(y) = W^\pi(y)U^\pi, \quad y\in\mathcal{Y}. \tag{11.69}$$

(4) *If there exists an operator* $U:\Gamma_{\rm s}(\mathcal{Z}^{\rm cpl})\to\mathcal{H}$ *such that* $UW(y)=W^\pi(y)U$, $y\in\mathcal{Y}$, *then* $\operatorname{Ran}U\subset\mathcal{H}^\pi$.

(5) $\mathcal{H}^\pi$ *depends on* j *only through the equivalence class of* j *w.r.t. the relation*

$$\mathrm{j}_1\sim\mathrm{j}_2 \ \Leftrightarrow\ \mathrm{j}_1-\mathrm{j}_2\in B^2(\mathcal{Y}). \tag{11.70}$$

Definition 11.47 *Introduce the equivalence relation (11.70) in the set of Kähler anti-involutions on* $\mathcal{Y}$. *Let* [j] *denote the equivalence class w.r.t. this relation. Then the subspace* $\mathcal{H}^\pi$ *defined in (11.68) is called the* [j]-Fock sector of a CCR representation W^π.

Proof of Thm. 11.46. Clearly, $\mathcal{H}^\pi$ is invariant under $W^\pi(y)$, $y\in\mathcal{Y}$. Now let $\Psi_i\in\mathcal{K}^\pi$, $y_i\in\mathcal{Y}$ for $i=1,2$. We have

$$\begin{aligned}\big(W^\pi(y_1)\Psi_1|W^\pi(y_2)\Psi_2\big) &= \mathrm{e}^{\frac{\mathrm{i}}{2}y_1\cdot\omega y_2}\big(\Psi_1|W^\pi(y_2-y_1)\Psi_2\big)\\ &= (\Psi_1|\Psi_2)\mathrm{e}^{\frac{\mathrm{i}}{2}y_1\cdot\omega y_2}\mathrm{e}^{-\frac14(y_2-y_1)\cdot(y_2-y_1)}.\end{aligned}$$

Similarly, if $(z_i,\overline{z_i})=y_i$, we have

$$\big(\Psi_1\otimes W(y_1)\Omega|\Psi_2\otimes W(y_2)\Omega\big) = (\Psi_1|\Psi_2)\mathrm{e}^{\frac{\mathrm{i}}{2}(z_1,\overline{z_1})\cdot\omega(z_2,\overline{z_2})}\mathrm{e}^{-\frac12\overline{(z_2-z_1)}\cdot(z_2-z_1)},$$

using (9.13). Using (11.58), we obtain that

$$\big(W^\pi(y_1)\Psi_1|W^\pi(y_2)\Psi_2\big) = \big(\Psi_1\otimes W(y_1)\Omega|\Psi_2\otimes W(y_2)\Omega\big). \tag{11.71}$$

Let us set

$$U^\pi\Psi\otimes W(y)\Omega := W^\pi(y)\Psi$$

and extend U^π to $\mathcal{K}^\pi\otimes\overset{\rm al}{\Gamma}_{\rm s}(\mathcal{Z}^{\rm cpl})$ by linearity. By the identity (11.71), we see that U^π is well defined and isometric. It extends as a unitary operator between $\mathcal{K}^\pi\otimes\Gamma_{\rm s}(\mathcal{Z}^{\rm cpl})$ and $\mathcal{H}^\pi$. Property (3) follows from the definition of U^π.

Finally, let $U:\Gamma_{\rm s}(\mathcal{Z}^{\rm cpl})\to\mathcal{H}$ be an operator such that $UW(y)=W^\pi(y)U$. Then, by the argument leading to (11.71), we see that $U\mathbb{C}\Omega\subset\mathcal{K}^\pi$. Therefore, $\operatorname{Ran}U\subset\operatorname{Ran}U^\pi$.

Let us prove (5). For $i=1,2$ denote by $\mathcal{H}_i$, $\mathcal{K}_i$ the spaces $\mathcal{H}^{\pi_i}$, $\mathcal{K}^{\pi_i}$ corresponding to the anti-involution j_i. It suffices to prove that $\mathcal{H}_2\subset\mathcal{H}_1$. We first claim that

$$\mathcal{K}_2\cap\mathcal{H}_1\neq\{0\}. \tag{11.72}$$

In fact, by Thm. 11.46 (1) we know that $\mathcal{H}_1$ is invariant under $W(y)$, $y\in\mathcal{Y}$. Besides, $y\mapsto W(y)\big|_{\mathcal{H}_1}$ is unitarily equivalent to a multiple of the Fock representation on $\Gamma_{\rm s}(\mathcal{Z}_1)$. Since $\mathrm{j}_1-\mathrm{j}_2\in B^2(\mathcal{Y})$, it follows from Thm. 11.45 that $y\mapsto W(y)\big|_{\mathcal{H}_1}$ contains vacua for j_2, hence (11.72) holds.

We claim now that $\mathcal{K}_2 \subset \mathcal{H}_1$, which implies that $\mathcal{H}_2 \subset \mathcal{H}_1$. If $\mathcal{K}_2 \not\subset \mathcal{H}_1$, then $\mathcal{K}_2 \cap \mathcal{H}_1^{\perp}$ is the non-trivial space of j_2-vacua for $y \mapsto W(y)\big|_{\mathcal{H}_1^{\perp}}$.

Applying the analog of (11.72) to $\mathcal{H}_1^{\perp}$, we see that $\mathcal{H}_1^{\perp}$ should contain vacua for j_1, which is absurd. Hence, $\mathcal{K}_2 \subset \mathcal{H}_1$, which completes the proof. □

Proposition 11.48 *If the CCR representation (11.59) is irreducible and $\mathcal{K}^{\pi} \neq \{0\}$, then it is unitarily equivalent to the* [j]*-Fock representation.*

Proof Since (11.59) is irreducible, we have $\mathcal{H}^{\pi} = \{0\}$ or $\mathcal{H}^{\pi} = \mathcal{H}$, which proves the proposition. □

11.4.5 Number operator of regular CCR representations

In this subsection we consider an arbitrary regular CCR representation (11.59) over a Kähler space $\mathcal{Y}$ with a Kähler anti-involution j. We now discuss the notion of the *number operator* N^{π} associated with (11.59). The number operator N^{π} allows for a direct description of the Fock sector $\mathcal{H}^{\pi}$. In some cases this description can be used to show that $\mathcal{H}^{\pi} = \mathcal{H}$. This is in particular the case for a finite-dimensional $\mathcal{Y}$, when Thm. 11.52 gives an alternative proof of the Stone–von Neumann theorem (Thm. 8.49).

Here is the first definition of N^{π}.

Definition 11.49 *Let N be the usual number operator on the bosonic Fock space. Let U^{π} be defined as in Thm. 11.46. Define* $\mathrm{Dom}\, N^{\pi} := U^{\pi}\, \mathcal{K}^{\pi} \otimes \mathrm{Dom}\, N$*, which is a dense subspace of $\mathcal{H}^{\pi}$. The* number operator in the representation π *is the operator on $\mathcal{H}$ with the domain* $\mathrm{Dom}\, N^{\pi}$ *defined by*

$$N^{\pi} := U^{\pi}(\mathbb{1} \otimes N)U^{\pi *}.$$

(Note that N^{π} need not be densely defined.)

Before we give an alternative definition of N^{π}, let us recall some facts about quadratic forms. We will assume that a positive quadratic form is defined on the whole space $\mathcal{H}$ and takes values in $[0, \infty]$. The domain of a positive quadratic form b is defined as

$$\mathrm{Dom}\, b := \big\{\Phi \in \mathcal{H} \ : \ b(\Phi) < \infty\big\}.$$

If the form b is closed, then there exists a unique positive self-adjoint operator B such that

$$\mathrm{Dom}\, b = \mathrm{Dom}\, B^{\frac{1}{2}}, \quad b(\Phi) = (\Phi|B\Phi).$$

If A is a closed operator, then $\|A\Phi\|^2$ is a closed form. The sum of closed forms is a closed form, and the supremum of a family of closed forms is a closed form.

Definition 11.50 *For each finite-dimensional subspace $\mathcal{V} \subset \mathcal{Z}$, set*

$$n_{\mathcal{V}}^{\pi}(\Phi) := \sum_{i=1}^{\dim \mathcal{V}} \|a^{\pi}(v_i)\Phi\|^2,$$

where $\{v_i\}_{i=1}^{\dim \mathcal{V}}$ is an o.n. basis of $\mathcal{V}$. (If $\Phi \notin \mathrm{Dom}\, a^{\pi}(v_i)$ for some i, set $n_{\mathcal{V}}^{\pi}(\Phi) = \infty$.)

The quadratic form $n_{\mathcal{V}}^{\pi}$ does not depend on the choice of the basis $\{v_i\}_{i=1}^{\dim \mathcal{V}}$ of $\mathcal{V}$. Moreover, by Thm. 8.29, $n_{\mathcal{V}}^{\pi}$ is densely defined.

Definition 11.51 *The* number quadratic form n^{π} *is defined by*

$$n^{\pi}(\Phi) := \sup_{\mathcal{V}} n_{\mathcal{V}}^{\pi}(\Phi), \quad \Phi \in \mathcal{H}.$$

The following theorem says that the number quadratic form of Def. 11.51 gives the number operator introduced in Def. 11.49.

Theorem 11.52 *Let n^{π} be the number quadratic form associated with W^{π} and* j. *Then $\mathrm{Dom}\, n^{\pi} = \mathrm{Dom}(N^{\pi})^{\frac{1}{2}}$ and*

$$n^{\pi}(\Phi) = (\Phi|N^{\pi}\Phi), \quad \Phi \in \mathrm{Dom}\, N^{\pi}.$$

In particular, $\mathcal{H}^{\pi} = (\mathrm{Dom}\, n^{\pi})^{\mathrm{cl}}$.

