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BAIRE CATEGORY AND LAURENT EXTENSIONS

DANIEL R. FARKAS

Based on a strategy of Kaplansky ([**3**]), Dixmier proved that a prime, separable C^*-algebra is primitive ([**1**]). As a consequence, when the C^*-closure of a countable discrete group is prime, it is primitive. The argument may be regarded as a clever application of the Baire Category Theorem to the spectrum of irreducible representations.

The present note is the first step in adapting this technique to abstract group algebras. For which groups G is the primitive ideal space of $k[G]$ a Baire space? One corollary of our main result is that the space is Baire when k is an uncountable field and G is a polycyclic-by-finite group. This gives an alternate proof of a special case of Passman's theorem that such a $k[G]$ will be primitive when its center is k ([**4**], p. 379).

Our proofs differ in style from group algebra methods. The predominantly ring theoretic arguments were inspired by ([**2**]). Basically, we show that when R is a well-behaved ring whose primitive ideal space is Baire then a twisted Laurent extension $R_\sigma[x, x^{-1}]$ also has a primitive ideal space which is Baire.

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1. The space of primitive ideals. Suppose R is any associative ring (with 1). The set of all (left) primitive ideals of R comes equipped with the Jacobson topology. Here a collection of primitive ideals is closed if it is precisely the set of all primitive ideals lying over some ideal of R. We shall denote this space by Priv R. Of particular interest is whether Priv R is a Baire space: Is the countable intersection of dense open sets always dense?

We begin with an algebraic version of Kaplansky's observation ([**3**]). By a *separating set* for R we mean a countable set of nonzero elements of R with the property that every nonzero ideal of R meets this set.

THEOREM 1. *Suppose R is a prime, semiprimitive ring. If R has a separating set and* Priv R *is Baire then R is a primitive ring.*

Proof. Let Y be the separating set for R. If $y \in Y$ then

$$U_y = \{P \in \text{Priv } R \mid y \notin P\}$$

is clearly an open subset of Priv R. Since R has no radical, $U_y \neq \emptyset$.

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Because R is prime, and semiprimitive Priv R is an irreducible topological space. That is, every nonempty open set is dense. Hence $\bigcap_{y \in Y} U_y \neq \emptyset$. Choose a primitive ideal Q in the intersection. If $Q \neq 0$ then $Q \cap Y$ is nonempty. But then we can find $y \in Y$ with $y \in Q$, i.e. $Q \notin U_y$. We conclude that $Q = 0$.

In honor of this theorem we define R to be a *Kaplansky ring* provided the primitive ideal space of every homomorphic image of R is a Baire space.

Recall that a noetherian ring is a Jacobson ring when each prime image is semiprimitive.

LEMMA 2. *Suppose R is a Jacobson ring which is (one-sided) noetherian. Then R is Kaplansky if and only if* Priv R/P *is Baire for every prime ideal P in R.*

Proof. One direction is obvious, so assume Priv R/P is Baire whenever P is prime. If I is an arbitrary ideal of R, there are only finitely many primes $P_1, \ldots, P_m$ minimal over I. If we set $R_i = R/P_i$ then we can regard each Priv R_i as a closed subspace of Priv R/I with

$$\text{Priv } R/I = (\text{Priv } R_1) \cup \ldots . \cup (\text{Priv } R_m).$$

Remove any component which is redundant (i.e. in the union of the remaining subspaces).

Now suppose $\{U_s | \, s \in \mathscr{S}\}$ is a countable collection of dense open subsets of Priv R/I. Then $U_s \cap \text{Priv } R_i \neq \emptyset$ for each $s \in \mathscr{S}$ and each i. Otherwise, U_s is contained in the union of a proper subcollection of the Priv R_j, a closed set. Since U_s is dense, this implies that Priv R_i can be deleted from the union.

Therefore $U_s \cap \text{Priv } R_i$ is a nonempty relatively open subset of the irreducible space Priv R_i. Hence $(\bigcap_{s \in \mathscr{S}} U_s) \cap \text{Priv } R_i$ is dense in Priv R_i. The result follows.

