9

The weak interaction: low energy phenomenology

In this chapter we review some of the early phenomenology of the weak interaction that played an important guiding role in the construction of the Standard Model. The phenomenology discussed is insensitive to the very small effects of neutrino mass. These effects will be ignored.

9.1 Nuclear beta decay

In early investigations of nuclear physics, the existence of a 'weak interaction' responsible for nuclear β decay was discerned. It was regarded as weak since the mean lives of decays such as

$$
\begin{aligned}
&{}^{17}_{9}\mathrm{F} \to {}^{17}_{8}\mathrm{O} + \mathrm{e}^+ + \nu_\mathrm{e}, \\
&\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}_\mathrm{e},
\end{aligned}
$$

are very long, minutes in these examples, compared with typical nuclear electromagnetic decays, which have a mean life of $\sim 10^{-15}$s.

Nuclear physicists have by careful and ingenious experimentation established the principal features of the weak interaction and the properties of the electron neutrino ν_e. To conserve electric charge the neutrino must be electrically neutral, and angular momentum is conserved if it is a Dirac spin $\frac{1}{2}$ fermion. If the electron neutrino has a mass, it is certainly very small.

The surprising feature of the weak interaction, which was established experimentally in 1957 by Wu following a suggestion by Lee and Yang, is that it *does not conserve parity.* Nature is not ambidextrous. Indeed, parity is maximally violated, in that only the left-handed components of both the electron and neutrino fields participate in the interaction.

This phenomenon is clearly illustrated if one examines the longitudinal electron polarisation of electrons produced in 'allowed' β decays. An electron of negative helicity $-\frac{1}{2}$ and velocity v is in a left-handed state with probability

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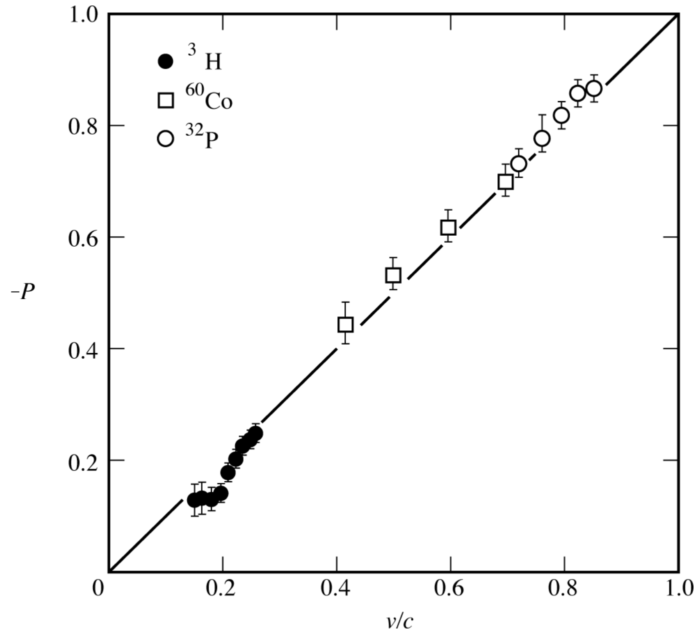

Figure 9.1 Measured degree of longitudinal polarisation P for allowed e^- decays. (Data from Koks and Van Klinken (1976).)

$\frac{1}{2}[1 + (v/c)]$; an electron of positive helicity $+\frac{1}{2}$ is in a left-handed state with probability $+\frac{1}{2}[1 - (v/c)]$ (Section 6.5). In allowed nuclear β decays there are no nuclear factors that favour one helicity state over another, so that if only the left-handed component of the electron field participates in the interaction, the degree of longitudinal polarisation of the emitted electron is

$$-\frac{1}{2}\left(1 + \frac{v}{c}\right) + \frac{1}{2}\left(1 - \frac{v}{c}\right) = -\frac{v}{c}.$$

For positrons, the probabilities are reversed (Section 6.5) and the longitudinal polarisation of a positron emitted in an allowed β decay is $+v/c$. Data from several such decays are shown in Fig. 9.1.

A direct measurement of the helicities of neutrinos emitted in β decay is almost impossible, but the helicities may be inferred from careful measurements of the angular momentum states of the participating nuclei. Within experimental error, only negative helicity neutrinos and positive helicity antineutrinos participate in the weak interaction.