To prepare for the proof of the above theorem, note that n^{π} defines a positive operator (with a possibly non-dense domain), which we temporarily denote $\tilde{N}^{\pi}$, such that $\mathrm{Dom}\, n^{\pi} = \mathrm{Dom}(\tilde{N}^{\pi})^{\frac{1}{2}}$ and

$$n^{\pi}(\Phi) = (\Phi|\tilde{N}^{\pi}\Phi), \quad \Phi \in \mathrm{Dom}\, \tilde{N}^{\pi}. \tag{11.73}$$

Our aim is to show that $\tilde{N}^{\pi} = N^{\pi}$.

Note also that

$$\mathrm{Dom}\, n^{\pi} \subset \mathrm{Dom}\, \phi^{\pi}(y), \quad y \in \mathcal{Y}. \tag{11.74}$$

Lemma 11.53 *If $\Phi \in \mathrm{Dom}(\tilde{N}^{\pi})^{\frac{1}{2}}$ and F is a Borel function, then*

$$a^{\pi}(z)F(\tilde{N}^{\pi} - 1\!\!1)\Phi = F(\tilde{N}^{\pi})a^{\pi}(z)\Phi. \tag{11.75}$$

Proof Let us suppress the superscript π to simplify notation. First we note that $W(y)$ maps $\mathrm{Dom}\, \tilde{N}^{\frac{1}{2}}$ into itself and have

$$n(W(y)\Phi) = n(\Phi) + \big(\Phi|\phi(\mathrm{j}y)\Phi\big) + \frac{1}{2}\|y\|^2\|\Phi\|^2. \tag{11.76}$$

In fact, using (8.21) we see that (11.76) is true if we replace n with $n_{\mathcal{V}}$, where $\mathcal{V}$ is a finite-dimensional subspace of $\mathcal{Y}$ containing y. Then (11.76) follows immediately.

By the polarization identity, (11.76) has the following consequence for $\Phi, \Psi \in \operatorname{Dom} \tilde{N}^{\frac{1}{2}}$:

$$\begin{aligned}&\big(\tilde{N}^{\frac{1}{2}} W(y)\Phi | \tilde{N}^{\frac{1}{2}} W(y)\Psi\big) \\ &\quad= \big(\tilde{N}^{\frac{1}{2}}\Phi | \tilde{N}^{\frac{1}{2}}\Psi\big) + \big(\Phi|\phi(\mathrm{j}y)\Psi\big) + \frac{1}{2}\|y\|^2(\Phi|\Psi).\end{aligned} \tag{11.77}$$

Replacing Φ by $W(y)^*\Phi$ and using the invariance of $\operatorname{Dom} \tilde{N}^{\frac{1}{2}}$ under $W(y)$, we can rewrite (11.77) as follows:

$$\begin{aligned}&\big(\tilde{N}^{\frac{1}{2}}\Phi | \tilde{N}^{\frac{1}{2}} W(y)\Psi\big) \\ &\quad= \big(\tilde{N}^{\frac{1}{2}} W(y)^*\Phi | \tilde{N}^{\frac{1}{2}}\Psi\big) + \big(W(y)^*\Phi|\phi(\mathrm{j}y)\Psi\big) + \frac{1}{2}\|y\|^2\big(W(y)^*\Phi|\Psi\big).\end{aligned} \tag{11.78}$$

Next assume in addition that $\Phi, \Psi \in \operatorname{Dom} \tilde{N}$. Then we can rewrite (11.78) as

$$\begin{aligned}&\big(\tilde{N}\Phi | W(y)\Psi\big) \\ &\quad= \big(W(y)^*\Phi|\tilde{N}\Psi\big) + \big(W(y)^*\Phi|\phi(\mathrm{j}y)\Psi\big) + \frac{1}{2}\|y\|^2(W(y)^*\Phi|\Psi).\end{aligned} \tag{11.79}$$

We replace y by ty and differentiate (11.79) w.r.t. t. (Differentiating is allowed by (11.74).) We obtain

$$(\tilde{N}\Phi|\phi(y)\Psi) = (\phi(y)\Phi|\tilde{N}\Psi) - \mathrm{i}(\Phi|\phi(\mathrm{j}y)\Psi). \tag{11.80}$$

Substituting $\mathrm{j}y$ for y in (11.80), we obtain

$$\big(\tilde{N}\Phi|\phi(\mathrm{j}y)\Psi\big) = -\big(\phi(\mathrm{j}y)\Phi|\tilde{N}\Psi\big) + \mathrm{i}\big(\Phi|\phi(y)\Psi\big). \tag{11.81}$$

Adding up (11.80) and (11.81), we get

$$\big(\tilde{N}\Phi|a(z)\Psi\big) = \big(a^*(z)\Phi|\tilde{N}\Psi\big) - \big(\Phi|a(z)\Psi\big). \tag{11.82}$$

Next let us assume that $\Phi \in \operatorname{Dom} \tilde{N}^{\frac{3}{2}}$. Then $\tilde{N}\Phi \in \operatorname{Dom} \tilde{N}^{\frac{1}{2}} \subset \operatorname{Dom} a(z)$. Hence, (11.82) implies

$$\big(\tilde{N}\Phi|a(z)\Psi\big) = \big(\Phi|a(z)(\tilde{N} - \mathbb{1})\Psi\big). \tag{11.83}$$

Therefore, $a(z)\Psi \in \operatorname{Dom} \tilde{N}$, and we have

$$\tilde{N} a(z)\Psi = a(z)(\tilde{N} - \mathbb{1})\Psi, \tag{11.84}$$

or equivalently

$$(\tilde{N} + \lambda\mathbb{1})a(z)\Psi = a(z)(\tilde{N} + \lambda\mathbb{1} - \mathbb{1})\Psi. \tag{11.85}$$

Now let $\Phi \in \operatorname{Dom} \tilde{N}^{\frac{1}{2}}$ and $\lambda > 1$. Then $(\tilde{N} + \lambda\mathbb{1} - \mathbb{1})^{-1}\Phi \in \operatorname{Dom} \tilde{N}^{\frac{3}{2}}$. Therefore, by (11.85),

$$(\tilde{N} + \lambda\mathbb{1})a(z)(\tilde{N} + \lambda\mathbb{1} - \mathbb{1})^{-1}v = a(z)\Phi. \tag{11.86}$$

Multiplying this by $(\tilde{N} + \lambda\mathbb{1})^{-1}$, we obtain

$$a(z)(\tilde{N} + \lambda\mathbb{1} - \mathbb{1})^{-1}\Phi = (\tilde{N} + \lambda\mathbb{1})^{-1}a(z)\Phi. \tag{11.87}$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

Since linear combinations of functions $(\tilde{N}+\lambda\mathbb{1})^{-1}$ with $\lambda>0$ are strongly dense in the von Neumann algebra of bounded Borel functions of $\tilde{N}$, and $a(z)$ is closed, (11.87) implies

$$a(z)F(\tilde{N}-\mathbb{1})\Phi = F(\tilde{N})a(z)\Phi,\ \Phi\in \operatorname{Dom}\tilde{N}^{\frac{1}{2}},$$

for any bounded Borel function F. □

Lemma 11.54 $\mathcal{K}^\pi=\{0\}$ *implies* $\operatorname{Dom}\tilde{N}^\pi=\{0\}$.

Proof Again we suppress the superscript π to simplify notation. Suppose that $\operatorname{Dom}\tilde{N}\neq\{0\}$. We know that $\tilde{N}\geq 0$. Therefore, $\operatorname{spec}\tilde{N}$ is non-degenerate and bounded from below. Hence, $\lambda_0:=\inf\operatorname{spec}\tilde{N}$ is a finite number, and

$$\operatorname{Ran}\mathbb{1}_{[\lambda_0,\lambda_0+1[}(\tilde{N})\neq\{0\}.$$

By Lemma 11.53, for any $z\in\mathcal{Z}$

$$a^\pi(z)\mathbb{1}_{[\lambda_0,\lambda_0+1[}(\tilde{N})=\mathbb{1}_{[\lambda_0-1,\lambda_0[}(\tilde{N})a^\pi(z). \tag{11.88}$$

But

$$\mathbb{1}_{[\lambda_0-1,\lambda_0[}(\tilde{N})=0.$$

Therefore, (11.88) is zero and

$$\operatorname{Ran}\mathbb{1}_{[\lambda_0,\lambda_0+1[}(\tilde{N})\subset\mathcal{K}=\{0\},$$

which is a contradiction. □

The following lemma is immediate:

Lemma 11.55 *Suppose that* $\mathcal{H}=\mathcal{H}^0\oplus\mathcal{H}^1$. *Suppose that*

$$\mathcal{Y}\ni y\mapsto W^\pi(y)\in\mathcal{H}$$

is a CCR representation and $W^\pi(y)$, $y\in\mathcal{Y}$, *leaves* $\mathcal{H}^0$ *invariant. Then* $W^\pi(y)$ *also leaves* $\mathcal{H}^1$ *invariant. Thus we have two CCR representations,*

$$\mathcal{Y}\ni y\mapsto W^\pi(y)\big|_{\mathcal{H}^0},\quad \mathcal{Y}\ni y\mapsto W^\pi(y)\big|_{\mathcal{H}^1}.$$

Let $\mathcal{K}^i$, $\tilde{N}^i$ *denote the corresponding spaces of vacua and the operators defined by (11.73) for the representations* $i=0,1$. *Then*

$$\mathcal{K}=\mathcal{K}^0\oplus\mathcal{K}^1,\quad \tilde{N}=\tilde{N}^0\oplus\tilde{N}^1. \tag{11.89}$$

Proof of Thm. 11.52. We are in the situation of Lemma 11.55: we have two CCR representations, in $\mathcal{H}^0=\mathcal{H}^\pi$ and in $\mathcal{H}^1=(\mathcal{H}^0)^\perp$.

