CONVECTIVE DYNAMOS

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1. INTRODUCTION

Convective dynamo theory can be regarded as combining two kinds of physical problems, each involving an electrically conducting fluid medium, but differing in the role of the magnetic field and in the physical processes described. On the one hand, if the fluid is taken to be permeated by a *prescribed* magnetic field $\vec{B}$, under suitable conditions, involving a sufficiently strong flux of heat for example, convective motion of the fluid will ensue. On the other hand, *kinematic* dynamo theory insures that a sufficiently complicated fluid motion $\vec{u}$ can sustain or excite a magnetic field. In a convective dynamo the origin of the magnetic field is internal and we must regard the applied and excited fields as one and the same (Figure 1). In the present paper we shall outline some of the current work on such systems. The research has been motivated primarily by the search for tractable models of planetary and solar magnetism, and the focus in this paper will be on models of the geodynamo. For simplicity we restrict attention to Boussinesq fluids and emphasize asymptotic solvable problems rather than a realistic description of the Earth's core. We shall, however, require that the dynamo be essentially convective, in that no auxiliary driving forces are needed. (The convective process could of course involve any advected, diffusing substance which changes the weight of a fluid element.)

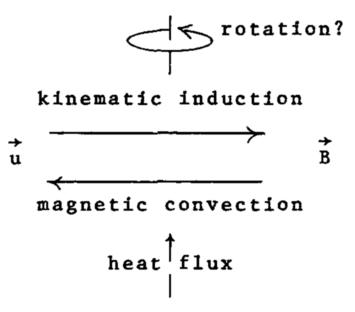

Figure 1. The Convective Dynamo Cycle.

The simplest physical system admitting a convective dynamo cycle is not obvious (to this author), although in the case of the geodynamo it would appear that large-scale rotation of the fluid is sufficient

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if not essential. In the models discussed below it is precisely the combination of bouyancy and Coriolis forces which create the necessary flow structure, so in this respect at least they may be relevant to the processes at work in the Earth's core.

2. BOUNDS AND ESTIMATES

One way to define dynamo action is simply to require that the mean magnetic field of the system, obtained by appropriate integrals over space and time, be positive. With this definition we can, in a sense, "prove" dynamo action by showing that the convective system with $\vec{B} = 0$ is unstable to magnetic fields; that is, small "seed" fields are always amplified. This property can be tested rather easily since the two parts of the system depicted in Figure 1 decouple when the magnetic field is weak. Now it is an essential feature of our problem that the mean magnetic energy of the ultimate state(s) of the dynamo is an internal property of the system, although we may assume that such a mean energy may be defined and that it will depend upon the various parameters, the geometry, etc. This being the case, it is of interest to determine, without studying the evolution of the system in detail, an *a priori* upper bound on mean magnetic energy.

This intriguing question was apparently first studied only recently by Kennett (1974) in the case of Bénard convection between free, perfectly conducting plane isothermal boundaries rotating about a vertical axis (cf. Section 5 below). We will use the following notation: μ = magnetic permeability, ρ = density, ν = kinematic viscosity, κ = thermal diffusivity, η = magnetic diffusivity, α = coefficient of thermal expansion, all of the above being taken to be constant, $P = \nu/\kappa$ = Prandtl number, $P_\eta = \eta/\kappa$, R = Rayleigh number (based on a temperature gradient γ) $= \alpha\gamma g L^4/\kappa\nu$, M = Hartmann number $= BL/(\mu\eta\rho\nu)^{1/2}$, Ta = Taylor number $= 4\Omega^2 L^4/\nu^2$, where Ω is the angular speed of the system. If E_B denotes the time and volume mean of B^2, Kennett's result may be written

$$E_B \leq 4R^{3/2}B_0^2/9\pi^2 P_\eta^2, \qquad B_0 = (\mu\eta\rho\nu)^{1/2}/L \ . \tag{1}$$

This estimate is obtained by equating the mean dissipation to the mean work done by the gravitational forces, and involves extensions of the familiar power integrals of the Bénard problem.

Although as a general rule analysis of this kind rather severely overestimates energies, (1) is interesting as an indication of the influence of the various material properties. The parameter

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P_η is evidently significant in determining the magnetic energy realized by convection at a given Rayleigh number. Incidentally, P_η has a value of about 10^6 in the earth's core, but may be as small as 10^{-5} in stars because of radiative cooling, so that ideally we would like a dynamo model to retain P_η as an arbitrary parameter. The fact that the bound (1) diverges as $\eta \to 0$ probably reflects the infinite amplification that can be achieved in a perfect conductor by the twisting and stretching of field lines. But note that the bound also diverges like $\nu^{-1/2}$ in the limit of small viscosity. While such a divergence might be expected for convection between isothermal boundaries in the limit of zero Prandtl number, it seems unlikely for systems driven by a fixed rate of heating; in this case E_B should be bounded independently of the viscosity.

Such a result is in fact implied by the interesting thermodynamic arguments of Malkus (1973), and Hewitt, McKenzie, and Weiss (1975). Following these authors we consider a spherical region of (current) conducting fluid surrounded by a rigid non-conductor. Let the boundary $r = L$ be held at a fixed temperature T_0 and the interior be heated uniformly at the rate q_0. We seek a bound for the magnetic energy in terms of the material constants, L, and q_0. Let E_B, q_j, q_v, and W now denote the time averages of global B^2, joule dissipation, viscous dissipation, and bouyancy work, respectively, all normalized by the volume V of the sphere. Assuming internal energy is bounded in time the first law requires

$$W = q_j + q_v , \quad q_j \geq 0, \quad q_v \geq 0 . \tag{2}$$

On the other hand, a well-known property of currents in a homogeneous spherical conductor is (see e.g. Backus 1958)

$$q_j \geq \eta\pi^2 E_B/\mu L^2 . \tag{3}$$

Then from (2) and (3)

$$E_B \leq \mu L^2 W/\eta\pi^2 . \tag{4}$$

To estimate W we use the temperature equation in the Boussinesq limit,

$$\frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T - \kappa\nabla^2 T = q_0/\rho c_p . \tag{5}$$

Let, with $\vec{g}\cdot\vec{u} = u_r r g_0/L$,

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$$W = (4\pi\alpha\rho g_0/LV) \int_0^L r^3 \, w(r) \, dr \qquad (6)$$

so that w is the time average of the spherical mean of $u_r T$. If θ denotes the same operation on T, (5) yields upon integration the flux balance

$$r^3 w - \kappa r^3 \frac{d\theta}{dr} = r^4 q_0/3\rho c_p \ . \qquad (7)$$

We may measure θ in $^\circ$K. Integrating (7) from $r = 0$ to $r = L$ and using $\theta \geq 0$ we have

$$\int_0^L r^3 w \, dr \leq L^5 q_0/15\rho c_p + \kappa L^3 T_0 \ . \qquad (8)$$

Combining (4), (6) and (8) there results

$$\eta E_B/\mu L^2 q_0 \leq (\beta/5\pi^2)(1 + 15\kappa\rho c_p T_0/L^2 q_0) \ . \qquad (9)$$