Nuclear β decays do not release sufficient energy to produce either of the two other lepton families known to exist: muons and muon neutrinos, and tau leptons

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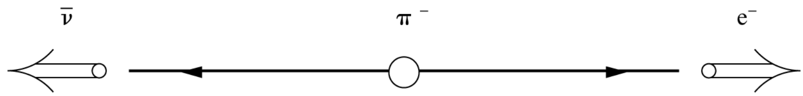

Figure 9.2 $\pi^- \to e^- + \bar{\nu}_e$. In this illustration the electron velocity is to the right, the antineutrino to the left, the spin directions are indicated Any orbital angular momentum is out of the plane of the page ($\mathbf{L} = \mathbf{r} \times \mathbf{p}$) and since the total angular momentum must be zero the spins have to be opposite.

and their partner neutrinos. We shall see in Chapter 13 that probably there are just these three, e, μ, τ, lepton families. Each family seems to play a similar role in Nature, an observation known as *lepton universality.* They differ only in the masses of the electrically charged leptons: $m_e \approx 0.511$ MeV, $m_\mu \approx 106$ MeV, $m_\tau = 1777$ MeV.

9.2 Pion decay

An important example that illustrates both the left-handedness of the lepton fields participating in β decay and lepton universality is provided by the decay of the charged pi mesons. These decays are common in the cosmic radiation and provide its principal component, muons, at ground level. Almost 100% of the pions decay through

$$\pi^- \to \mu^- + \bar{\nu}_\mu, \quad \pi^+ \to \mu^+ + \nu_\mu,$$

with a decay rate $1/\tau\ (\pi \to \mu\bar{\nu}_\mu) = 2.53 \times 10^{-14}$ MeV. The corresponding decays to electrons have much smaller decay rates: $1/\tau\ (\pi \to e\bar{\nu}_e) = 1.23 \times 10^{-4}(1/\tau\,(\pi \to \mu\bar{\nu}_\mu))$.

The decay rate to electrons is suppressed because only the left-handed fields of the electron and neutrino take part. Consider the π^- decay in a frame in which the pion is at rest (Fig. 9.2). The π^- has zero spin, the antineutrino has positive helicity. Hence to conserve angular momentum in this two-body decay the electron also must have positive helicity. The probability of its being in the left-handed state is $\frac{1}{2}[1 - (v_e/c)] = m_e^2/(m_\pi^2 + m_e^2) = 1.34 \times 10^{-5}$ (Problem 9.1). The μ^- decay is similarly inhibited, but the muon's much larger mass makes the factor less effective: $\frac{1}{2}[1 - (v_\mu/c)] = 0.36$.

An effective interaction Lagrangian density that incorporates these features is

$$\mathcal{L}_{int} = \alpha_\pi[j^\mu \partial_\mu \Phi_\pi + j^{\mu\dagger} \partial_\mu \Phi_\pi^\dagger], \tag{9.1}$$

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where

$$j^{\mu} = e_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\upsilon_{\mathrm{eL}} + \mu_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\upsilon_{\mu\mathrm{L}} + \tau_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\upsilon_{\tau\mathrm{L}}, \tag{9.2}$$

and α_{π} is an effective (real) coupling constant.

Φ_{π} is a complex scalar field describing the charged $\pi^{\pm}$ mesons (Section 7.6). Φ_{π} destroys negative pions, and creates positive pions. It is not a fundamental field of the Standard Model, since it ignores the internal structure of the pions. The four-vector $e_{\mathrm{L}}^{\dagger}\tilde{\sigma}^{\mu}\upsilon_{\mathrm{eL}}$ is the simplest Lorentz structure we can construct from the two left-handed spinor fields, e_{L}, υ_{eL}, belonging to the electron and its neutrino (see Problem 5.3). Lepton universality is then incorporated in the model, the three lepton families contributing in a similar way to the 'current' j^{μ}; this structure survives in the Standard Model. A Lorentz invariant $\mathcal{L}_{\mathrm{int}}$ is obtained by taking the scalar product of j^{μ} with $\partial_{\mu}\Phi$, and, finally, we make $\mathcal{L}_{\mathrm{int}}$ real. Note that $\mathcal{L}_{\mathrm{int}}$ is a 'point' interaction: j^{μ} and $\partial_{\mu}\Phi$ are evaluated at the same point x in space-time. Since the pion is an extended object, this point interaction must be an approximation, not to be taken too seriously.