Finally, we shall need some technical results. If σ is an automorphism of R then σ extends to a topological automorphism of Priv R. A subset $X \subseteq \text{Priv } R$ is *σ-invariant* when $\sigma(X) \subseteq X$ and *σ-stable* when $\sigma(X) = X$. A similar definition can be made for the ideals of R. We say that R is *σ-prime* provided the product of two nonzero σ-invariant ideals is nonzero.

LEMMA 3. *Suppose R is a σ-prime semiprimitive ring. If* Priv R *is Baire then the countable union of proper σ-stable closed subspaces remains proper.*

Proof. It suffices to show that if U is a nonempty σ-stable open subset of Priv R then U is dense. So suppose V is open and $U \cap V = \emptyset$. Then $W = \bigcup_{d \in \mathbf{Z}} \sigma^d(V)$ is still open and $U \cap W = \emptyset$. Let I be the intersection of all primitive ideals lying over the complement of U and J be the intersection of all primitives over the complement of W. Both ideals are σ-stable and $I \cap J = 0$. But $I \neq 0$ since $U \neq \emptyset$. Hence $J = 0$; $V = \emptyset$.

Notice that when σ is the identity automorphism, the converse of Lemma 3 is true. It is useful to remember the ideal-theoretic interpretation of this lemma.

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If $\{I_s | s \in \mathscr{S}\}$ is a countable collection of σ-stable ideals of R such that every primitive ideal lies over some I_s then there is a $t \in S$ with $I_t = 0$.

LEMMA 4. *Suppose R is a prime, semiprimitive noetherian ring. Assume that whenever $\{P_s | s \in \mathscr{S}\}$ is a countable collection of nonzero prime ideals of R there is a primitive ideal which does not lie over any P_t. Then* Priv R *is Baire.*

Proof. Apply the argument of Lemma 2 to the "converse" of Lemma 3.

2. Twisted Laurent extensions. For the remainder of this paper R will always denote a (one-sided) noetherian ring and σ will be an automorphism of R. When R is a k-algebra we will assume that σ fixes all elements of k.

By the twisted Laurent extensions $S = R_\sigma[x, x^{-1}]$ we mean the collection of all finite sums

$$\sum_{i \in \mathbf{Z}} r_i x^i, \; r_i \in R$$

with the obvious addition and

$$xrx^{-1} = \sigma(r) \text{ for all } r \in R.$$

The usual argument for the Hilbert Basis Theorem shows that S is noetherian.

The properties of S which we shall need can be found in [**2**]. Although that paper deals with Ore extensions (twisted polynomial rings) the results we summarize below carry over to Laurent extensions without difficulty.

If P is a prime ideal of S then $P \cap R$ is a σ-prime ideal of R. This, in turn, forces $P \cap R$ to be a semiprime ideal of R. In addition, $(P \cap R)S$ is a prime ideal of S. If P is nonzero and $P \cap R = 0$ then any ideal which properly contains P meets the regular elements (i.e. nonzero divisors) of R. Consequently such a P is minimal among the nonzero prime ideals of S.

For our purposes, the major theorem of [**2**] states that if R is a Jacobson ring then so is S.

We are now able to state the main theorem of this paper.

THEOREM 5. *Suppose R is a noetherian Jacobson algebra over the uncountable field k. If R is a Kaplansky ring then so is $S = R_\sigma[x, x^{-1}]$.*

The proof will be found in the next section.

One lemma of [**2**] requires a bit of tinkering before it can be used. If $\psi = \sum_{j=u}^{v} r_j x^j$ is a nonzero element of S with both r_u and r_v nonzero, define $\deg(\psi) = v - u$. Set $\deg(0) = -\infty$, for good measure. When I is a nonzero ideal of S it is easy to see that the elements of minimal degree in I have the same degree as the elements of minimal degree in $I \cap R_\sigma[x]$ with nonzero constant term. Call this degree, $\deg(I)$. We define $\tau^*(I)$ (respectively $\tau_*(I)$) to be 0 together with the leading (resp. constant) coefficients of the elements in $I \cap R_\sigma[x]$ with degree $\deg(I)$. Clearly $\tau^*(I)$ and $\tau_*(I)$ are σ-stable ideals of R. Notice that if $I \neq 0$ then $\tau^*(I)$ and $\tau_*(I)$ are nonzero.