Here $\beta = \alpha g_0 L/c_p$ is the ratio of L to the temperature scale height and is necessarily a small number in the Boussinesq approximation. (The inclusion of the dissipation terms on the right of (5) changes (9) by terms $O(\beta^2)$.) But the left-hand side of (9) should be independent of the origin of the temperature scale and, indeed, it can be shown that for uniform heat addition (9) holds with $T_o = 0$. We therefore have

$$\eta E_B/\mu L^2 q_0 \leq \beta/5\pi^2 \ . \qquad (10)$$

This provides us with a useful (and small) measure of the efficiency of a convective dynamo. Other estimates of this kind are contained in the references cited above.

If we introduce the Rayleigh number

$$R_q = \alpha g_0 q_0 L^5/\rho c_p \kappa^2 \nu \ ,$$

then (10) may be rewritten

$$E_B \leq R_q B_0^2/5\pi^2 P_\eta^2 \qquad (11)$$

and thus has the form of (1) reduced by a factor $9/(20\ R^{1/2})$, the bound now being independent of the viscosity.

In the above we have dropped viscous dissipation because of the

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inequality (3), but if this term is now retained one obtains

$$L^4 q_v/\rho\nu\eta^2 + \pi^2 E_B/B_0^2 \leq R_q/5P_\eta^2 \tag{11*}$$

in place of (11). If, relative to a rotating frame, the no-slip condition is satisfied on the boundary, q_v can be bounded from below by a multiple of $\rho\nu L^{-2}E_u$, in which case the terms on the left of the inequality would be comparable provided the ratio of kinetic to magnetic energy is roughly P_η/P. It is known (Childress 1969a) that L^2E_u/η^2 must exceed a fixed positive bound for a dynamo effect to be possible in a given domain of fluid. Consequently in the $E_B^{1/2} - E_u^{1/2}$ plane a convective dynamo must lie within the first quadrant of an ellipse, and to the right of a vertical line determined by the dynamo condition. In reality, of course, the radius of the ellipse should be altered to express the existence of a critical Rayleigh number, and it would be of interest to extend (11*) to account for this shift, perhaps by applying the method used by Kennett (1974).

Since the above arguments completely ignore the dynamical process by which the dynamo effect is realized, (11) tells us little about the behavior of any given system. Suppose, however, that some dynamics allows the bound (11) to be obtained, and take $1 \sim P_\eta > 1$. (Throughout we use the symbol $\sim$ as follows: $a \sim b$ if $a = O(b)$ and $b = O(a)$.) Then (11) implies $M^2 \sim R_q$, which is reminiscent of the relation $R_c \sim M^2$ $(M >> 1)$ obtained for the *critical* Rayleigh number for convection between isothermal planes dominated by the magnetic field (Chandrasekhar 1961). That is, the "optimal state" of the convective dynamo is close to marginal in the context of linear stability theory. If the process by which the optimal state were reached involved rapid rotation (Ta >> 1), the analogous stability results of Eltayeb and Roberts (1970) and Eltayeb (1972) show that if $P > .67659$, then $R_c(M,Ta)$ is minimized when $M^2 \sim Ta^{1/2}$, $R_c \sim Ta^{1/2}$, which is again compatible with (11) when $P_\eta > 1$. (And note that M^4/Ta is also independent of viscosity.) The Eltayeb-Roberts ordering may be loosely interpreted to imply that magnetic energy with Hartmann number $T_a^{1/2}$ would be acquired by a rapidly rotating body once the Rayleigh number was raised to a value $\sim T_a^{1/2}$. But again the argument assumes the necessary dynamo action by the convection. In particular there is no implied critical rotation rate.

If both fluid inertia and viscous stresses can be neglected (as seems to be the case in the Earth's core outside Ekman layers, cf. Roberts and Soward 1972), the dimensionless parameters of the heated convective system may be reduced to a "Rayleigh number"

$$\tilde{R} = R_q/P_\eta^2 Ta^{1/2} \tag{12}$$

together with P_η, by the choice of η/L, $(2\Omega\mu\rho\eta)^{1/2}$, and $q_0 L^2/\rho c_p \eta$ as units of speed, magnetic field strength, and temperature respectively. The dimensionless equations are then

$$\nabla p + \Omega^{-1}\vec{\Omega}\times\vec{u} + \vec{B}\times(\nabla\times\vec{B}) = -\tilde{R}g_0^{-1}\vec{g}T, \qquad \nabla\cdot\vec{u} = 0, \tag{13}$$

$$\frac{\partial\vec{B}}{\partial t} - \nabla^2\vec{B} = \nabla\times(\vec{u}\times\vec{B}), \qquad \nabla\cdot\vec{B} = 0 , \tag{14}$$

$$\frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T - P_\eta^{-1}\nabla^2 T = 1 . \tag{15}$$

For given P_η, $\tilde{R}$ we can seek solutions of (13)-(15) which are compatible with boundary conditions, and from these determine the one with maximum E_B (now dimensionless). By (11) this value cannot exceed $\tilde{R}/5\pi^2$. Such an operating state, where the mean magnetic energy is as large as possible for a given heating rate, may be taken as "optimal", since it is presumably stable locally and can only lead to smaller energy under a finite perturbation. Of course it is not clear that the system admits *any* nontrivial solutions ($\vec{B} \neq 0$). Note that if such a solution existed for some $\tilde{R}$, and if it were known that rotation was essential for a dynamo effect, then it would be necessary that solutions terminate for sufficiently small *and* sufficiently large values of $\tilde{R}$.

The existing theory of convective dynamos has concentrated on cases which are probably far from optimal in the above sense. Viscous effects are freely admitted, parameters and geometry chosen to allow the convective modes to be determined by considerations at marginal stability, and various devices are used to simplify the analysis of electromagnetic induction. In the following sections we study various aspects of these highly idealized models, but return to some of the questions raised above at the end of the paper.