By the definition of N^π, we have

$$N^\pi=N^0\oplus N^1,$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

where $\mathrm{Dom}\, N^1 = \{0\}$. From $U^\pi \mathbb{1} \otimes W(y) = W^\pi(y) U^\pi$, $y \in \mathcal{Y}$, we get $U^\pi \mathbb{1} \otimes a(z) = a^\pi(z) U^\pi$, $z \in \mathcal{Z}$, and hence $\tilde{N}^0 = N^0$.

We know that $\mathcal{K} \subset \mathcal{H}^0$, hence $\mathcal{K}^1 = \{0\}$. By Lemma 11.54, this implies $\mathrm{Dom}\, \tilde{N}^1 = \{0\}$. Therefore, $\tilde{N}^\pi = N^\pi$. □

11.4.6 Relative continuity of Gaussian measures

In this subsection we prove Thm. 5.78, the Feldman–Hajek theorem, which says that the Gaussian measures with covariances A_1, A_2 are relatively continuous iff $A_1^{-\frac{1}{2}} A_2 A_1^{-\frac{1}{2}} - \mathbb{1}$ is Hilbert–Schmidt.

Proof of Thm. 5.78. To conform with the notation used in this chapter we denote the covariances A_1, A_2 of Thm. 5.78 by a_1, a_2.

Without loss of generality we can assume that $a_1 = \mathbb{1}$. In fact, note that

$$(\xi|a_1\xi)_{\mathcal{X}} = (\xi|\xi)_{a_1^{-\frac{1}{2}}\mathcal{X}}, \quad (\xi|a_2\xi)_{\mathcal{X}} = (\xi|b\xi)_{a_1^{-\frac{1}{2}}\mathcal{X}},$$

for $b = a_1^{-1} a_2$. Since $a_1^{\frac{1}{2}}$ is unitary from $a_1^{-\frac{1}{2}}\mathcal{X}$ to $\mathcal{X}$, we see that $\mathbb{1} - a_1^{-1} a_2 \in B^2(a_1^{-\frac{1}{2}}\mathcal{X})$ iff $\mathbb{1} - a_1^{-\frac{1}{2}} a_2 a_1^{-\frac{1}{2}} \in B^2(\mathcal{X})$. Hence, replacing $\mathcal{X}$ by $a_1^{-\frac{1}{2}}\mathcal{X}$ and a_2 by $a_1^{-1} a_2$, we may assume that $a_1 = \mathbb{1}$ and denote a_2 simply by a.

Let us consider the real-wave representations for the covariances $\mathbb{1}$ and a, as in Sect. 9.3. From Prop. 9.16 we know that they are unitarily equivalent to the Fock representations for the symplectic space $(\mathcal{X} \oplus \mathcal{X}, \omega)$ with the Kähler anti-involutions

$$\mathrm{j} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}, \quad \mathrm{j}_1 = \begin{bmatrix} 0 & -(2a)^{-1} \\ 2a & 0 \end{bmatrix}.$$

We set

$$k = -\mathrm{j}\mathrm{j}_1 = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}.$$

It is easy to see that $\mathbb{1} - k$ is Hilbert–Schmidt for the real scalar product on $\mathcal{X} \oplus \mathcal{X}$ coming from the Kähler structure (ω, j) iff $\mathbb{1} - a$ is Hilbert–Schmidt on $\mathcal{X}$.

Proof of $\Rightarrow$. Assume that $\mathbb{1} - a \in B^2(\mathcal{X})$. For simplicity let us denote the two Gaussian $\mathbf{L}^2$ spaces by $L^2(\mathcal{X}, \mathrm{d}\mu)$ and $L^2(\mathcal{X}, \mathrm{d}\tilde{\mu})$.

By Thm. 11.45, we know that the two real-wave representations above are unitarily equivalent. In particular, there exists a unitary operator U intertwining these representations. By restriction to the position operators, we deduce that

$$U : L^2(\mathcal{X}, \mathrm{d}\mu) \to L^2(\mathcal{X}, \mathrm{d}\tilde{\mu}), \quad UF(x)U^{-1} = F(x),$$

for all cylinder continuous functions F on $\mathcal{X}$. By monotone convergence, this identity extends first to all bounded $\mathfrak{B}_{\mathrm{cyl}}$-measurable functions, and then to all measurable functions (see Subsect. 5.2.1). We note then that if $A \in \mathfrak{B}$, then $\mu(A) = 0$, resp. $\mu_1(A) = 0$ iff $\mathbb{1}_A(x) = 0$ as a multiplication operator on $L^2(\mathcal{X}, \mathrm{d}\mu)$, resp. $L^2(\mathcal{X}, \mathrm{d}\tilde{\mu})$, which shows that μ and μ_1 are mutually absolutely continuous.

Proof of $\Leftarrow$. Assume now that μ and $\tilde{\mu}$ are mutually absolutely continuous. Then we have

$$\mathrm{d}\tilde{\mu} = F\mathrm{d}\mu, \quad \text{for } F \geq 0 \text{ a.e., and } \int_{\mathcal{X}} F\mathrm{d}\mu = 1.$$

Clearly, $\Psi := F^{\frac{1}{2}}$ is a unit vector in $L^2(\mathcal{X}, \mathrm{d}\mu)$. We will show that

$$a_{\mathrm{rw}}(w)\Psi - a_{\mathrm{rw}}^*(c\overline{w})\Psi = 0, \quad w \in \mathbb{C}\mathcal{X}, \tag{11.90}$$

in the weak sense, where $c = \frac{a-\mathbb{1}}{a+\mathbb{1}}$ and $a_{\mathrm{rw}}^*(\cdot)$, $a_{\mathrm{rw}}(\cdot)$ are the creation and annihilation operators in the real-wave representation on $L^2(\mathcal{X}, \mathrm{d}\mu)$ defined in Subsect. 9.3.1.

We claim that (11.90) implies that $\mathbb{1} - a \in B^2(\mathcal{X})$. In fact, by Prop. 9.16, the real-wave representation on $L^2(\mathcal{X}, \mathrm{d}\mu)$ with the anti-involution j above is unitarily equivalent to the Fock representation on $\Gamma_{\mathrm{s}}(\mathbb{C}\mathcal{X})$. Applying Thm. 11.28, we deduce from (11.90) that $c \in B^2(\mathbb{C}\mathcal{X})$, i.e. $\mathbb{1} - a \in B^2(\mathcal{X})$.

Note that if $\mathcal{X}$ is finite-dimensional, then

$$\begin{aligned} F(x) &= (\det a)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}(x|a^{-1}x) + \frac{1}{2}(x|x)}, \\ \Psi(x) \;=\; F^{\frac{1}{2}}(x) &= (\det a)^{-\frac{1}{4}} \mathrm{e}^{-\frac{1}{4}(x|a^{-1}x) + \frac{1}{4}(x|x)}. \end{aligned} \tag{11.91}$$

Hence,

$$w \cdot \nabla_x \Psi(x) = \frac{1}{2}(\mathbb{1} - a^{-1}) w \cdot x \Psi(x),$$

which is equivalent to (11.90). In the general case we will approximate $\mathcal{X}$ by an increasing family of finite-dimensional subspaces $\mathcal{X}_n$ on which (11.91) is valid, and pass to the limit $n \to +\infty$.

We choose an increasing sequence $(\pi_n)_{n\in\mathbb{N}}$ of rank n orthogonal projections in $\mathcal{X}$ such that $\mathrm{s} - \lim\limits_{n\to\infty} \pi_n = \mathbb{1}$. We set

$$\mathcal{X}_n := \pi_n \mathcal{X}, \quad a_n := \pi_n a \pi_n, \quad \mathfrak{B}_n := \mathfrak{B}^{\mathcal{X}_n},$$

where we recall from Sect. 5.2.1 that $\mathfrak{B}^{\mathcal{Y}}$ is the σ-algebra of cylinder sets based on $\mathcal{Y}$.

We note that $\mathfrak{B}$ is generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$. This follows from the fact that the polynomials based on $\bigcup_{n\in\mathbb{N}} \mathcal{X}_n$ are dense in $L^2(\mathcal{X}, \mathrm{d}\mu)$, which is a consequence of Thm. 5.56.

We denote by μ_n, resp. $\tilde{\mu}_n$ the Gaussian measures on $\mathcal{X}_n$ with covariances $\mathbb{1}$, resp. a_n. For $\xi \in \mathcal{X}_n$ we have

$$\int_{\mathcal{X}_n} \mathrm{e}^{\mathrm{i}\xi\cdot x} \mathrm{d}\mu_n = \int_{\mathcal{X}} \mathrm{e}^{\mathrm{i}\xi\cdot x} \mathrm{d}\mu, \qquad \int_{\mathcal{X}_n} \mathrm{e}^{\mathrm{i}\xi\cdot x} \mathrm{d}\tilde{\mu}_n = \int_{\mathcal{X}} \mathrm{e}^{\mathrm{i}\xi\cdot x} \mathrm{d}\tilde{\mu},$$

which by a density argument implies that

$$\int_{\mathcal{X}_n} u(x)\mathrm{d}\mu_n = \int_{\mathcal{X}} u(x)\mathrm{d}\mu, \qquad \int_{\mathcal{X}_n} u(x)\mathrm{d}\tilde{\mu}_n = \int_{\mathcal{X}} u(x)\mathrm{d}\tilde{\mu}, \tag{11.92}$$

if u is a $\mathfrak{B}_n$-measurable integrable function.