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Set $\mu(I) = \tau^*(I) \cap \tau_*(I)$. If S is prime and $I \neq 0$ then σ-primality forces $\mu(I) \neq 0$.

LEMMA 6. *Suppose I is an ideal of S with* $\deg(I) > 0$. *Fix $c \in \mu(I)$. If $\varphi \in S$ there is a positive integer w and an $r \in R_\sigma[x]$ with* $\deg r < \deg I$ *such that*

$$c^w\varphi \equiv r \pmod{I}.$$

Proof. Write $\varphi = \sum_{j=u}^{v} \varphi_j x^j$ where $\varphi_j \in R$. We first assume that $u \geqq 0$ and argue by induction on v. Clearly we may assume $v \geqq \deg I$. If $c\varphi_v = 0$ then we are done. Otherwise, choose $p \in I \cap R_\sigma[x]$ of degree $\deg(I)$ with leading coefficient $c\varphi_v$. Apply induction to $c\varphi - px^{v-\deg I}$.

Now assume $u \leqq 0$ and argue by induction on $-u$. The case $-u = 0$ is covered by the first paragraph. So suppose $-u < 0$. If $c\varphi_u = 0$ then we are done. Otherwise, choose $q \in I \cap R_\sigma[x]$ of degree $\deg(I)$ with constant coefficient $c\varphi_u$. Apply induction to $c\varphi - qx^u$.

3. Kaplansky rings.

LEMMA 7. *Assume $S = R_\sigma[x, x^{-1}]$ is a prime ring and H is a nonzero prime ideal of S with $H \cap R = 0$. If* Priv R *is Baire then* Priv S/H *is Baire.*

Proof. Let $\mathscr{M}$ be the collection of all maximal left ideals M of R such that $SM + H = S$. Set $\mathscr{O} = \{\mathscr{P} \in \text{Priv } R \mid \mathscr{P} = \text{ann } R/M \text{ for some } M \in \mathscr{M}\}$. It is easy to check that $\mathscr{M}$ and $\mathscr{O}$ are σ-stable.

We first show that $\mu(H) \subseteq M$ for all $M \in \mathscr{M}$. Choose $a_i \in M$ such that

$$\sum x^i a_i \equiv 1 \pmod{H}.$$

Since $H \cap R = 0$ and $H \neq 0$, we have $\deg(H) > 0$; Lemma 6 applies. Given $c \in \mu(H)$ there is a positive integer w and polynomials $r_i(x)$ with degree smaller than $\deg(H)$ such that $c^w x^i \equiv r_i \pmod{H}$ for all i appearing in the original sum. Hence $\sum r_i(x)a_i \equiv c^w \pmod{H}$. Since M is a left ideal of R, we can rewrite this as

$$\sum_{0 \leqq j < \deg H} x^j b_j \equiv c^w \pmod{H}$$

where $b_j \in M$. Looking at degrees, we see that $\sum x^j b_j - c^w = 0$. In particular $c^w \in M$. We now apply a technique of [**2**]. If $c \notin M$ then $1 = tc + m$ for some $t \in R$ and $m \in M$. Repeat the argument with tc replacing c: then $(tc)^v \in M$ for some $v > 0$. But then

$$(tc)^{v-1} = (tc)^v + (tc)^{v-1}m$$

so we eventually conclude that $tc \in M$. This forces $1 \in M$, so we conclude that $c \in M$, as desired.

Now set $J = \bigcap_{\mathscr{P} \in \mathscr{O}} \mathscr{P}$. Then $\mu(H) \subseteq J$ while $\mu(H) \neq 0$ by the remark preceding Lemma 6. Thus J is a nonzero σ-stable ideal of R.