3. KINEMATIC INDUCTION

This aspect of the problem has a large recent literature (see e.g. the reviews of Roberts 1971, Weiss 1971, and Gubbins 1974). One of the more direct evaluations of the regenerative effect is possible if the fields are taken to be periodic in space and time. (The spatially-periodic case was treated by Childress 1967, 1969b, 1970; the theory in its most general setting was developed by G. O. Roberts 1969, 1970, 1972.) This situation arises naturally in planar or almost-planar models involving simple boundary conditions. The analysis is facilitated (and can be made explicit) if the two dimensionless numbers

$$r_\omega = \omega/\eta k^2, \qquad r_k = U/\eta k, \tag{16}$$

where U, k , and ω are the speed, wavenumber, and frequency characteristic of the velocity field, satisfy

$$r_k = o(1), \qquad r_\omega = O(1). \tag{17}$$

That is, the magnetic Reynolds number of a fluid eddy must be small, and the time scale of the motion should be of the order of the decay time of a magnetic field structure of the same size. With (17) it becomes rather easy to demonstrate self-excitation of a magnetic field which is slowly-varying relative to the scales k, ω. (It is unlikely that the first of (17) is satisfied in the Earth's core, but the basic inductive mechanism, which goes back to the pioneering paper of Parker (1955), can in fact be deduced without such a restriction (G. O. Roberts 1970).)

We return to the dimensional induction equations, which are

$$\frac{\partial \vec{B}}{\partial t} - \eta\nabla^2\vec{B} = \nabla\times(\vec{u}\times\vec{B}) , \tag{18}$$

$$\nabla\cdot\vec{B} = 0 . \tag{19}$$

Consider the solenoidal velocity field

$$\vec{u}(\sigma) = U(0, \sin\sigma, \sin(\sigma+\phi)), \qquad \sigma = kx + \omega t, \tag{20}$$

and suppose that $\vec{B}$ has the decomposition

$$\vec{B} = \vec{f} + \vec{g}, \qquad \vec{g} = o(1), \qquad \vec{f} \text{ slowly-varying.} \tag{21}$$

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Using (20) and (21) in (18), (19), one sees that the part $\vec{g}$ will approximately satisfy

$$\frac{\partial \vec{g}}{\partial t} - \eta\nabla^2\vec{g} = \vec{f}\cdot\nabla\vec{u} \;, \tag{22}$$

so that

$$\vec{g} \simeq \frac{f_1 k}{\eta^2 k^4 + \omega^2} \left(\eta k^2 \frac{d\vec{u}}{d\sigma} + \omega\vec{u}\right) \;. \tag{23}$$

The slowly-varying component will then satisfy

$$\frac{\partial \vec{f}}{\partial t} - \eta\nabla^2\vec{f} = \nabla\times(\overline{\vec{u}\times\vec{g}}) \simeq \nabla\times(A\cdot\vec{f}), \qquad \nabla\cdot\vec{f} = 0, \tag{24}$$

where the overbar denotes the σ-average and A is a constant pseudo-tensor. For (20) the only non-zero component of A is

$$A_{11} = -\left(\eta k^3 U^2(\sin\phi)/(\eta^2 k^4+\omega^2)\right) \equiv \alpha \;. \tag{25}$$

This additional contribution to mean electromotive force is usually referred to as the "α-effect". The most general α-effect, involving arbitrary symmetric A, can be created by suitably combining linearly independent modes of the form (20). It is easily seen that, by examining the case of diagonal A, that (24) can be made to admit exponentially-growing spatially periodic solutions (and note that (17) insures that they will be slowly-varying).

Let us look more closely at the underlying inductive mechanism when $\phi = \pi/2$. From (22) it is clear that the source of small-scale magnetic structure is proportional to the x-derivative of $\vec{u}$, i.e. to the *shear* of the flow. Now trigonometric spatial modes of the diffusion equation decay without change of shape, but there is a phase shift between the solution and *moving* sources. Combining this shift with that introduced by differentiation, we see that $\vec{g}$ is proportional to $\vec{u}(\sigma + \psi)$ where $\eta k^2/\omega = \tan\psi$. As σ varies, $\vec{u}$ and $\vec{g}$ rotate in the yz plane while maintaining this phase difference, so the induced current, obtained as a cross product, is independent of σ and proportional to $\sin\psi$.

For a given mode (20) the corresponding entry in A is maximized when $\phi = \psi = \pi/2$, i.e. the motion is both quasi-steady and Beltrami (vorticity and velocity everywhere parallel). This maximizes the mean *helicity* (Moffatt 1968), defined as the volume average of $\vec{u}\cdot\nabla\times\vec{u}$, for a given mean kinetic energy. Note that the mean helicity is opposite in sign to α for these elementary Beltrami modes.

To summarize, time-independent velocity modes having the property that the velocity is orthogonal to the wavenumber vector and the two orthogonal components are 90° out of phase, provides a basic element of a particularly efficient kinematic dynamo process, characterized by a constant mean helicity. A variety of other, less efficient dynamo mechanisms (involving an A which either vanishes or has rank 1, see e.g. case IV in G. O. Roberts 1972) can be studied by a refinement of these procedures, but in the present context it is rather a slightly different point of view which is needed, since the relevant convective modes cannot be regarded as exclusively small-scale. We accordingly consider next the dynamo mechanisms which are compatible with the dynamics of convection.

4. DYNAMICS

The efficient kinematic dynamos considered above are very special in that the helicity of the flow may be averaged over space and time to obtain a non-zero pseudo-scalar H. It appears to be difficult, however, to find physical systems which will exhibit this property. For example, owing to dissipative processes we expect a rotating sphere of heated fluid to settle down so that H can be defined independently of the initial conditions. Now restart the system but with initial conditions $T(-\vec{r},0)$, $-\vec{u}(-\vec{r},0)$. If the magnetic field is zero the Boussinesq equations are invariant under this reflection (recall $g = g_0 r$), so the system will evolve toward a mean helicity $-H = H$, implying $H = 0$. To argue this in a different way, the mean state of the system should depend only upon the mean heating rate and various material parameters (all scalars) and the pseudo-vector $\vec{\Omega}$, from which it is impossible to construct a pseudo-scalar. If the system is endowed with a magnetic field having mean dipole moment $\vec{m}$ (a vector), H could be expressed as an odd function of $\vec{m}\cdot\vec{\Omega}$, but the record of magnetic reversals suggests that for the Earth there is no preferred polarity and therefore $\vec{m} = 0$!

If, nevertheless, rotation is to be regarded as essential to the convective dynamo, its action must be not to create mean helicity, but rather to "polarize" helicity in space (or time) in such a way that the resulting pattern of induced currents can be self-excited. This self-excitation is difficult to visualize and compute when the length and time scales are unique (as in the system (13)-(15)) since the induced currents resulting from the polarization are bound up closely with the "eddy" currents which dissipate the field. For the purpose of analysis of the effect it is therefore fortunate that sufficiently rapid rotation of the fluid introduces two spatial scales into the marginal stability problem, at least for sufficiently weak magnetic fields (Chandrasekhar 1961), so that large-scale induction and small-scale dissipation can be clearly distinguished.