An effective interaction Lagrangian is to be used only in low orders of perturbation theory. It is not suitable for calculating high order corrections. One should not therefore demand high accuracy when comparing the results of a calculation with experiment.

Using our $\mathcal{L}_{\mathrm{int}}$ to lowest order, the partial decay rates for pions at rest are (Problem 9.4)

$$\frac{1}{\tau(\pi \to \mathrm{e}\bar{\nu}_{\mathrm{e}})} = \frac{\alpha_{\pi}^{2}}{4\pi}\left(1 - \frac{\upsilon_{\mathrm{e}}}{c}\right) p_{\mathrm{e}}^{2}E_{\mathrm{e}}, \qquad \frac{1}{\tau(\pi \to \mu\bar{\nu}_{\mu})} = \frac{\alpha_{\pi}^{2}}{4\pi}\left(1 - \frac{\upsilon_{\mu}}{c}\right) p_{\mu}^{2}E_{\mu}. \tag{9.3}$$

In these equations, E_{e}, E_{μ} and p_{e}, p_{μ} are the charged lepton's energy and momentum, and are determined by energy and momentum conservation. The factors $p_{\mathrm{e}}^{2}E_{\mathrm{e}}$, $p_{\mu}^{2}E_{\mu}$ come from the density of states factor in the expression for the transition probability (Problem 9.2). The factors $(1 - \upsilon_{\mathrm{e}}/c)$ and $(1 - \upsilon_{\mu}/c)$ are a consequence of the participation of left-handed fields only.

The ratio

$$\frac{\tau\,(\pi \to \mu\bar{\nu}_{\mu})}{\tau\,(\pi \to e\bar{\nu}_{\mathrm{e}})} = \frac{m_{\mathrm{e}}^{2}(m_{\pi}^{2} - m_{\mathrm{e}}^{2})^{2}}{m_{\mu}^{2}(m_{\pi}^{2} - m_{\mu}^{2})^{2}} = 1.28 \times 10^{-4} \tag{9.4}$$

(Problem 9.3). This lowest order calculation, which neglects the effects of nonlocality and electromagnetic corrections, agrees well with the experimental value of 1.23×10^{-4}, and gives strong support for lepton universality.

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The observations give $1/\tau\,(\pi \to \mathrm{e}\bar{\nu}_{\mathrm{e}}) = 3.11 \times 10^{-18}$ MeV, $1/\tau\,(\pi \to \mu\bar{\nu}_{\mu}) = 2.53 \times 10^{-14}$ MeV, from which we may estimate

$$\alpha_\pi = 2.09 \times 10^{-9}\ \mathrm{MeV}^{-1}.$$

The smallness of α_π reflects the weakness of the weak interaction.

Although the pion does not have enough mass to decay to tau leptons, the effective Lagrangian (9.1) also described the decays

$$\tau^+ \to \pi^+ + \bar{\nu}_\tau, \quad \tau^- \to \pi^- + \nu_\tau,$$

and in lowest order of perturbation theory, predicts

$$\frac{1}{\tau\,(\tau \to \pi\nu_\tau)} = \frac{\alpha_\pi^2}{32\pi} m_\tau^3 [1 - (m_\pi/m_\tau)^2]^2. \tag{9.5}$$

Using the estimate of α_π from $\pi^\pm$ decay to calculate $1/\tau\,(\tau \to \pi\nu_\tau)$ provides a further test of lepton universality: the predicted value 2.42×10^{-10} MeV compares quite well with the experimental value, $(2.6 \pm 0.1) \times 10^{-10}$ MeV.

9.3 Conservation of lepton number

In the model Lagrangian discussed so far, a single lepton can change only to another of the same family, and a lepton and antilepton of the same family can only be created or destroyed together. There is thus a conservation law, *the conservation of lepton number* (antileptons being counted negatively), for each separate family, exemplified in the decays we have so far considered.