We denote by $F_n := E_{\mathfrak{B}_n}(F)$ the *conditional expectation* of F w.r.t. $\mathfrak{B}_n$. We recall that if $(Q, \mathfrak{B}, \mu)$ is a probability space and $\mathfrak{B}_0 \subset \mathfrak{B}$ is a σ-algebra, then $E_{\mathfrak{B}_0}$ is defined on $L^2(Q, \mathrm{d}\mu)$ as the orthogonal projection on the subspace of $\mathfrak{B}_0$-measurable L^2 functions. $E_{\mathfrak{B}_0}$ extends to a contraction on $L^1(Q, \mathrm{d}\mu)$ with $\int_Q E_{\mathfrak{B}_0}(u)\mathrm{d}\mu = \int_Q u\mathrm{d}\mu$. In our case, since $\mathfrak{B}$ is generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$, we know that

$$F_n \to F \quad \mu \text{ a.e. and in } L^1(\mathcal{X}, \mathrm{d}\mu). \tag{11.93}$$

If Φ is $\mathfrak{B}_n$-measurable, we have, using (11.92),

$$\int_{\mathcal{X}_n} \Phi(x)\mathrm{d}\tilde{\mu}_n = \int_{\mathcal{X}} \Phi(x)\mathrm{d}\tilde{\mu} = \int_{\mathcal{X}} \Phi(x)F\mathrm{d}\mu = \int_{\mathcal{X}} \Phi(x)F_n\mathrm{d}\mu = \int_{\mathcal{X}_n} \Phi(x)F_n\mathrm{d}\mu_n,$$

which shows that

$$\mathrm{d}\tilde{\mu}_n = F_n\mathrm{d}\mu_n, \quad n \in \mathbb{N}. \tag{11.94}$$

For $P \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X}_n)$ and $w \in \mathbb{C}\mathcal{X}_n$, we have

$$\int_{\mathcal{X}_n} \overline{w \cdot \nabla_x P(x)} F(x)\mathrm{d}\mu_n = \int_{\mathcal{X}_n} \overline{w \cdot \nabla_x P(x)}\mathrm{d}\tilde{\mu}_n,$$

which we can rewrite as

$$(w \cdot \nabla_x P|F)_{L^2(\mathcal{X}_n, \mathrm{d}\mu_n)} = (w \cdot \nabla_x P|1)_{L^2(\mathcal{X}_n, \mathrm{d}\tilde{\mu}_n)}.$$

On $L^2(\mathcal{X}, \mathrm{d}\mu_n)$, we have

$$(w\cdot\nabla_x)^* = -\overline{w}\cdot\nabla_x + \overline{w}\cdot x, \tag{11.95}$$

and on $L^2(\mathcal{X}, \mathrm{d}\tilde{\mu}_n)$, we have

$$(w\cdot\nabla_x)^* = -\overline{w}\cdot\nabla_x + a_n^{-1}\overline{w}\cdot x. \tag{11.96}$$

This yields

$$\int_{\mathcal{X}_n} \overline{w \cdot \nabla_x P} F_n\mathrm{d}\mu_n = \int_{\mathcal{X}_n} a_n^{-1}\overline{w \cdot xP} F_n\mathrm{d}\mu_n. \tag{11.97}$$

Since $\mathcal{X}_n$ is n-dimensional, μ_n and $\tilde{\mu}_n$ can be realized on $\mathcal{X}_n$ with

$$\mathrm{d}\mu_n = (2\pi)^{-n/2}\mathrm{e}^{-\frac{1}{2}(x|x)}\mathrm{d}x, \quad \mathrm{d}\tilde{\mu}_n = (2\pi)^{-n/2}(\det a_n)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}(x|a_n^{-1}x)}\mathrm{d}x,$$

so that

$$F_n(x) = (\det a_n)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}(x|a_n^{-1}x)+\frac{1}{2}(x|x)}.$$

It follows from (11.97) that F_n satisfies in the usual sense the identity

$$w \cdot \nabla_x F_n(x) = (\mathbb{1} - a_n^{-1}) w \cdot x F_n(x).$$

Let us set

$$\Psi_n := F_n^{\frac{1}{2}},$$

so that $\Psi_n \in L^2(\mathcal{X}_n, \mathrm{d}\mu_n)$, $\|\Psi_n\| = 1$. From $\nabla_x \Psi_n = \frac{1}{2} F_n^{-\frac{1}{2}} \nabla_x F_n$, we get that

$$w \cdot \nabla_x \Psi_n(x) = \frac{1}{2}(\mathbb{1} - a_n^{-1}) w \cdot x \Psi_n(x), \quad w \in \mathbb{C}\mathcal{X}_n.$$

Considering now Ψ_n as a function on $\mathcal{X}$, using (11.95) we can rewrite this identity as

$$\int_{\mathcal{X}} \big(-\overline{w} \cdot \nabla_x + \frac{1}{2}(\mathbb{1} + a_n^{-1}) \overline{w} \cdot x\big) \overline{P} \Psi_n \mathrm{d}\mu = 0, \quad P \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X}_n), \quad w \in \mathbb{C}\mathcal{X}_n,$$

or equivalently

$$\int_{\mathcal{X}} \big(-a_n \overline{w} \cdot \nabla_x + \frac{1}{2}(\mathbb{1} + a_n) \overline{w} \cdot x\big) \overline{P} \Psi_n \mathrm{d}\mu = 0, \quad P \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X}_n), \quad w \in \mathbb{C}\mathcal{X}_n. \tag{11.98}$$

We note now that if $w \in \mathcal{X}$ and $w_n := \pi_n w$, then for all $P \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$

$$\lim_{n \to \infty} (a_n w_n - a w) \cdot \nabla_x P = 0,$$
$$\lim_{n \to \infty} \big((a_n + \mathbb{1}) w_n \cdot x - (a + \mathbb{1}) w \cdot x\big) P = 0 \quad \text{in } L^2(X,\ \mu).$$

Since Ψ_n is uniformly bounded in $L^2(\mathcal{X}, \mathrm{d}\mu)$, we deduce from (11.98) that

$$\lim_{n \to \infty} \int_{\mathcal{X}} \big(-a \overline{w} \cdot \nabla_x + \frac{1}{2}(\mathbb{1} + a) \overline{w} \cdot x\big) P \Psi_n \mathrm{d}\mu = 0, \quad P \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X}_m),\ w \in \mathbb{C}\mathcal{X} \quad \text{for some } m. \tag{11.99}$$

We claim now that

$$\mathrm{w} - \lim_{n \to \infty} \Psi_n = \Psi. \tag{11.100}$$

In fact, since Ψ_n is uniformly bounded in L^2, it suffices to show that, for all $G \in L^\infty(\mathcal{X}, \mathrm{d}\mu)$,

$$\lim_{n \to \infty} \int_{\mathcal{X}} (\Psi_n - \Psi) G \mathrm{d}\mu = 0. \tag{11.101}$$

Let $\Phi_n = (\Psi_n - \Psi) G \in L^2 \subset L^1$. We know from (11.93) that $\Phi_n \to 0$ μ a.e.. Moreover the sequence (Φ_n) is bounded in L^2, since $G \in L^\infty$. It follows from Subsect. 5.1.9 that the sequence (Φ_n) is equi-integrable, which using the fact that $\Phi_n \to 0$ μ a.e. implies (11.101).

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

Passing to the limit in (11.99) and using (11.100), we finally get

$$\int_{\mathcal{X}} \big(-a\overline{w}\cdot\nabla_x + \frac{1}{2}(\mathbb{1}+a)\overline{w}\cdot x\big)P\Psi \mathrm{d}\mu = 0, \quad w\in\mathcal{X}, \tag{11.102}$$

first for $P\in\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X}_m)$ for some m, and then by density for all $P\in\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$.

Using the definition of $a_{\mathrm{rw}}(w)$, $a_{\mathrm{rw}}^*(w)$ on $L^2(\mathcal{X},\mathrm{d}\mu)$, we see that (11.102) implies (11.90), which completes the proof of the $\Leftarrow$ part of the theorem. □

Proof of Prop. 5.79. We use the notation of the proof of Thm. 5.78. We have seen that Ψ, resp. Ψ_n are the bosonic Gaussian vectors for $c=(a-\mathbb{1})(a+\mathbb{1})^{-1}$, resp. $c_n=\pi_n c\pi_n$. Since $c_n\to c$ in $B^2(\mathcal{X})$, it follows from Thm. 11.28 that $\Psi_n\to\Psi$ in L^2, hence $F_n\to F$ in L^1. This proves (1) and (3). (2) is left to the reader. □

11.5 Coherent sector of CCR representations

This section is devoted to coherent representations, that is, translations of Fock CCR representations. It is to a large extent parallel to the previous section about Fock CCR representations.

We keep the same notation as in the previous section. In particular, $(\mathcal{Y},\omega)$ is a symplectic space,

$$\mathcal{Y}\ni y\mapsto W^\pi(y)\in U(\mathcal{H})$$

is a regular representation of CCR and $\mathrm{j}\in L(\mathcal{Y})$ is a Kähler anti-involution, so that we obtain a Kähler space $(\mathcal{Y},\cdot,\omega,\mathrm{j})$.

$\mathcal{Z}$ is the holomorphic subspace of $\mathbb{C}\mathcal{Y}$. As usual, we identify $\mathbb{C}\mathcal{Y}$ with $\mathcal{Z}\oplus\overline{\mathcal{Z}}$. We do not assume that $\mathcal{Y}$, or equivalently $\mathcal{Z}$, is complete.

$\mathcal{Y}^\#$ denotes the algebraic dual of $\mathcal{Y}$. Similarly, $\mathcal{Z}^*$ denotes the algebraic anti-dual of $\mathcal{Z}$. We have the identification $\mathcal{Y}^\# = \mathrm{Re}(\mathcal{Z}^*\oplus\overline{\mathcal{Z}}^*)$.