To take a concrete example of a zero mean helicity dynamo, consider the solenoidal field

$$\vec{u} = (-\tfrac{a}{k}\sin kx \cos az,\ \sin kx \cos az,\ \cos kx \sin az). \quad (26)$$

If $k \gg a$ the induction problem can be solved approximately as in Section 3, with similar results except that now there is an additional factor $\frac{1}{2}\sin 2z$ in A. But note that (26) can also be written

$$\vec{u} = (0,\tfrac{1}{2},\tfrac{1}{2})\sin(kx+az) + (0,\tfrac{1}{2},-\tfrac{1}{2})\sin(kx-az) + O(\tfrac{a}{k}) \quad , \quad (27)$$

that is, as a sum of two O(1) rapidly-varying fields, each having components *in phase*, so that each fails as an "efficient" dynamo of the kind considered above. An analogous calculation can be carried out for the standing wave

$$\vec{u} = (0,\ \sin kx \cos at,\ \cos kx \sin at),\ a \ll \eta k^2 \quad (28)$$

to obtain helicity varying as $\sin 2at$, and (28) can be expressed as a sum of two progressive waves, moving in opposite directions with phase speed a/k.

More generally, let

$$\vec{u} = \vec{\mu}\sin\sigma + \vec{\mu}'\sin\sigma', \qquad \sigma = \vec{k}\cdot\vec{r} + \omega t\ , \quad (29)$$

and assume $|k - k'| \ll k$, $|\omega - \omega'| \ll \eta k^2$. One then finds, using the notation of Section 3,

$$A = \frac{\eta k^2 \sin(\sigma-\sigma')}{\eta^2 k^4 + \omega^2}\,(\vec{\mu}'\times\vec{\mu})\circ\vec{k} \quad (30)$$

The operation $k \to k'$ can be thought of as a reflection across a plane normal to $\vec{k} - \vec{k}'$, and if $\omega = \omega'$ the dispersion relation for the modes must be invariant under this reflection; in addition, from (30) it is clear that the two corresponding amplitudes must not be parallel. The special case $\vec{k} = \vec{k}'$, $\omega' = -\omega$, which could arise in a conservative system, gives the time-periodic induction.

These properties pertain to infinite fields having the proper structure. In *contained* rotating fluids helicity can also be polariized as a result of "Ekman pumping" into a quasi-geostrophic flow. If the latter has wavenumber k, the secondary flow set up by the Ekman layer is of magnitude $\sim Ta^{-1/4}kL$, L being the length scale for the container, and the resulting helicity may or may not be comparable to that introduced by other processes, depending upon the magnitudes of k and Ta. Under certain conditions it can be demonstrated that the Ekman layers are essential to a convective dynamo effect (see Roberts and Soward 1972), but in certain idealized models (Section 5) they definitely are not.

The polarity of the resulting helicity is fixed by the direction of rotation and is easily computed. At a point on the boundary with outer normal $\vec{n}$, the nearby helicity has the sign of $-\vec{\Omega}\cdot\vec{n}$. For the rapidly rotating Benard layer (case I below), the polarity is the same as that introduced globally by the convection mode for free boundaries, but the distribution is different and (as just noted) it is smaller, by a factor $Ta^{-1/12}$.

We turn now to the situation in rotating convection. We have in mind, of course, convection in a heated sphere or spherical annulus, but to get a qualitative picture it is helpful to consider several planar "approximations" to parts of a spherical annulus, which we indicate in Figure 2.

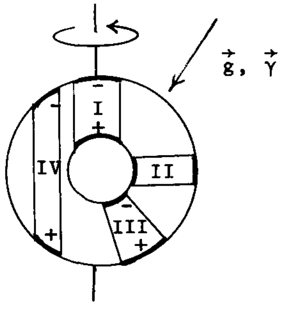

Figure 2. Planar "approximations". The heavy lines indicate isothermal boundaries to be represented by tangent planes. In IV the included angle is made small. The polarization of helicity is indicated for rotation in the direction shown at the top. In a homogeneous sphere, region I can be regarded as extending from top to bottom.

We consider these regions in turn. Letting $\vec{g}$ and $\vec{\gamma}$ be parallel and constant, and choosing units of length, time, and speed to be L, L^2/ν and κ/L, we find the dimensionless linearized equations to be

$$\frac{\partial \vec{u}}{\partial t} + \nabla p + Ta^{1/2}\, \vec{i}_\Omega \times \vec{u} - \nabla^2 \vec{u} = -R\vec{i}_g T', \qquad \nabla\cdot\vec{u} = 0 \tag{31}$$

$$P\frac{\partial T'}{\partial t} + \vec{u}\cdot\vec{i}_g - \nabla^2 T' = 0 \tag{32}$$

where $\vec{i}_g$ and $\vec{i}_\Omega$ are unit vectors, T' is the temperature perturbation, and we take $Ta >> 1$.

Case I. This classical problem is treated by Chandrasekhar (1961). In the limit $Ta \to \infty$, provided P exceeds about 0.68, convection ensues as a small-scale pattern. The critical wavenumber vectors (L is now the layer thickness) are almost perpendicular to $\vec{i}_\Omega$, the motion is quasi-geostrophic, and under reflection across the plane of the layer the vertical velocity component changes sign. There is accordingly polarization of the helicity along the axis of rotation, in a manner similar to that obtained for the motion (26). The critical parameters are

$$R_c \simeq 3(\pi^2/2)^{2/3} Ta^{2/3}, \qquad k_c \simeq (\pi^2 Ta/2)^{1/6}, \tag{33}$$

Using the term "roll" to denote the convection field corresponding to a given wavenumber vector $\vec{k}$ in the plane of the layer, and setting $\vec{i}_\Omega = (0,0,1)$, a single roll has, in the case of free boundaries, the form

$$\vec{u} = \pi \sin \pi z \cos \vec{k}\cdot\vec{r}\ \vec{i}_3 + (\pi^2 Ta^{1/2}/k^4)\ \vec{i}_3\times\vec{k} \cos \pi z \sin \vec{k}\cdot\vec{r} \tag{34}$$

Each such roll contributes an entry in the upper left 2 × 2 submatrix of A, which is a negative multiple of $\sin 2\pi z$. The corresponding helicity has the polarity shown in Figure 2. For rigid boundaries the results are similar over the interior, the secondary Ekman flow being smaller by a factor $k_c Ta^{-1/4} \sim Ta^{-1/12}$ according to (33).