We saw in Section 7.1 that particle conservation follows from a $U(1)$ symmetry of the Lagrangian, and it is interesting to see how this is accomplished with our model Lagrangian. We have

$$\mathcal{L} = \mathcal{L}_{\mathrm{free}} + \mathcal{L}_{\mathrm{int}}$$

where, using Dirac spinors for the lepton fields,

$$\begin{aligned}
\mathcal{L}_{\mathrm{free}} = {} & \partial_\mu \Phi^\dagger \partial^\mu \Phi - m_\pi^2 \Phi^\dagger \Phi \\
& + \bar{\psi}_{\mathrm{e}}(\gamma^\mu \mathrm{i}\partial_\mu - m_{\mathrm{e}})\psi_{\mathrm{e}} + \bar{\nu}_{\mathrm{e}}\gamma^\mu \mathrm{i}\partial_\mu \nu_e \\
& + \bar{\psi}_{\mu}(\gamma^\mu \mathrm{i}\partial_\mu - m_{\mu})\psi_{\mu} + \bar{\nu}_{\mu}\gamma^\mu \mathrm{i}\partial_\mu \nu_\mu \\
& + \bar{\psi}_{\tau}(\gamma^\mu \mathrm{i}\partial_\mu - m_{\tau})\psi_{\tau} + \bar{\nu}_{\tau}\gamma^\mu \mathrm{i}\partial_\mu \nu_\tau, \\
\mathcal{L}_{\mathrm{int}} = {} & \alpha_\pi [j^\mu \partial_\mu \Phi_\pi + j^{\mu\dagger}\partial_\mu \Phi_\pi^\dagger],
\end{aligned}$$

and, in terms of Dirac spinors, the current j_μ of equation (9.2) can be written

$$j^\mu = \bar{\psi}_{\mathrm{e}}\gamma^\mu \frac{1}{2}(1-\gamma^5)\nu_{\mathrm{e}} + \bar{\psi}_{\mu}\gamma^\mu \frac{1}{2}(1-\gamma^5)\nu_{\mu} + \bar{\psi}_{\tau}\gamma^\mu \frac{1}{2}(1-\gamma^5)\nu_{\tau}. \tag{9.6}$$

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By itself, $\mathcal{L}_{\text{free}}$ has seven $U(1)$ symmetries: seven independent phases on the seven free fields. Including $\mathcal{L}_{\text{int}}$ reduces these to four, which can be written

$$\begin{aligned}
\psi_{\mathrm{e}} &\to e^{i\beta} e^{i\alpha_e} \Psi_{\mathrm{e}}, & \nu_{\mathrm{e}} &\to e^{i\alpha_e} \nu_{\mathrm{e}}; \\
\psi_{\mu} &\to e^{i\beta} e^{i\alpha_\mu} \Psi_{\mu}, & \nu_{\mu} &\to e^{i\alpha_\mu} \nu_{\mu}; \\
\psi_{\tau} &\to e^{i\beta} e^{i\alpha_\tau} \Psi_{\tau}, & \nu_{\tau} &\to e^{i\alpha_\tau} \nu_{\tau}; \\
\Phi_{\pi} &\to e^{i\beta} \Phi_{\pi}.
\end{aligned}$$

The phase factors $\alpha_e, \alpha_\mu, \alpha_\tau$ are associated with the conserved lepton currents (Problem 9.6). If we require $\mathcal{L}$ to be invariant under a *local* gauge symmetry, with $\beta = \beta(x)$ arbitrarily space and time dependent, we are led to the introduction of the electromagnetic field A^μ, as in Section 5.5. We shall see that not all these features of our effective Lagrangian survive the introduction of neutrino mass into the Standard Model.

9.4 Muon decay

The analysis of the muon decays

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu, \quad \mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu, \tag{9.7}$$

has played a very important role in establishing the Standard Model. The decays involve lepton fields only, so that the physics is not obscured by the phenomenology of strong interaction fields as was our example of pion decay.

An effective Lagrangian density that describes the decays again couples the participating particles into currents. In fact all decays seen so far that involve just leptons are well described by the effective interaction Lagrangian density

$$\mathcal{L}_{\text{lepton}} = -2\sqrt{2} G_F g_{\mu\nu} j^\mu j^{\nu\dagger}, \tag{9.8}$$

with j^μ again defined by (9.2) or (9.6). A similar form for nuclear β decay was introduced by Fermi, and G_{F} is called the Fermi constant. The $2\sqrt{2}$ is a related accident of history.