11.5.1 Coherent vectors in a CCR representation

Let $f\in\mathcal{Z}^*$, that is, f is an anti-linear functional on $\mathcal{Z}$, possibly unbounded. We also introduce a symbol for the corresponding (possibly unbounded) linear functional on $\mathcal{Y}$, $v=(f,\overline{f})\big|_{\mathcal{Y}}\in\mathcal{Y}^\#$. Clearly,

$$v\cdot(z,\overline{z}) = (f|z)+(z|f).$$

Definition 11.56 *We define the* space of j,f-coherent vectors

$$\mathcal{K}_f^\pi := \big\{\Psi\in\mathcal{H} \ :\ \Psi\in\mathrm{Dom}\,a^\pi(z),\quad a^\pi(z)\Psi=(z|f)\Psi,\ z\in\mathcal{Z}\big\},$$

where the j*-creation, resp.* j*-annihilation operators* $a^{\pi*}(z)$*, resp.* $a^\pi(z)$ *are defined in Subsect. 8.2.4.*

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

Proposition 11.57 (1) *$\mathcal{K}_f^\pi$ is a closed subspace of $\mathcal{H}$.*
(2) *$\Psi \in \mathcal{K}_f^\pi$ iff $(\Psi|W^\pi(y)\Psi) = \mathrm{e}^{-\frac{1}{4}y^2 + \mathrm{i}v\cdot y}$, $y \in \mathcal{Y}$.*
(3) *Elements of $\mathcal{K}_f^\pi$ are analytic vectors for $\phi^\pi(y)$, $y \in \mathcal{Y}$.*
(4) *If $\Phi, \Psi \in \mathcal{K}_f^\pi$, then*

$$
\begin{aligned}
(\Phi|W^\pi(y)\Psi) &= (\Phi|\Psi)\mathrm{e}^{-\frac{1}{4}y^2 + \mathrm{i}v\cdot y},\\
(\Phi|\phi^\pi(y)\Psi) &= (\Phi|\Psi)v\cdot y,\\
(\Phi|\phi^\pi(y_1)\phi^\pi(y_2)\Psi) &= \frac{1}{2}\left(y_1\cdot y_2 + \mathrm{i}y_1\cdot\omega y_2\right)(\Phi|\Psi) + (v\cdot y_1)(v\cdot y_2)(\Phi|\Psi).
\end{aligned}
$$

Proof We will suppress the superscript π to simplify notation.

(1) $\mathcal{K}_f$ is closed as an intersection of kernels of closed operators.

Let us prove (2) $\Leftarrow$. Let $\Psi \in \mathcal{H}$ such that $(\Psi|W(y)\Psi) = \|\Psi\|^2\mathrm{e}^{-\frac{1}{4}y^2+\mathrm{i}v\cdot y}$. Without loss of generality we can assume that $\|\Psi\| = 1$. Taking the first two terms in the Taylor expansion of

$$t \mapsto (\Psi|W(ty)\Psi) = \mathrm{e}^{-\frac{1}{4}t^2y^2 + \mathrm{i}tv\cdot y},$$

we obtain

$$(\Psi|\phi(y)\Psi) = v\cdot y, \quad (\Psi|\phi(y)^2\Psi) = \frac{1}{2}y^2 + (v\cdot y)^2. \tag{11.103}$$

If $z = \frac{1}{2}(y - \mathrm{i}\mathrm{j}y) \in \mathcal{Z}$, we have

$$
\begin{aligned}
&\big(a^*(z) - (f|z)\mathbb{1}\big)\big(a(z) - (z|f)\mathbb{1}\big)\\
&\quad = \tfrac{1}{4}\phi(y)^2 + \tfrac{1}{4}\phi(\mathrm{j}y)^2 + \tfrac{\mathrm{i}}{4}[\phi(y), \phi(\mathrm{j}y)]\\
&\qquad - \tfrac{1}{2}(z|f)\phi(y) + \tfrac{\mathrm{i}}{2}(z|f)\phi(\mathrm{j}y) - \tfrac{1}{2}(f|z)\phi(y) - \tfrac{\mathrm{i}}{2}(f|z)\phi(\mathrm{j}y) + (f|z)(z|f)\mathbb{1}.
\end{aligned}
$$

Using (11.103), we obtain that

$$\|\big(a(z) - (z|f)\mathbb{1}\big)\Psi\|^2 = 0.$$

Let us prove (2) $\Rightarrow$. Consider $y \in \mathcal{Y}$. Note that if $\Psi \in \mathcal{K}_f$, then $\Psi \in \mathrm{Dom}\,\phi(y)$. We consider the function

$$\mathbb{R} \ni t \mapsto (\Psi|W(ty)\Psi),$$

as in the proof of Prop. 11.42. For $z = \frac{1}{2}(y - \mathrm{i}\mathrm{j}y) \in \mathcal{Z}$, we get

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}F(t) &= \mathrm{i}\big(\Psi|\phi(y)W(ty)\Psi\big)\\
&= \tfrac{\mathrm{i}}{2}\big(a(z)\Psi|W(ty)\Psi\big) + \tfrac{\mathrm{i}}{2}\big(\Psi|W(ty)a(z)\Psi\big) - \tfrac{t}{2}y^2\big(\Psi|W(ty)\Psi\big)\\
&= \mathrm{i}\big((f|z) + (z|f)\big)F(t) - \tfrac{t}{2}y^2F(t),
\end{aligned}
$$

which yields $F(t) = \|\Psi\|^2\mathrm{e}^{-\frac{1}{4}t^2y^2 + \mathrm{i}tv\cdot y}$.

(2) immediately implies (3).

(4) follows from (2) and (11.103) by polarization. □

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

11.5.2 Coherent vectors in Fock spaces

We consider the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$ and the usual creation and annihilation operators $a^*(z)$, $a(z)$.

Theorem 11.58 *Let $\Psi \in \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$ be an f-coherent vector for the Fock representation over $\mathcal{Y}$, that is, for any $z \in \mathcal{Z}$, $\Psi \in \mathrm{Dom}\, a(z)$ and*

$$a(z)\Psi = (z|f)\Psi.$$

Then the following is true:

(1) *If f is continuous, i.e. $f \in \mathcal{Z}^{\mathrm{cpl}}$, then Ψ is proportional to $W(-\mathrm{i}f, \mathrm{i}\overline{f})\Omega$.*
(2) *If f is not continuous, then $\Psi = 0$.*

Proof By induction we show that, for $z_1, \dots, z_n \in \mathcal{Z}$,

$$a(z_{n-1}) \cdots a(z_1)\Psi \in \mathrm{Dom}\, a(z_n), \quad a(z_n) \cdots a(z_1)\Psi = (z_1|f) \cdots (z_n|f)\Psi.$$

This implies

$$(a^*(z_1) \cdots a^*(z_n)\Omega|\Psi) = (z_1|f) \cdots (z_n|f)(\Omega|\Psi). \tag{11.104}$$

In particular,

$$(z|\Psi) = (a^*(z)\Omega|\Psi) = (z|f)(\Omega|\Psi), \quad z \in \mathcal{Z}.$$

Using the fact that $\mathcal{Z}$ is dense in $\mathcal{Z}^{\mathrm{cpl}}$, we see that $(\Omega|\Psi)f$ is a bounded functional on $\mathcal{Z}$, hence it belongs to $\mathcal{Z}^{\mathrm{cpl}}$. Thus either $f \in \mathcal{Z}^{\mathrm{cpl}}$ or $(\Omega|\Psi) = 0$. In the latter case, (11.104) implies that $\Psi = 0$. □

11.5.3 Coherent representations

Consider the usual Weyl operators $W(y)$, $y \in \mathcal{Y}$, on the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$. Set

$$\mathcal{Y} \ni y \mapsto W_f(y) := W(y)\mathrm{e}^{2\mathrm{i}\mathrm{Re}(f|z)} \in U\big(\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})\big). \tag{11.105}$$

Clearly, (11.105) is a regular CCR representation,

Definition 11.59 *(11.105) is called the* j, f-coherent CCR representation.

Theorem 11.60 (1) *(11.105) is the translation of the Fock representation by the vector $(f, \overline{f}) \in \mathcal{Y}^{\#}$ (see Def. 8.21).*
(2) *If $f \in \mathcal{Z}^{\mathrm{cpl}}$ (equivalently, $v \in \mathcal{Y}^{\mathrm{cpl}}$), then*

$$W_f(y) = W(\mathrm{i}f, -\mathrm{i}\overline{f})W(y)W(-\mathrm{i}f, \mathrm{i}\overline{f}).$$

(3) *If $f \notin \mathcal{Z}^{\mathrm{cpl}}$, then (11.105) is not unitarily equivalent to the Fock representation $\mathcal{Y} \ni y \mapsto W(y)$.*

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

Proof (3) Let $U \in U(\mathcal{H})$ intertwining $W_f(\cdot)$ and $W(\cdot)$. Since $U\Omega$ satisfies the assumptions of Thm. 11.58 we obtain that $U\Omega = 0$, which is a contradiction. □

11.5.4 Coherent sector

Let us go back to an arbitrary CCR representation (11.59) over a Kähler space $\mathcal{Y}$. We will describe how to determine the largest sub-representation of (11.59) unitarily equivalent to a multiple of a coherent representation over $\mathcal{Y}$ in $\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$.