From the point of view of dynamo action, the essential feature of this case is the high degeneracy of the geostrophic flow, allowing rolls of arbitrary direction. The fact that A can be made to have rank 2 implies a relatively efficient dynamo process based exclusively on small scale motions (see Section 5). This realizes in a simple planar geometry the so-called "α^2" kinematic dynamo (Roberts 1971). On the other hand region I can hardly be regarded as typical of the sphere as a whole, especially since, strictly speaking, the geostrophic contours have this degeneracy only at the poles!

Case II. Here gravity and $\vec{i}_\Omega = (0,0,1)$ are orthogonal, and the choice of geostrophic velocity $\vec{u} = \nabla\times\psi(x,y)\vec{i}_3$ reduces the problem to classical Bénard convection without rotation. There is no dynamo effect from these rolls since particle paths lie in planes. (Indeed from (18) it follows that B_3 decays, and the resulting two-dimensional non-dynamo can be regarded as a special instance of Cowling's theorem, cf. Roberts 1967.)

However, these special solutions completely neglect the presence of "sidewalls" which might represent the effect of the sloping spherical boundary. As Busse (1970) has emphasized, the sidewall constraint drastically upsets the geostrophic balance and the near-equatorial region is best approached through case IV below.

Case III. We let $\vec{i}_g = (0,0,-1)$, $\vec{i}_\Omega = (\sin\psi, 0, \cos\psi)$. For steady convection the Rayleigh number is given by

$$R = [k^6 + Ta(k_1 \sin\psi + k_3 \cos\psi)^2]/(k_1^2+k_2^2)$$

where $\vec{k}$ is now an arbitrary vector. To make this expression an even function of k_3 and therefore obtain modes of the kind needed to satisfy conditions at the plane boundaries, it is seen that k_1 must vanish, in which case the problem reduces to case I above but with Ta replaced by $Ta\cos^2\psi$. The effect of the obliqueness is therefore to reduce the critical Rayleigh number somewhat, and to restrict the locally horizontal wavenumber vector to be nearly perpendicular to the plane of $\vec{\Omega}$ and $\vec{g}$. Thus the α^2 dynamo of Case I is reduced to an incomplete or near incomplete α-effect, strongly biased toward inducing $\vec{i}_2$ current from $\vec{i}_2$ field. In kinematic dynamo theory this induction is nevertheless essential to the success of the "$\alpha\omega$" dynamo (Roberts 1971), as originally envisaged by Parker (1955). In the $\alpha\omega$ mechanism the α-effect is supplemented by large-scale shear, which here replenishes the $\vec{i}_2$ field. Thus case III, which might be taken to be typical of a large fraction of spherical annulus, suggests a natural mechanism for obtaining "one-half" of the dynamo effect from convection.

Case IV. Here one seeks to represent the effect of the sloping sidewalls. A class of such models was studied by Busse (1970) and subsequently used in a convective cycle (Busse 1975, see Section 6 below). The model has the advantage of describing rather closely, in a simple geometry, the essential physics of the convective instability in a rapidly rotating, heated homogeneous sphere (Busse 1970).

The sidewalls upset geostrophy through Ekman pumping as well as by their inclination to the axis of rotation, but the latter effect can be made to dominate. In this case Busse's results can be obtained rather easily by inverting a device of the oceanographer and replacing a slow decrease of the depth by a slow increase of Ω. Let $\vec{i}_g = (-1,0,0)$, $\vec{i}_\Omega = (0,0,1)$, and replace $Ta^{1/2}$ by $Ta^{1/2}(1 + \lambda x)$ in (31). Letting

$$\vec{u} = \nabla \times \psi(x,y,t)\vec{i}_3 , \qquad p = -(1 + \lambda x)Ta^{1/2}\,\psi + p'$$

and eliminating p' by cross-differentiation we obtain

$$\left(\frac{\partial}{\partial t} - \nabla_2^2\right)\nabla_2^2\psi = \frac{\partial}{\partial y}\left(\lambda Ta^{1/2}\psi + RT'\right) , \qquad (35)$$

$$P\frac{\partial T'}{\partial t} - \nabla_2^2 T' = \frac{\partial \psi}{\partial y} , \qquad \nabla_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} . \qquad (36)$$

If we look for modes proportional to $\exp(i\omega t + iax + iby)$ we have

$$(Pi\omega + c^2)(i\omega c^2 + c^4 + ia\lambda Ta^{1/2}) = Ra^2 , \qquad (37)$$

where $c^2 = a^2 + b^2$. Thus

$$\omega = -(a\lambda Ta^{1/2})/(P + 1)c^2 , \qquad a^2R = c^6 + (P/(P+1))^2a^2\lambda Ta/c^2$$

and the conditions for marginal stability become

$$a_c = 2^{-1/6}\left(\lambda P/(1+P)\right)^{1/3}Ta^{1/6}, \quad b_c = 0, \quad R_c = 3(\lambda P/(1+P))^{4/3}(Ta/2)^{2/3}. \qquad (38)$$

If λ is regarded as small, the perturbation is on the strict geostrophic equilibrium of case II, but it is a singular perturbation with strong roll selection. If λ is regarded as ~ 1, the ordering (38) is in accord with that of cases I and III, although the mode in the present case has some features of a Rossby wave. In Busse's formulation λ is a typical sidewall slope and should be taken as positive, so the phase speed given by (38) is "eastward". (If one regards the observed "westward drift" of the non-dipole geomagnetic field as a phase speed of a flow field associated with the α-effect, this result is disappointing. However, as Busse notes the *group* velocity *is* westward, and this raises the question of whether sufficiently supercritical convection in this system might not take the form of intermittent wave packets.)

As we have noted the model can be interpreted as an appropriate local section of a sphere, parametrized by the latitude of the intersection with the boundary and, if this is done, reasonable agreement is obtained with the stability analysis of a rapidly rotating heated sphere (Roberts 1968, Busse 1970). Roberts found that convection begins on a cylindrical annulus oriented parallel to the axis of rotation, with radius about half that of the sphere, and consists of vertical rolls of dimension $\sim Ta^{-1/6}$ along the aximuth, the radial thickness of the convection zone being $\sim Ta^{-1/9}$, both relative to the radius of the sphere. It would appear that in the planar model the radial structure is lost, or rather reflected only in the vanishing of b_c in (38). Certainly the spherical case should have a local approximation with radial structure, which could be incorporated into a dynamo model. Soward's remarks at this meeting concerning his current calculations of the localized α-effect point toward such a possibility, which would in effect open the way to an asymptotic analysis of the spherical dynamo.