Suppose $\{\bar{P}_s \mid s \in \mathscr{S}\}$ is a countable collection of prime ideals of S/H over which lie the primitive ideals of S/H. (Here we let P_s denote the preimage of $\bar{P}_s$ in S.) By Lemma 4 we are done once we show that some $\bar{P}_t$ is zero.

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We claim that every primitive ideal of R either lies over J or over some $\bar{P}_s \cap R$. Obviously if $\mathscr{P} \in \mathscr{O}$, it lies over J. When $\mathscr{P} \notin \mathscr{O}$ there exists a maximal left ideal M of R and a maximal left ideal N of S such that $\mathscr{P} = \text{ann}\, R/M$ and $SM + H \subseteq N$. If Q is the annihilator of S/N then $Q \in \text{Priv}\, S$, it contains H, and $Q \cap R$ annihilates R/M. Hence $Q \cap R \subseteq \mathscr{P}$. But $Q \supseteq P_s$ for some $s \in \mathscr{S}$. Therefore $\mathscr{P}$ lies over $\bar{P}_s \cap R$.

Since $\bar{P}_s \cap R$ is σ-stable and $\text{Priv}\, R$ is Baire, we are done by Lemma 3.

Proof of Theorem 5. Suppose the theorem is false. By the noetherian condition we can find an ideal H maximal with respect to $\text{Priv}\, S/H$ not being a Baire space. According to Lemma 2, H is a prime ideal of S. Now

$$S/(H \cap R)S \simeq (R/H \cap R)_\sigma[x, x^{-1}]$$

and $(H \cap R)S$ is a prime ideal. Thus we can replace R by $R/H \cap R$ and assume that S is a prime ring and H is a prime ideal of S with $H \cap R = 0$.

If $H \neq 0$, then $\text{Priv}\, S/H$ is Baire by Lemma 7. So we are reduced to proving that $S = R_\sigma[x, x^{-1}]$ has a primitive ideal space which is Baire under the assumptions that S is prime and $\text{Priv}\, \bar{S}$ is Baire for all proper homomorphic images $\bar{S}$ of S.

We first handle the special case in which every nonzero prime ideal of S meets R nontrivally. Suppose $\mathscr{P} \in \text{Priv}\, R$ is the annihilator of R/M. It is easy to see that SM is a proper left ideal of S. Now one of the arguments in the previous lemma produces a $Q \in \text{Priv}\, S$ such that $Q \cap R \subseteq \mathscr{P}$. Consequently, if $\{P_s | \, s \in \mathscr{S}\}$ is a countable collection of prime ideals of S over which lie $\text{Priv}\, S$, $\{P_s \cap R | \, s \in \mathscr{S}\}$ is a countable collection of σ-stable ideals of R over which lie $\text{Priv}\, R$. By Lemma 3, $P_t \cap R = 0$ for some $t \in \mathscr{S}$. The hypothesis of this paragraph implies $P_t = 0$. By Lemma 4, $\text{Priv}\, S$ is Baire.

We now assume that S possesses a nonzero prime ideal P such that $P \cap R = 0$. Let U be 0 together with the polynomials in $P \cap R_\sigma[x]$ whose ordinary degree is $\deg(P)$. If $0 \neq \alpha \in k$ (with R an algebra over k) and

$$0 \neq q = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in U;\ n = \deg(P)$$

define

$$E^\alpha(q) = \alpha a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0.$$

Set $U_\alpha = E^\alpha(U)$. We leave it to the reader to verify that U_α is a σ-stable R-bimodule. Therefore $U_\alpha S = \sum U_\alpha x^i$ is a two-sided ideal of S. Because each nonzero element of U_α has the same degree, each member in $U_\alpha S$ can be written in a unique way as $\sum q(j)x^j$ where $q(j) \in U_\alpha$. One consequence is that $U_\alpha S \cap R = 0$. Since R is σ-prime, there must be a minimal prime Q_α over $U_\alpha S$ such that $Q_\alpha \cap R = 0$.

We claim that $Q_\alpha = Q_\beta$ implies $\alpha = \beta$. For when the ideals are equal, we have $E^\alpha(q) - E^\beta(q) \in Q_\alpha$ for any $0 \neq q \in U$. That is, $(\alpha - \beta)a_n x^n \in Q_\alpha$. But $a_n \notin Q_\alpha$ since $Q_\alpha \cap R = 0$.