Definition 11.61 *Introduce the equivalence relation on the set of Kähler anti-involutions on $\mathcal{Y}$ and anti-linear functionals on the corresponding holomorphic space $\mathcal{Z}$:*

$$(\mathrm{j}_1, f_1) \sim (\mathrm{j}_2, f_2) \ \Leftrightarrow\ \mathrm{j}_1 - \mathrm{j}_2 \in B^2(\mathcal{Y}), \quad f_1 - f_2 \in \mathcal{Z}^{\mathrm{cpl}}.$$

Let $[\mathrm{j}, f]$ denote the equivalence class of (j, f) w.r.t. this relation.

The $[\mathrm{j}, f]$-coherent sector of the representation W^π *is the subspace of $\mathcal{H}$ defined as*

$$\mathcal{H}^\pi_{[f]} := \mathrm{Span}^{\mathrm{cl}}\big\{W^\pi(y)\Psi \ :\ \ \Psi \in \mathcal{K}^\pi_f, \quad y \in \mathcal{Y}\big\}.$$

The CCR representation W^π is $[\mathrm{j}, f]$-coherent *if $\mathcal{H}^\pi_{[f]} = \mathcal{H}$.*

Theorem 11.62 (1) *$\mathcal{H}^\pi_{[f]}$ is invariant under $W^\pi(y)$, $y \in \mathcal{Y}$.*
(2) *There exists a unique unitary operator*

$$U^\pi_f : \mathcal{K}^\pi_f \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}}) \to \mathcal{H}^\pi_{[f]}$$

satisfying

$$U^\pi_f \ \Psi \otimes W_f(y)\Omega = W^\pi(y)\Psi, \quad \Psi \in \mathcal{K}^\pi_f, \quad y \in \mathcal{Y}.$$

(3)

$$U^\pi_f \ 1\!\!1 \otimes W_f(y) = W^\pi(y) U^\pi_f, \quad y \in \mathcal{Y}. \tag{11.106}$$

(4) *If there exists an operator $U : \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}}) \to \mathcal{H}$ such that $UW_f(y) = W^\pi(y)U$ for $y \in \mathcal{Y}$, then* $\mathrm{Ran}\, U \subset \mathcal{H}^\pi_{[f]}$.

11.6 van Hove Hamiltonians

In this section we will study self-adjoint operators on bosonic Fock spaces of the form

$$\begin{aligned} H &= \int h(\xi) a^*(\xi) a(\xi) \mathrm{d}\xi + \int \overline{w}(\xi) a(\xi) \mathrm{d}\xi + \int w(\xi) a^*(\xi) \mathrm{d}\xi + c \\ &= \mathrm{d}\Gamma(h) + a(w) + a^*(w) + c 1\!\!1 \end{aligned} \tag{11.107}$$

(first written in the "physicist's notation" and then in the "mathematician's notation"). Note that this expression may have only a formal meaning. In some

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

cases, the constant c is actually infinite. Following Schweber (1962), we call (11.107) *van Hove Hamiltonians*. We will see that in the case of an infinite number of degrees of freedom these Hamiltonians have a surprisingly rich theory. We will discuss both classical and quantum van Hove Hamiltonians. Their theories are parallel to one another.

Throughout this section, $\mathcal{Z}$ is a Hilbert space and h is a positive operator on $\mathcal{Z}$ with $\operatorname{Ker} h = \{0\}$. (It is, however, easy to generalize the theory of van Hove Hamiltonians to non-positive h.)

In addition to h, van Hove Hamiltonians depend on the choice of w. The choice $w \in \mathcal{Z}$ turns out to be too narrow.

In order to explain the nature of w, it will be convenient to use the notation introduced in Subsect. 2.3.4. In particular, we will consider the spaces $(h^\alpha + h^\beta)\mathcal{Z}$ for $0 \le \alpha \le \beta$. Note that

$$w \in (h^\alpha + h^\beta)\mathcal{Z} \;\Leftrightarrow\; 1\!\!1_{]0,1]}(h)w \in h^\alpha \mathcal{Z}, \qquad 1\!\!1_{[1,+\infty[}(h)w \in h^\beta \mathcal{Z}. \tag{11.108}$$

For $w \in (h^\alpha + h^\beta)\mathcal{Z}$, the behavior of w near $h = 0$ (resp. $h = +\infty$), i.e. at low (resp. high) energies, is encoded by the exponent α (resp. β) and connected with the so-called *infrared* (resp. *ultraviolet*) *problem*. We will always assume that

$$w \in (1\!\!1 + h)\mathcal{Z}. \tag{11.109}$$

Note that if $w \in (1\!\!1 + h)\mathcal{Z}$, then $(\mathrm{e}^{\mathrm{i}th} - 1\!\!1)h^{-1}w \in \mathcal{Z}$ for any $t \in \mathbb{R}$.

11.6.1 Classical van Hove dynamics

Definition 11.63 *The* classical van Hove dynamics *is defined for* $t \in \mathbb{R}$ *as*

$$\alpha^t(z) := \mathrm{e}^{\mathrm{i}th} z + (\mathrm{e}^{\mathrm{i}th} - 1\!\!1)h^{-1}w, \quad z \in \mathcal{Z}, \quad t \in \mathbb{R}.$$

It is easy to see that $\mathbb{R} \ni t \mapsto \alpha^t$ is a one-parameter group of affine transformations of $\mathcal{Z}$ preserving the scalar product.

Let us note the following property of the dynamics α. Let p_1, p_2 be two complementary orthogonal projections commuting with h. For $i = 1, 2$, let $\mathcal{Z}_i := \operatorname{Ran} p_i$. Thus we have a direct sum decomposition $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$.

Set $h_i := p_i h$, treated as a self-adjoint operator on $\mathcal{Z}_i$. Let α_i be the dynamics on $\mathcal{Z}_i$ defined by h_i, w_i. Then the dynamics α splits as

$$\alpha^t(z_1, z_2) = (\alpha_1^t(z_1), \alpha_2^t(z_2)).$$

In particular, we can take

$$p_1 := 1\!\!1_{[0,1]}(h), \quad p_2 := 1\!\!1_{]1,\infty[}(h). \tag{11.110}$$

Then h_1 is bounded and $h_2 \ge 1\!\!1$. In the case of h_1 the ultraviolet problem is absent, but the infrared problem can show up. In the case of h_2 we have the opposite situation: the infrared problem is absent, but we can face the ultraviolet problem.

Assume for a moment that $w \in h\mathcal{Z}$. Then the van Hove dynamics is equivalent to the free van Hove dynamics:

$$\alpha^t = \tau^{-1} \circ \alpha_0^t \circ \tau, \tag{11.111}$$

where

$$\tau(z) := z + h^{-1}w, \quad \alpha_0^t(z) := \mathrm{e}^{\mathrm{i}th}z. \tag{11.112}$$

11.6.2 Classical van Hove Hamiltonians

$\mathcal{Z}$ can be interpreted as a charged symplectic space with the form $\overline{z}_1 \cdot \omega z_2 := \mathrm{i}(z_1|z_2)$. In the case of finite dimensions we know that for every charged symplectic dynamics $t \mapsto \alpha_t$ there exists a real function H on $\mathcal{Z}$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha^t(z) = \mathrm{i}\nabla_{\overline{z}}H(\alpha^t(z)). \tag{11.113}$$

This function is called a Hamiltonian of α. It is unique up to an additive constant.

In the case of an infinite number of degrees of freedom the situation is more complicated. It is even unclear how to give a general definition of a Hamiltonian of an arbitrary charged symplectic dynamics. It may, for instance, turn out that natural candidates for a Hamiltonian are defined only on a subset of $\mathcal{Z}$, and differentiable on a smaller subset.

The classical van Hove dynamics is an example of a charged symplectic dynamics. If the dimension is finite, it is easy to see that its Hamiltonian is

$$H(z) = (z|hz) + (z|w) + (w|z) + c,$$

where c is an arbitrary real constant.

In infinite dimensions we will see that the van Hove dynamics possesses natural Hamiltonians. Clearly, these Hamiltonians will be defined only up to an arbitrary additive constant. One can ask whether it is possible to fix this constant in a natural way. We will argue that there are two ways to do so, both under some additional assumptions on w in addition to (11.109).

Definition 11.64 *Assume*

$$w \in (1\!\!1 + h^{\frac{1}{2}})\mathcal{Z}. \tag{11.114}$$

Set $\mathcal{D}_{\mathrm{I}} := \mathrm{Dom}\, h^{\frac{1}{2}}$, *and*

$$H_{\mathrm{I}}(z) := (z|hz) + (z|w) + (w|z), \quad z \in \mathcal{D}_{\mathrm{I}}.$$

We will say that H_{I} *is the* classical van Hove Hamiltonian of the first kind.