5. BÉNARD-TYPE MODELS

We consider in this section two examples of a convective dynamo cycle based upon a classical Bénard layer. Busse (1973) considers Bénard convection without rotation. It is assumed that only one set of rolls is present, so helicity is created by adding a shear flow along the axis of the rolls. Such a flow is somewhat artificial for a Bénard layer, but it can in principle be driven by a modification of the mean temperature profile. Moreover, such distortions might well arise in a different geometry through convective heat transport oblique to the direction of gravity. To effect a scale separation it is assumed that the (spatial) mean magnetic field is dominated by a component orthogonal to the roll axis and slowly-varying along it.

In this model the mean helicity vanishes, since the unidirectional shear flow passes down rolls of alternating sign. The kinematic dynamo effect is therefore not the first order α-effect of Section 3, but rather a higher-order mechanism involving the spatial derivatives of the mean magnetic field. (In the terminology of Roberts 1971 the dynamo is of "βω" type.) Busse uses a numerical method to study the equilibration of the system and the partitioning of internal energy. While his approach, being essentially quasi-steady, does not deal with the dynamics of equilibration, it does reveal an interesting balance, similar to that suggested by (11*), which is perhaps typical of near-critical convective dynamos. Namely, if $R - R_c > 0$ is sufficiently small, one finds that $aE_u + bE_B$ = constant, where a and be are positive constants, but that E_u must exceed a critical value E_u^* in order

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to have a dynamo effect. The conclusion is that for $E_u > E_u^*$ the field energy will increase as the kinetic energy falls, while for $E_u < E_u^*$ the kinetic energy will rise as the field decays. Ultimately the magnetic field will be sustained at the maximum energy compatible with the above dynamic constraint as well as stationary dynamo action.

The second model, put forward by Childress and Soward (1973) and worked out by Soward (1974) for equilibrium at minimum field energy, is based upon the rotating Bénard layer considered as case I in Section 4. This model utilizes the rotation of the fluid to effect the scale separation, in a manner which permits the dynamics of equilibration to be followed in detail. We take $P_\eta \sim 1$.

The "weak-field" case studied by Soward (1974) assumes that the Hartmann number of the induced field is ~ 1. This *a priori* hypothesis on field intensity is then justified by exhibiting consistent, apparently stable, operating states. The rolls have the form (34) with $\vec{k}$ an arbitrary vector in the plane of the layer. The length L is here the thickness of the layer, so that for regeneration of the field we must have (cf. (25) with $\omega = 0$)

$$R_m^2 \sim k_c \sim Ta^{1/6} ,$$

where $Rm = UL/\eta$ is a magnetic Reynolds number based on roll amplitude U. Thus $Rm \sim Ta^{1/12}$ and the small-scale field satisfies $g \sim Ta^{-1/12} f$. This ordering generates a series solution in powers of $Ta^{-1/12}$.

The mean field equations are easily obtained for a discrete or continuous distribution of rolls, and in the former case, with $\vec{f} = \left(B_1(z,t), B_2(z,t), 0\right)$, take the form

$$\frac{\partial B_i}{\partial t} + 2\pi\Lambda \frac{\partial}{\partial z} [\sin 2\pi z \; M_{ij}(t) B_j] - \frac{\partial^2 B_i}{\partial z^2} = 0 \tag{39}$$

where, in terms of the A in (24),

$$M = \begin{bmatrix} -A_{21} & -A_{22} \\ A_{11} & A_{12} \end{bmatrix} \tag{40}$$

Here the unit of time is L^2/η. Note that, since we have in mind that roll structure is to be determined by auxiliary equations of evolution, the discreteness or continuity of the roll pattern is determined by the initial conditions. In (39) and (40) the normalization is such that $A_{11} + A_{22} = 2$ so that $\Lambda(t)$ is a parameter proportional to the kinetic energy of the flow.

For the weak-field solutions, it turns out that near marginal instability Λ is fixed by the quantity $(R - R_c)/R_c$. Thus the dynamics of the model reduces to the study of how the magnetic field determines the partitioning of a fixed constant kinetic energy among the various rolls. Soward was able to show that the roll structure indeed evolves on the same time scale as the field. If $q(t,\vec{K})$ denotes the kinetic energy in a given roll, where $\vec{K} = \vec{k}/Ta^{1/6}$, the equation for q takes the form

$$\frac{dq}{dt} = 2 \sum_{\vec{K}'} Q(\vec{K},\vec{K}') \, q(t,\vec{K}) \, q(t,\vec{K}') - [\beta(t,\vec{K}) - \frac{1}{2} \sum_{\vec{K}'} \beta(t,\vec{K}') \, q(t,\vec{K}')] \, q(t,\vec{K}) = 0 , \tag{41}$$

where $Q(\vec{K},\vec{K}')$ and $\beta(t,\vec{K})$ are given explicitly, the former being skew in its arguments. The magnetic field is contained in the quantity β, which takes the form

$$\beta = f(\vec{K}) + g(\vec{K}) \sum_{i,j} K_i K_j \, \mathscr{L}(B_i B_j) , \tag{42}$$

$$\mathscr{L}(f) = \int_0^1 (\pi^2 \cos^2 \pi z - K^2 \sin^2 \pi z) f \, dz .$$

Finally, the matrix A in (40) is obtained from the q's by

$$A_{ij} = \sum_{\vec{K}} \frac{K_i K_j}{K^2} \, q(t,\vec{K}) \tag{43}$$

where we may set $K = K_c$.

The weak-field model is then given by (39)-(43) with Λ as parameter. Soward examines 2 and 3-roll solutions of this system, as well as an interesting continuous-roll solution, and finds a tendency for the kinetic energy to localize itself at any one time in rolls near a single direction, but the direction itself changes with time. In fact, as the number of admitted discrete rolls is increased, the solutions tend increasingly to resemble a single rotating roll, a property then explicitly exhibited in the continuous roll example, where energy is dispersed about an orientation which rotates with uniform speed. Physically, the magnetic field, at each instant, favors rolls with a certain orientation (determined by the term β in (41)). From (39) one sees that the α-effect then feeds energy into the component of the field *parallel* to the preferred roll axis. If the field were independent of z, the analysis of Eltayeb (1972) would apply and it could be concluded that rolls with axis *orthogonal* to the field are most unstable. Thus both field and roll axis rotate, as the α-effect keeps

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up with the destabilizing effect of the field. This argument would imply that rotation is opposite to the large-scale rotation of the layer (cf. M in (39) when A is diagonal), a property that was always obtained in Soward's calculations.

The continuous-roll solution consists of at least two branches when E_B is plotted as a function of Λ, with subcritical bifurcation occurring from $E_B = 0$, $\Lambda = 1.5974$, and it seems likely from the 2 and 3-roll calculations that some of these are stable on the time scale of the model. (Soward establishes dynamic stability on the relevant short time scale.) On the other hand, it is by no means obvious that these solutions are stable to finite increases of initial field energy, and indeed the Eltayeb-Roberts ordering mentioned in Section 2 would suggest that they are not.