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In summary, since k is uncountable, we have an uncountable collection Q_α of distinct nonzero prime ideals of S with $Q_\alpha \cap R = 0$. According to a result quoted immediately before the statement of Theorem 5 in the second section of this paper, if P is a prime ideal of S with $P \subsetneq Q_\alpha$ then $P = 0$.

Let $\{P_s | s \in S\}$ be a countable collection of nonzero prime ideals in S over which lie Priv S. Since $P_s \subseteq Q_\alpha$ implies $P_s = Q_\alpha$, cardinality considerations force the existence of nonzero $\alpha \in k$ such that $P_s \not\subseteq Q_\alpha$ for all $s \in \mathscr{S}$. Notice that every primitive ideal of S/Q_α lies over some $P_s + Q_\alpha/Q_\alpha$. On the other hand, Priv S/Q_α is a Baire space. Therefore $P_s + Q_\alpha = Q_\alpha$ for some s, a contradiction. We conclude that some primitive ideal of S does not lie over any P_t. By Lemma 4, Priv S is Baire.

The reader may justifiably ask why the main theorem was proved for twisted Laurent extensions rather than twisted polynomial extensions. All of the arguments generalize without great difficulty except for the special case handled at the beginning of the proof of Theorem 5. We do not know the structure of $R_\sigma[x]$ when every nonzero prime ideal of $R_\sigma[x]$ not containing x meets R nontrivally and yet there is a primitive ideal in R which does not lie over any $P \cap R$ with $0 \neq P \in$ Priv $(R_\sigma[x])$ and $x \notin P$. It would be pleasant if $R_\sigma[x]$ was itself a primitive ring in this instance.

4. Group algebras.

COROLLARY 8. *Suppose k is an uncountable field.*

(i) *If G is a polycyclic group then $k[G]$ is a Kaplansky ring.*

(ii) *If G is a polycyclic-by-finite group and $k[G]$ is a prime ring then* Priv $(k[G])$ *is a Baire space.*

Proof. (i) Observe that if B is a subgroup of G and x normalizes B, then $\langle B, x \rangle$ is the homomorphic image of some semidirect product of B by $\mathbf{Z}$. It is easy to see now how (i) follows by Theorem 5 and induction.

(ii) G contains a normal polycyclic subgroup H with G/H finite. Let $\mathscr{P} \in$ Priv $(k[H])$. Since maximal left ideals of $k[H]$ are always contained in maximal left ideals of $k[G]$, there exists a $P \in$ Priv $(k[G])$ such that $P \cap k[H] \subseteq \mathscr{P}$. The argument in the special case at the beginning of Theorem 5 establishes the corollary, since every nonzero ideal of $k[G]$ meets $k[H]$ nontrivally ([**4**], p. 359).

COROLLARY 9. ([**4**], p. 379) *Suppose k is an uncountable field and G is a polycyclic-by-finite group. If $\Delta(G) = 1$ then $k[G]$ is a primitive ring.*

Proof. Let $K \subseteq k$ be the prime field in k. A theorem in ([**4**], p. 355) states that the nonzero elements of $K[G]$ constitute a separating set for $k[G]$. Now apply Corollary 8 and Theorem 1.

REFERENCES

1. J. Dixmier, *Sur les C*-algèbres*, Bull. Soc. Math. France *88* (1960), 95–112.

https://doi.org/10.4153/CJM-1979-077-4 Published online by Cambridge University Press

2. A. Goldie and G. Michler, *Ore extensions and polycyclic group rings*, J. London Math. Soc. (2) *9* (1974/75), 337–345.

3. I. Kaplansky, *The structure of certain operator algebras*, T.A.M.S. *70* (1951), 219–255.

4. D. S. Passman, *The algebraic structure of group rings* (Wiley-Interscience, N.Y., 1977).

Virginia Polytechnic Institute and State University,
Blacksburg, Virginia

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