Definition 11.65 *Assume*

$$w \in h^{1/2}\mathcal{Z} + h\mathcal{Z}. \tag{11.115}$$

Set $\mathcal{D}_{\mathrm{II}} := \{z \in \mathcal{Z} \ : \ h^{1/2}z + h^{-1/2}w \in \mathcal{Z}\}$, *and*

$$H_{\mathrm{II}}(z) := (h^{1/2}z + h^{-1/2}w|h^{1/2}z + h^{-1/2}w), \quad z \in \mathcal{D}_{\mathrm{II}}.$$

We will say that H_{II} *is the* classical van Hove Hamiltonian of the second kind*.*

Clearly, both H_{I} and H_{II} are well defined iff

$$w \in h^{1/2}\mathcal{Z} = (\mathbb{1} + h^{\frac{1}{2}})\mathcal{Z} \cap (h^{\frac{1}{2}} + h)\mathcal{Z},$$

and then

$$H_{\mathrm{II}} = H_{\mathrm{I}} + (w|h^{-1}w).$$

Definition 11.66 *Let* w *be a functional satisfying (11.109). We split the dynamics* α *into* $\alpha_1 \oplus \alpha_2$*, the functional* w *into* $w_1 \oplus w_2$ *as explained in Subsect. 11.6.1, with the splitting given by (11.110). Then, by (11.108),* $w_1 \in \mathcal{Z}_1 = (\mathbb{1} + h_1^{\frac{1}{2}})\mathcal{Z}_1$ *and* $w_2 \in h_2\mathcal{Z}_2 = (h_2^{\frac{1}{2}} + h_2)\mathcal{Z}_2$*. So we can define the Hamiltonian* $H_{1,\mathrm{I}}$ *for the dynamics* α_1 *on the domain* $\mathcal{D}_{1,\mathrm{I}}$*, and the Hamiltonian* $H_{2,\mathrm{II}}$ *for the dynamics* α_2 *on the domain* $\mathcal{D}_{2,\mathrm{II}}$*.*

Set $\mathcal{D} := \mathcal{D}_{1,\mathrm{I}} \oplus \mathcal{D}_{2,\mathrm{II}}$*. A function* H *on* $\mathcal{D}$ *will be called* a classical van Hove Hamiltonian *if it is of the form*

$$H(z_1, z_2) := H_{1,\mathrm{I}}(z_1) + H_{2,\mathrm{II}}(z_2) + c, \quad (z_1, z_2) \in \mathcal{D},$$

where $c \in \mathbb{R}$ *is arbitrary.*

Note that, in general, there exist $w \in (\mathbb{1} + h)\mathcal{Z}$ that do not belong to $(\mathbb{1} + h^{1/2})\mathcal{Z} \cup (h^{1/2} + h)\mathcal{Z}$. For such w, the dynamics α is well defined but neither H_{I} nor H_{II} are well defined.

The following proposition says that van Hove Hamiltonians are in a certain sense Hamiltonians of the van Hove dynamics. Recall that the Gâteaux differentiability was defined in Def. 2.50.

Proposition 11.67 *Let* $w \in (\mathbb{1} + h)\mathcal{Z}$*. Let* H *be the corresponding van Hove Hamiltonian with the domain* $\mathcal{D}$*. Then*

(1) *The function* H *is Gâteaux differentiable at* $z \in \mathcal{Z}$ *iff* $hz + w$ *belongs to* $\mathcal{Z}$*, and then*

$$\nabla_{\overline{z}} H(z) = hz + w.$$

(2) *The dynamics* $t \mapsto \alpha^t(z)$ *is differentiable w.r.t.* t *iff* $h\alpha^t(z) + w \in \mathcal{Z}$*, and then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha^t(z) = \mathrm{i}(h\alpha^t(z) + w),$$

which can be written in the form (11.113).

(3) α^t *leaves* $\mathcal{D}$ *invariant and* H *is constant along the trajectories.*

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

The following theorem discusses various special features of van Hove Hamiltonians.

Theorem 11.68 *Let H be a van Hove Hamiltonian.*

(1) *0 belongs to $\mathcal{D}$ iff $w \in (\mathbb{1} + h^{1/2})\mathcal{Z}$. If this is the case, then $H = H_{\mathrm{I}} + H(0)$.*
(2) *H is bounded from below iff $w \in (h^{1/2} + h)\mathcal{Z}$. If this is the case, then $H = H_{\mathrm{II}} + \inf H$.*
(3) *H has a minimum iff $w \in h\mathcal{Z}$. This minimum is at $-h^{-1}w$, and then*

$$H_{\mathrm{II}}(z) = (\tau(z)|h\tau(z)),$$

where τ was defined in (11.112).

Proof We split the dynamics, the functional and the vectors in $\mathcal{Z}$. Then the proofs are immediate. □

Formally,

$$\begin{aligned} H_{\mathrm{I}}(z) &= (z|hz) + (w|z) + (z|w), \\ H_{\mathrm{II}}(z) &= (z|hz) + (w|z) + (z|w) + (w|h^{-1}w). \end{aligned}$$

11.6.3 Quantum van Hove dynamics

We again assume $w \in (\mathbb{1} + h)\mathcal{Z}$. Many quantum objects are analogous to their classical counterparts. Typically, in such cases we will use the same symbols in the classical and quantum case, which should not lead to any confusion.

Definition 11.69 *For $B \in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$, we set*

$$\alpha^t\,(B) := V(t)BV(t)^*,$$

where $V(t)$ is a family of unitary operators on $\Gamma_{\mathrm{s}}(\mathcal{Z})$

$$V(t) := \Gamma(\mathrm{e}^{\mathrm{i}th}) \exp\left(a^*\big((\mathbb{1} - \mathrm{e}^{-\mathrm{i}th})h^{-1}w\big) - a\big((\mathbb{1} - \mathrm{e}^{-\mathrm{i}th})h^{-1}w\big)\right).$$

$t \mapsto \alpha^t$ will be called a quantum van Hove dynamics*.*

It is easy to check that $V(t)$ is strongly continuous and, for any $t_1, t_2 \in \mathbb{R}$,

$$V(t_1)V(t_2) = c(t_1, t_2)V(t_1 + t_2),$$

for some $c(t_1, t_2) \in \mathbb{C}$, $|c(t_1, t_2)| = 1$. Hence, α is a one-parameter group of $*$-automorphisms of $B\big(\Gamma_{\mathrm{s}}(\mathcal{Z})\big)$, continuous in the strong operator topology.

In order to make the relationship with the classical dynamics clearer, one can note that

$$\alpha^t\,(a^*(z)) = \big((\mathrm{e}^{\mathrm{i}th} - \mathbb{1})h^{-1}w|z\big) + a^*(\mathrm{e}^{\mathrm{i}th}z), \quad z \in \mathcal{Z}.$$

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

Let $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$, as in Subsect. 11.6.1. Then we have the identification $\Gamma_{\mathrm{s}}(\mathcal{Z}) = \Gamma_{\mathrm{s}}(\mathcal{Z}_1) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_2)$. The dynamics α factorizes as

$$\alpha^t(B_1 \otimes B_2) = \alpha_1^t(B_1) \otimes \alpha_2^t(B_2), \quad B_i \in B\big(\Gamma_{\mathrm{s}}(\mathcal{Z}_i)\big), \; i = 1, 2. \tag{11.116}$$

11.6.4 Quantum van Hove Hamiltonians

Definition 11.70 *We say that a self-adjoint operator H is* a quantum van Hove Hamiltonian for the dynamics $t \mapsto \alpha^t$ *if*

$$\alpha^t(B) = \mathrm{e}^{\mathrm{i}tH} B \mathrm{e}^{-\mathrm{i}tH}. \tag{11.117}$$

By Prop. 6.68, such a Hamiltonian always exists and is unique up to an additive real constant.

Assume for a moment that $w \in h\mathcal{Z}$. Then, up to a constant, van Hove Hamiltonians are unitarily equivalent to the free van Hove Hamiltonian:

$$H := U \mathrm{d}\Gamma(h) U^* + c\mathbb{1},$$

where U is the "dressing operator"

$$U := \exp\big(-a^*(h^{-1}w) + a(h^{-1}w)\big). \tag{11.118}$$

In the general case, (11.118) can be ill defined, and the construction of van Hove Hamiltonians is more complicated.

Definition 11.71 *Let $w \in (\mathbb{1} + h^{1/2})\mathcal{Z}$. Define*

$$U_{\mathrm{I}}(t) := \mathrm{e}^{\mathrm{i}\mathrm{Im}(h^{-1}w|\mathrm{e}^{\mathrm{i}th}h^{-1}w) - \mathrm{i}t(w|h^{-1}w)} V(t).$$

We easily check that $U_{\mathrm{I}}(t)$ is a one-parameter strongly continuous unitary group. Therefore, by the Stone theorem there exists a unique self-adjoint operator H_{I} such that

$$U_{\mathrm{I}}(t) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{I}}}.$$

We will say that H_{I} is the quantum van Hove Hamiltonian of the first kind.

Definition 11.72 *Let $w \in (h^{1/2} + h)\mathcal{Z}$. Define*

$$U_{\mathrm{II}}(t) := \mathrm{e}^{\mathrm{i}\mathrm{Im}(h^{-1}w|\mathrm{e}^{\mathrm{i}th}h^{-1}w)} V(t).$$

We easily check that $U_{\mathrm{II}}(t)$ is a one-parameter strongly continuous unitary group. Therefore, by the Stone theorem there exists a unique self-adjoint operator H_{II} such that

$$U_{\mathrm{II}}(t) = \mathrm{e}^{\mathrm{i}tH_{\mathrm{II}}}.$$

We will say that H_{II} is the quantum van Hove Hamiltonian of the second kind.

Clearly, both H_{I} and H_{II} are well defined iff $w \in h^{1/2}\mathcal{Z}$, and then

$$H_{\mathrm{II}} = H_{\mathrm{I}} + (w|h^{-1}w).$$

Theorem 11.73 *Let H be a quantum van Hove Hamiltonian of a dynamics $t \mapsto \alpha^t$. Then the following statements are true:*

(1) *Ω belongs to $\mathrm{Dom}\,|H|^{1/2}$ (the form domain of H) iff $w \in (\mathbb{1} + h^{1/2})\mathcal{Z}$. Under this condition $H = H_{\mathrm{I}} + (\Omega|H\Omega)$.*
(2) *The operator H is bounded from below iff $w \in (h^{1/2} + h)\mathcal{Z}$. Under this condition $H = H_{\mathrm{II}} + \inf H$, where $\inf H$ denotes the infimum of the spectrum of H.*
(3) *The operator H has a ground state ($\inf H$ is an eigenvalue of H) iff $w \in h\mathcal{Z}$. Then, using the dressing operator defined in (11.118), we can write*

$$H_{\mathrm{II}} = U \mathrm{d}\Gamma(h) U^*. \tag{11.119}$$

Proof We write α as $\alpha_1 \otimes \alpha_2$, w as $w_1 \oplus w_2$, with $w_1 \in \mathcal{Z}_1$, $w_2 \in h_2\mathcal{Z}_2$; see (11.116).