The term in (9.8) that describes μ^- decay is

$$\mathcal{L} = -2\sqrt{2} G_{\mathrm{F}} g_{\mu\nu} \left[e_{\mathrm{L}}^\dagger \tilde{\sigma}^\mu \nu_{\mathrm{eL}} \nu_{\mu\mathrm{L}}^\dagger \tilde{\sigma}^\nu \mu_{\mathrm{L}} \right]. \tag{9.9}$$

The most ready supply of muons comes from pion decays and these, as we have seen, are almost 100% polarised. The interaction Lagrangian density (9.9) implies a strong correlation between the angle θ made by the direction of the electron with the direction of the muon spin, and the energy E_{e} of the electron. In the muon rest frame, to lowest order of perturbation theory, and neglecting terms in $(m_{\mathrm{e}}/m_\mu)^2$, the decay rate into an angular interval $\mathrm{d}\theta$ and energy interval $\mathrm{d}E_{\mathrm{e}}$ is (see Donoghue

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et al. 1992, p. 138)

$$R(\theta, E_{\mathrm{e}})\,\mathrm{d}\theta\,\mathrm{d}E_{\mathrm{e}} = \frac{m_\mu G_{\mathrm{F}}^2}{6\pi^3}\left[\left(\frac{3}{4}m_\mu - E_{\mathrm{e}}\right) + \cos\theta\left(\frac{1}{4}m_\mu - E_{\mathrm{e}}\right)\right] E_e^2\,\mathrm{d}E_{\mathrm{e}}\sin\theta\mathrm{d}\theta. \tag{9.10}$$

Integrating (9.10) over θ and E_{e} gives the total decay rate for this process

$$\frac{1}{\tau(\mu \to \mathrm{e}\bar{\nu}_{\mathrm{e}}\nu_\mu)} = \frac{m_\mu^5 G_{\mathrm{F}}^2}{192\pi^3}. \tag{9.11}$$

The total muon decay rate, which includes also decays with photons in the final state, for example the decays

$$\mu^- \to \mathrm{e}^- + \gamma + \bar{\nu}_{\mathrm{e}} + \nu_\mu,$$

has been very accurately measured, giving

$$\tau_\mu = (2.19703 \pm 0.00004) \times 10^{-6}\ \mathrm{s}.$$

A corresponding accurate theoretical expression that corrects (9.11) by including terms in $(m_{\mathrm{e}}/m_\mu)^2$ and electromagnetic effects, gives

$$G_{\mathrm{F}} = 1.16639(2) \times 10^{-5}\ \mathrm{GeV}^{-2}, \tag{9.12}$$

which is the presently accepted value of this important constant.

Further tests of lepton universality are provided by the decays

$$\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau, \quad \tau^- \to \mathrm{e}^- + \bar{\nu}_{\mathrm{e}} + \nu_\tau,$$

and their charge conjugates. These, like muon decay, are described by appropriate terms in the interaction Lagrangian (9.8). Since both $(m_{\mathrm{e}}/m_\tau)^2$ and $(m_\mu/m_\tau)^2$ are small, the first-order formula (9.11) with m_μ replaced by m_τ predicts these decay rates to be equal and $\approx 4 \times 10^{-10}$ MeV. They are indeed so within experimental error. Also from this formula

$$\frac{\tau\,(\tau \to \mathrm{e}\bar{\nu}_{\mathrm{e}}\nu_\tau)}{\tau\,(\mu \to \mathrm{e}\bar{\nu}_{\mathrm{e}}\nu_\mu)} \approx \left(\frac{m_\mu}{m_\tau}\right)^5.$$

The ratio of the decay rates is 7.36×10^{-7} and the ratio of the fifth power of the masses is 7.43×10^{-7}.

It should be noted that the coupling constant G_{F} has the dimension of $(\mathrm{mass})^{-2}$. The effective interaction (9.8) cannot be elevated into a quantum field interaction; see Section 8.4.

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9.5 The interactions of muon neutrinos with electrons

In the 1960s, intense muon neutrino beams were engineered at Brookhaven and at CERN. Muon neutrinos (or antineutrinos) were produced as secondary particles from the decay of π^+ (or π^-) mesons in flight. It was from the observation that these neutrino beams produced almost exclusively muons rather than electrons, when in interaction with a target, that the distinction between electron neutrinos and muon neutrinos was established.

The centre of mass energy $\sqrt{s}$ available in a collision of a neutrino with an electron at rest is relatively small, because of the smallness of the electron mass. If E_ν is the neutrino energy,

$$s = m_e(2E_\nu + m_e), \tag{9.13}$$

(Problem 9.8). For example, if $E_\nu = 30$ GeV then $s = (175\ \text{MeV})^2$, which will produce no more than a muon. Most neutrino interactions will be with the atomic nuclei in the target. However, here we consider only the interactions with electrons.