Preliminary attempts by the author to test the stability of the dynamo by starting it with magnetic energy corresponding to a Hartmann number $\sim Ta^{1/12}$ (the intermediate-field regime) have in fact uncovered several kinds of instability, and recent unpublished calculations of Yves Fautrell in the strong-field regime $Ta^{1/6} < O(M) < Ta^{1/4}$ also indicate instability under certain conditions. It is not definitely known at this time whether or not there are regimes other than that of weak field where local stability is obtained. It is possible that stability is regained only at the "very strong field" level $M \sim Ta^{1/4}$, but there the multiple-scale procedure is ineffective since $k_c \sim 1$.

At the intermediate level one finds, first, that dynamic stability on the fast time scale, shorter than L^2/η by a factor $Ta^{-1/12}$, is upset. Examination of some two-roll solutions show that this instability represents collapse onto a single roll (without the dispersion of energy about a preferred direction which characterized the weak-field continuous-roll solution). Single-roll solutions are dynamically stable on the fast time scale at the intermediate level. Single-roll solutions are found to be unstable, however, on the time scale L^2/η ! To see how this happens we write out the equation for q at the intermediate-field level:

$$Ta^{-1/6} \frac{dq}{dt} = q[c_1 + c_2\Theta(t) + c_3\mathcal{L}(\vec{B}\cdot\vec{K})^2)] , \qquad (44)$$

where the c's are positive constants. Here $\Theta Ta^{-1/6}$ is an amplitude of the perturbation of the mean temperature, satisfying

$$\frac{d\Theta}{dt} + c_4\Theta = - c_5\Lambda , \qquad (45)$$

where again the c's are positive constants. In particular, at the intermediate level Λ must be retained as a function of time even at fixed Rayleigh number. Let us assume that the system (39), (44), (45) is started "most stably" by adjusting the roll angle and Θ to make the right-hand side of (44) take on its maximum value, and that this value is zero. Assuming that this configuration (stable on short time scales) is maintained, we may neglect the left-hand side of (44), solve for Θ, and use this expression in (45) to obtain an expression for Λ in terms of $\vec{B}$, which expression can then be used in (39) to obtain an equation for $\vec{B}$.(This procedure also fixes the roll angle as a function of $\vec{B}$.) Numerical studies clearly show that in general this system allows solutions which diverge in a finite time, owing to the quadratic dependence of Λ upon $\vec{B}$, and quite apart from the dynamo or non-dynamo property of a single roll. In effect it appears that the field has destabilized the convection to the extent that divergent behavior of the field is caused by the rapid increase of the kinetic energy and heat flux, rather than by dynamo action.

One reason for these difficulties may lie in the degeneracy allowing multiple-roll solutions. Indeed, if a single roll direction could be fixed by other considerations (as in Cases III and IV in Section 4), the component of the field which is amplified by dynamo action is orthogonal to $\vec{K}$, and so does not enter into (44). So long as more than one roll direction is permitted, however, the quadratic growth of Λ with $\vec{B}$ would probably have the instability of the rotating roll.

It can also be asked whether the instability is not an indirect result of the isothermal boundary condition, which more realistically should be replaced by a condition of constant mean heat *flux*. Some tentative work at the intermediate level did not indicate a stabilizing effect, but the question remains undecided at the higher field levels.

6. THE ANNULUS MODEL

As an outgrowth of his quasi-planar analysis of convection in case IV above, Busse (1975) has considered a corresponding convective dynamo model. His approach, as in the stability analysis, is to consider a simultaneous expansion for large Ta and small λD, where λ is a typical boundary slope and $D >> 1$ is the width to height ratio to the annulus. The geometry is summarized in Figure 3.

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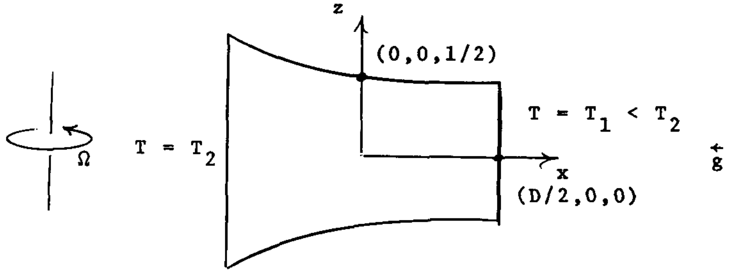

Figure 3. Busse's annulus model; lengths are in units of the mean height. The almost horizontal boundaries are given by $2z = \pm \exp(-\lambda x)$.

The top boundaries are treated as rigid interfaces between conductor and non-conductor, and on the vertical ones the temperature perturbation as well as the x-component of velocity vanish.

The latter conditions are particularly important, since they enforce a fixed roll structure on the convection, effectively removing the degeneracy of case I. Indeed, from the analysis of Section 4, we see that a_0 and R_c again have the asymptotic expressions (38), but now $b_c = \pi/D$, so the most unstable velocity mode has the form

$$\vec{u} = a_c A Ta^{1/2}(\sin(\pi x/D + \pi/2) \sin a_c y, \quad (\pi/a_c D) \cos(\pi x/D + \pi/2) \cos a_c y, 0). \tag{46}$$

Of course, once R exceeds R_c a band of wavenumbers with continuously varying b will grow, but rather than introduce a Fourier transform we follow Busse and take (46), which can be expressed as a sum of two rolls with wavenumber vectors $a_c \pm \pi/D$, as typical of the convective mode.

The helicity corresponding to (46) vanishes identically, so the kinematic dynamo action rests on the combined effects of Ekman pumping and sidewall slope. For suitable parameter values the former effect can be made to predominate, and an α-effect is achieved, but one which is strongly biased, the induced current associated with a y-component of the mean field being $(a_c/b_c)^2 \equiv 1/\varepsilon^2$ times that associated with a comparable x-component. The steady-state mean field is thus found to satisfy an equation obtainable from (39) by replacing $\sin 2\pi z$ by z, and M by

$$m\begin{pmatrix} \varepsilon & 0 \\ 0 & 1/\varepsilon \end{pmatrix} \tag{47}$$

where

$$m = [(A^2 a_c^5 Ta^{3/4})/8\sqrt{2}\ (\omega_c^2 + a_c^4 P_\eta^2/P^2](\pi/D). \tag{48}$$

These expressions can be derived using the familiar results for the flow induced by the Ekman layer, together with (25). Since $\omega_c/a_c^2 \sim P^{-1}$ we see from (47) and (48) that the quantity

$$\Gamma = P^2 A^2 a_c Ta^{3/4}/D(1+P_\eta^2) \tag{49}$$

must exceed a positive number of order unity if we are to have a dynamo effect.