The operator $\mathrm{d}\Gamma(h_1) + a^*(w_1) + a(w_1)$ is essentially self-adjoint on $\mathrm{Dom}\,N_1$, by Nelson's commutator theorem with the comparison operator N_1; see Thm. 2.74 (1). We set

$$H_{1,\mathrm{I}} := \big(\mathrm{d}\Gamma(h_1) + a^*(w_1) + a(w_1)\big)^{\mathrm{cl}}.$$

Clearly, $H_{1,\mathrm{I}}$ is a Hamiltonian of α_1.

Next we set

$$H_{2,\mathrm{II}} := U_2 \mathrm{d}\Gamma(h_2) U_2^*, \quad U_2 := \exp\big(-a^*(h_2^{-1}w_2) + a(h_2^{-1}w_2)\big),$$

which is a Hamiltonian of α_2^t. Hence, any Hamiltonian of α is of the form

$$H = H_{1,\mathrm{I}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{2,\mathrm{II}} + c\mathbb{1}.$$

We drop the subscripts I, II in the rest of the proof. Since $\Omega_1 \in \mathrm{Dom}\,H_1$ we see that

$$\begin{aligned}\Omega = \Omega_1 \otimes \Omega_2 \in \mathrm{Dom}\,|H|^{\frac{1}{2}} &\Leftrightarrow \Omega_2 \in \mathrm{Dom}\,H_2^{\frac{1}{2}} \\ \Leftrightarrow \quad U_2^*\Omega \in \mathrm{Dom}\,\mathrm{d}\Gamma(h_2)^{\frac{1}{2}} &\Leftrightarrow w_2 \in h_2^{\frac{1}{2}}\mathcal{Z}_2.\end{aligned}$$

This proves (1).

Let us now prove (2). Since $H_2 \geq 0$, H is bounded below iff H_1 is bounded below. Since $\mathrm{Dom}\,N_1$ is a core for H_1, we have

$$\inf \mathrm{spec}\, H_1 = \inf_{\Psi_1 \in \mathrm{Dom}\,N_1,\ \|\Psi_1\|=1} (\Psi_1|H_1\Psi_1).$$

Set $w_1^\epsilon = 1\!\!1_{[\epsilon,1]}(h)w_1$, $U_\epsilon = \exp(-a^*(h_1^{-1}w_1^\epsilon) + a(h_1^{-1}w_1^\epsilon))$. Then

$$\begin{aligned}(\Psi_1|H_1\Psi_1) &= \lim_{\epsilon\to 0}\Big(\Psi_1|\big(\mathrm{d}\Gamma(h_1) + a^*(w_1^\epsilon) + a(w_1^\epsilon)\big)\Psi_1\Big)\\ &= \lim_{\epsilon\to 0}\Big(\big(U_\epsilon^*\Psi_1|\mathrm{d}\Gamma(h_1)U_\epsilon^*\Psi_1\big) - \big(w_1^\epsilon|h_1^{-1}w_1^\epsilon\big)\|\Psi_1\|^2\Big).\end{aligned}$$

It follows that if $w_1 \in h_1^{\frac{1}{2}}\mathcal{Z}_1$, then $H_1 \geq -(w_1|h_1^{-1}w_1)$.

Conversely, assume that H_1 is bounded below. Then, for

$$\Omega_{z_1} = \exp(a^*(z_1) - a(z_1))\Omega_1,$$

we have, by Subsect. 9.1.4,

$$(\Omega_{z_1}|H_1\Omega_{z_1}) = (z_1|h_1z_1) + (w_1|z_1) + (z_1|w_1).$$

By Thm. 11.68 (2), this implies that $w_1 \in h_1^{\frac{1}{2}}\mathcal{Z}_1$. This completes the proof of (2).

To prove (3), we note that H_2 has the ground state $U_2\Omega_2$. Hence, H has a ground state iff H_1 has one. If $w_1 \in h_1\mathcal{Z}_1$, then $H_1 = U_1\mathrm{d}\Gamma(h_1)U_1^*$ for $U_1 = \exp\big(-a^*(h_1^{-1}w_1) + a(h_1^{-1})w_1\big)$, hence it has a ground state.

Assume now that H_1 has a ground state Ψ. We again split $\mathcal{Z}_1$ into $\mathcal{Z}_1^\epsilon \oplus \mathcal{Z}_1^{\epsilon\perp}$, for $\mathcal{Z}_1^\epsilon = 1\!\!1_{[0,\epsilon]}(h)\mathcal{Z}_1$. Then H_1 splits into $H_1^\epsilon \otimes 1\!\!1 + 1\!\!1 \otimes H_1^{\epsilon\perp}$, w_1 into $w_1^\epsilon \oplus w_1^{\epsilon\perp}$ and Ψ into $\Psi^\epsilon \otimes \Psi^{\epsilon\perp}$. Since $w_1^{\epsilon\perp} \in h_1^{-1}\mathcal{Z}_1^{\epsilon\perp}$, we have

$$\Psi^{\epsilon\perp} = \exp\big(-a^*(h_1^{-1}w_1^{\epsilon\perp}) - a(h_1^{-1}w_1^{\epsilon\perp})\big)\Omega,$$

and therefore

$$a(z)\Psi = (z|h^{-1}w_1)\Psi, \quad z \in \mathcal{Z}_1^{\epsilon,\perp}.$$

We apply Thm. 11.58 with $\mathcal{Z} = \bigcup\limits_{\epsilon>0}\mathcal{Z}_1^{\epsilon\perp}$, so that $\mathcal{Z}^{\mathrm{cpl}} = \mathcal{Z}_1$, and we obtain that $w_1 \in h_1\mathcal{Z}_1$. □

Formally,

$$\begin{aligned}H_{\mathrm{I}} &= \mathrm{d}\Gamma(h) + a^*(w) + a(w),\\ H_{\mathrm{II}} &= \mathrm{d}\Gamma(h) + a^*(w) + a(w) + (w|h^{-1}w)1\!\!1.\end{aligned}$$

11.6.5 Nine classes of van Hove Hamiltonians

We can sum up the theory of van Hove Hamiltonians by dividing them into three classes based on the infrared behavior and three classes based on the ultraviolet behavior. Altogether we obtain $3 \times 3 = 9$ classes.

Infrared regularity

1. $\big(w|1\!\!1_{[0,1]}(h)h^{-2}w\big) < \infty$.
In the classical case, H has a minimum; and in the quantum case, H has a ground state.

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press

2. $\big(w|\mathbb{1}_{[0,1]}(h)h^{-1}w\big) < \infty$, $\big(w|\mathbb{1}_{[0,1]}(h)h^{-2}w\big) = \infty$.
H is bounded from below and H_{II} is well defined, but in the classical case H has no minimum, and in the quantum case H has no ground state.

3. $\big(w|\mathbb{1}_{[0,1]}(h)w\big) < \infty$, $\big(w|\mathbb{1}_{[0,1]}(h)h^{-1}w\big) = \infty$.
H is unbounded from below; H_{II} is ill defined.

Ultraviolet regularity

1. $\big(w|\mathbb{1}_{[1,\infty[}(h)w\big) < \infty$.
In the classical case the perturbation is bounded; in the quantum case the perturbation is a closable operator.

2. $\big(w|\mathbb{1}_{[1,\infty[}(h)h^{-1}w\big) < \infty$, $\big(w|\mathbb{1}_{[1,\infty[}(h)w\big) = \infty$.
H_{I} is well defined, but in the classical case the perturbation is not bounded, and in the quantum case the perturbation is not a closable operator.

3. $\big(w|\mathbb{1}_{[1,\infty]}(h)h^{-2}w\big) < \infty$, $\big(w|\mathbb{1}_{[1,\infty]}(h)h^{-1}w\big) = \infty$.
The constant c in (11.107) is infinite; H_{I} is ill defined.

11.7 Notes

The existence of many inequivalent representations of CCR was noticed in the 1950s, e.g. by Segal (1963) and Gårding–Wightman (1954). Shale's theorem was first proven in Shale (1962). Among early works describing implementations of symplectic transformations on Fock spaces let us mention the books by Friedrichs (1953) and by Berezin (1966). They give concrete formulas for the implementation of Bogoliubov transformations in bosonic Fock spaces. Related problems were discussed, often independently, by other researchers, such as Ruijsenaars (1976, 1978) and Segal (1959, 1963).

Infinite-dimensional analogs of the metaplectic representation seem to have been first noted by Lundberg (1976).

The book by Neretin (1996) and the review article by Varilly–Gracia-Bondia (1992) describe the infinite-dimensional metaplectic group.

The Fock sector of a CCR representation is discussed e.g. in Bratteli–Robinson (1996). It is, in particular, useful in the context of scattering theory; see Chap. 22 and Dereziński–Gérard (1999, 2000, 2004).

Coherent representations appeared already in the book by Friedrichs (1953), and were used by Roepstorff (1970). Our presentation follows Dereziński–Gérard (2004).

The ultraviolet problem of van Hove Hamiltonians is discussed e.g. in the books of Berezin (1966), Sect. III.7.4, and of Schweber (1962), following earlier papers by van Hove (1952), Edwards–Peierls (1954) and Tomonaga (1946). The name "van Hove model" is used in Schweber (1962).

The understanding of the infrared problem of van Hove Hamiltonians can be traced back to the papers by Bloch–Nordsieck (1937) and by Kibble (1968).

Our presentation of the theory of van Hove Hamiltonians follows Dereziński (2003).

https://doi.org/10.1017/9781009290876.012 Published online by Cambridge University Press