The interaction

$$\nu_\mu + e^- \to \mu^- + \nu_e$$

is included in the effective interaction Lagrangian density (9.8). In first-order perturbation theory and averaging over electron polarisations, this Lagrangian predicts an isotropic differential cross-section in the centre of mass system:

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{4\pi^2}\frac{\left(s - m_\mu^2\right)^2}{s}, \qquad \sigma_{\text{tot}} = \frac{G_F^2}{\pi}\frac{\left(s - m_\mu^2\right)^2}{s} \tag{9.14}$$

with s the square of the centre of mass energy. (See Okun 1982, p. 134.)

At the low energies available experimentally, the cross-section appears to be consistent with the theoretical form. The high energy structure is not easily explored experimentally, because of (9.13), but clearly the theoretical formulae become inadequate at high energies: the expressions (9.14) increase without limit as s increases, and for a 'point' interaction this is inconsistent with unitarity. Nor is it possible to improve the expressions within this framework, since the effective Lagrangian does not give a renormalisable theory.

The most significant result to come from the experiments on neutrino–electron interactions was the observation of elastic scattering for both ν_μ and $\bar{\nu}_\mu$:

$$\nu_\mu + e^- \to \nu_\mu + e^-,$$
$$\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-,$$

with cross-sections of a magnitude similar to those for muon production. Such elastic scattering is *not* included in our $\mathcal{L}_{\text{int}}$ (though there are terms corresponding to

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$e\nu_e \to e\nu_e$ and $e\bar{\nu}_e \to e\bar{\nu}_e$). Thus another weak interaction must exist. The experimental investigation of this is difficult because of the smallness of the cross-sections at the available energies. We shall see from the Standard Model that the effective interaction Lagrangian required is again of current–current form,

$$\mathcal{L}_{\text{int}} = \frac{-G_F}{\sqrt{2}}(j_{\text{neutral}})_\mu (j_{\text{neutral}})^\mu, \tag{9.15}$$

where, in terms of Dirac spinors,

$$\begin{aligned}(j_{\text{neutral}})^\mu = \bar{\nu}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)\nu_e + \bar{\psi}_e \gamma^\mu (c_V - c_A \gamma^5)\psi_e \\ + \text{similar terms for the } \mu \text{ and } \tau \text{ lepton families,}\end{aligned} \tag{9.16}$$

and c_V and c_A are parameters. The current is called a *neutral current* because it does not induce a change of charge as do the currents (9.2). (Note that it will also contribute to the scattering $e\nu_e \to e\nu_e$.)

Rewriting (9.16) with two-component spinors,

$$\begin{aligned}(j_{\text{neutral}})^\mu = (\nu_{eL})^\dagger \tilde{\sigma}^\mu \nu_{eL} + (c_V + c_A) e_L^\dagger \tilde{\sigma}^\mu e_L \\ + (c_V + c_A) e_R^\dagger \sigma^\mu e_R + \text{similar } \mu \text{ and } \tau \text{ terms.}\end{aligned} \tag{9.17}$$

In this form it is evident that right-handed lepton fields as well as left-handed are involved in the neutral currents. The parameters c_V and c_A are related to the Weinberg angle θ_w, which appears in the Standard Model, as we shall see in Chapter 12 (equation (12.24)). The subscripts V and A refer, respectively, to the vector and axial vector nature of the terms in (9.16). (See Section 5.5.)

One might anticipate that neutral currents are also present in atomic physics, and indeed they are. However, their effects are hard to discern experimentally. For example, they induce parity violation in atoms, but at atomic energies the weak interaction gives a very small effect. Indeed the decay of an unstable nuclear or atomic system through the neutral current must always compete with faster electromagnetic decays, and for this reason neutral current decays in these systems have never been observed.

Problems

9.1 In the decay of the π^- at rest, $\pi^- \to e^- + \bar{\nu}_e$, show that

$$\frac{1}{2}\left(1 - \frac{\upsilon_e}{c}\right) = \frac{m_e^2}{m_\pi^2 + m_e^2}.$$

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9.2 Show that the density of final states for the decay of Problem 9.1 is

$$\rho(E) = \frac{V}{(2\pi)^3} 4\pi p_{\mathrm{e}}^2 \frac{\mathrm{d}p_{\mathrm{e}}}{\mathrm{d}E}$$

where V is the normalisation volume and

$$\frac{\mathrm{d}p_{\mathrm{e}}}{\mathrm{d}E} = \frac{E_{\mathrm{e}}}{m_\pi}.$$

9.3 Obtain the ratio of decay rates given by equation (9.4).

9.4 The term in $\mathcal{L}_{\mathrm{int}}$ describing the decay $\pi^- \to \mathrm{e}^- + \bar{\nu}_{\mathrm{e}}$ is

$$\mathcal{L}_{\mathrm{int}} = \alpha_\pi e_{\mathrm{L}}^\dagger \tilde{\sigma}^\mu \nu_{\mathrm{eL}} \partial_\mu \Phi_\pi .$$

Assume that this gives a corresponding term $V(0)$ in the effective Hamiltonian,

$$V(0) = -\alpha_\pi \int e_{\mathrm{L}}^\dagger \tilde{\sigma}^\mu \nu_{\mathrm{eL}} \partial_\mu \Phi_\pi d^3\mathbf{x}.$$

(This assumption will be justified in Chapter 12.)

The transition probability per unit time for the decay is to lowest order

$$2\pi |\langle \mathrm{e}_{\mathrm{p}}, \bar{\nu}_{\mathrm{p}'} | V(0) | \pi^-(\mathrm{rest})\rangle|^2 \rho(E)$$

where $\rho(E)$ is given by Problem 9.2.

Use the free field expansions given in equations (3.35) and (6.24), and Problem 6.5, to evaluate the matrix element above and hence verify equation (9.3).

9.5 Verify the equivalence of the expressions (9.2) and (9.6) for the current j^μ.

9.6 Taking the effective Lagrangian of Section 9.3, show that the conserved current associated with the $U(1)$ symmetry $\psi_{\mathrm{e}} \to \mathrm{e}^{\mathrm{i}\alpha}\psi_{\mathrm{e}}$, $\nu_e \to \mathrm{e}^{\mathrm{i}\alpha}\nu_{\mathrm{e}}$, is the electron electron–neutrino current

$$j^\mu = \bar{\psi}_{\mathrm{e}} \gamma^\mu \psi_{\mathrm{e}} + \bar{\nu}_{\mathrm{e}} \gamma^\mu \nu_{\mathrm{e}}.$$

Show that the conserved current associated with $\mathrm{e}^{\mathrm{i}\beta}$ in the transformations (9.7) is

$$\bar{\psi}_{\mathrm{e}}\gamma^\mu\psi_{\mathrm{e}} + \bar{\psi}_\mu\gamma^\mu\psi_\mu + \bar{\psi}_\tau\gamma^\mu\psi_\tau + \mathrm{i}[(\Phi^\dagger\partial^\mu\Phi - \Phi\partial^\mu\Phi^\dagger) + \alpha_\pi(j^{\mu\dagger}\Phi^\dagger - j^\mu\Phi)].$$

Construct the Lagrangian density that results, when the electromagnetic field is introduced by elevating the global $U(1)$ symmetry of the phase factor $\mathrm{e}^{\mathrm{i}\beta}$ into a local gauge symmetry.

9.7 Estimate G_{F} from the expression (9.11) and the experimental lifetime τ_μ.

9.8 Using a suitable Lorentz invariant, obtain equation (9.13).

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9.9 Pick out the term in the effective Lagrangian density (9.8) that contributes to the scattering

$$\mathrm{e}^- + \nu_e \to \mathrm{e}^- + \nu_e,$$

and the term in (9.15) that contributes to the scattering

$$\mathrm{e}^- + \nu_\mu \to \mathrm{e}^- + \nu_\mu.$$

9.10 The K^- is like the π^-, but with an s quark replacing the d. An effective interaction with leptons is similar in form to equation (9.1), with Φ_{K} replacing Φ_π and α_{K} replacing α_π. Use the analogue of equation (9.4) to estimate the ratio $\tau\,(\mathrm{K} \to \mu\bar{\nu}_\mu)/\tau\,(\mathrm{K} \to \mathrm{e}\bar{\nu}_\mathrm{e})$, and compare with the observed value $(2.44 \pm 0.1) \times 10^{-5}$ ($m_{\mathrm{K}} = 493.68\,\mathrm{MeV}$).

The mean life $\tau\,(\mathrm{K}^- \to \mu^-\bar{\nu}_\mu)$ is measured to be 1.948×10^{-8} s. Estimate $\alpha_{\mathrm{K}}/\alpha_\pi$.

9.11 Obtain the decay rate (9.5).

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