A second condition is imposed by the two-scale expansion. The small-scale magnetic field can be estimated from (22) as follows: The dimensionless velocity amplitude is $a_c A\ Ta^{1/2}$, and since the x-component of the field predominates (see below), the relevant shear is 1/D times this amplitude. Thus

$$g/f \sim a_c A\ Ta^{1/2} D^{-1}\ (\omega_c^2 + P_\eta^2\ /P^2 a_c^4)^{-1/2} << 1$$

is a condition on the expansion. Combining (50) with the condition on Γ and using (38) we have

$$Ta^{1/4}\lambda D >> 1\ , \tag{50}$$

and this inequality is easily met by the assumed ordering. On the basis of his solution of the kinematic dynamo problem Busse concludes that if

$$a_c b_c/P_\eta Ta^{1/4} >> 1 \tag{51}$$

(the inequality following equation (5.3) of Busse 1975), the expansion is consistent. This adds a much stronger condition, which can only be met, with a_c given by (38) and $b_c \sim 1/D$, by making P_η *small*. Since it is important to retain P_η as a large parameter in a geodynamo model, it will be of interest to know if (51) can be relaxed while maintaining a consistent expansion, or if (50) by itself might be sufficient.

The equilibration of the system is studied using an equilibrium calculation as for the non-rotating layer model of Section 5, with similar results: For slightly supercritical convection the magnetic and kinetic energies are linearly related and the system equilibrates as the dynamo effect becomes stationary.

The model has a number of advantages over those of Bénard type, the foremost being that it is constructed to represent the convection within a region of a homogeneous rotating sphere. Roll structure is independently fixed by the boundary conditions, rather than evolving in response to the mean magnetic field. The α-effect is of a new type, induced by the response of the domain to a Rossby-like wave.

As Busse notes, the rather stringent conditions on the parameters can probably be considerably relaxed without affecting the qualitative features of the model. Moreover, the behavior of the system as a weak α^2-type dynamo is probably of secondary importance compared to the insight it gives into the possible origin of the α-effect in a sphere. In this connection it should be mentioned that the effect of the boundary (absent in the realization of the case IV considered in Section 4) enters into α as $O(\lambda^2)$, and thus is a reflection of boundary *curvature*. This boundary contribution is independent of viscosity and can be made to predominate over that due to Ekman pumping, although Busse does not investigate the full dynamo cycle in that case.

On the other hand, certain features of the solution, imposed by its asymptotic form, should be noted. First, the α-effect is such that $B_1 \sim \varepsilon^{-2} B_2$, and since B_1 here represents the "meridional" component of the field, the dynamo is characterized by a small "toroidal" component. As Busse notes, the implication is, if one accepts the model when $\varepsilon \sim 1$, that the two components are comparable, but it is disturbing that this state is approached through an asymptotic ordering that is usually regarded as improbable in the Earth's core. A second point concerns the possibility of subcritical instabilities. We have seen in the case of the rotating Bénard model that the locally stable weak-field case may not be stable to finite-amplitude perturbations in the magnetic field, and the question arises as to whether or not a similar state of affairs prevails here. If one examines the stability in case IV with an applied *uniform* magnetic field of the form $B(\vec{i} + \varepsilon^2\vec{j})$, it can be seen that (37) is replaced by

$$(iP\omega + c^2)(i\omega c^2 + c^4 + M^2b^2 + ia\lambda Ta^{1/2}) = Ra^2 \qquad (52)$$

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where M is the Hartmann number based on B. From (52) it is easily seen that convection at Rayleigh numbers $\sim \lambda Ta^{1/2}$ can be realized provided that $a \sim b \sim 1$ and that $M^2 \sim \lambda Ta^{1/2}$. This is fully analogous to the Eltayeb-Roberts ordering mentioned earlier, and we suggest that there may be similar implications for the present model at higher field energies.

7. MODAL EXPANSIONS

Numerical calculations utilizing truncated expansions in fundamental modes have played a prominent role in the kinematic dynamo theory (we mention in particular the pioneering paper of Bullard and Gellman (1954) and the recent study of Gubbins (1973)) as well as in the simulation of thermal convection (Gough, Spiegel, and Toomre 1975). It is natural to consider the application of these methods to the convective dynamo.

One immediate difficulty is the choice of appropriate "fundamental modes", capable of representing the system at a rather low level of truncation. The asymptotic models of the kind discussed above, which have something of a "modal" character near the critical Rayleigh number, can be helpful here. The practical problem is, of course, that if the asymptotic solution were to represent a globally stable state, its finite-amplitude modal counterpart offers a modest and perhaps unnecessary extension. On the other hand, the value of the modal approach lies in *simulation* of the dynamo, and there the structure of the asymptotic solutions may be misleading. To take a specific example, in rapidly rotating non-magnetic Bénard convection the roll structure is given by (34). As we have seen, however, the appropriate horizontal scale of the convection may increase dramatically once a magnetic field is developed and $M^2 \sim Ta^{1/2}$. Generally the horizontal scales of the modes are prescribed at the outset and it is not obvious, *a priori*, what value should be used.

In a rotating Bénard layer, the fundamental modes for velocity or magnetic field will generally consist of a "poloidal" part

$$\vec{P} = F_z(z,t)\nabla f(x,y) - F(z,t)\nabla^2 f(x,y)\vec{i}_3$$

and a "toroidal" part

$$\vec{T} = G(z,t)\vec{i}_3 \times \nabla g(x,y)$$

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where f and g are functions chosen to represent the horizontal structure. The fields are built up from a finite number of such terms, each corresponding to a choice of a,b in the equations $\nabla^2 f + a^2 f = 0$, $\nabla^2 g + b^2 g = 0$.

This approach has been applied by Baker (1973) to a Bénard-type convective dynamo. Baker focuses on a "2-mode" closure (the number of distinct F and G) and specifically on square convection cells generated by the choice $f = \cos(ax) \cdot \cos(ay)$. If the system does not rotate, the model can be further reduced to "1-1/2 modes" by expressing the magnetic field in terms of one poloidal and one toroidal component, and the velocity field in terms of two poloidal modes. One then obtains six equations, second order in z and first order in t, for the undetermined functions. The full 2-mode system includes additional toroidal parts of the velocity field and takes account of the influence of the Coriolis force, so it would appear to be the simplest modal realization of a rotating dynamo.

In the 1-1/2-mode closure dynamo action was found to occur over a range of parameter values and for various boundary conditions. In Figure 4 we show the energies developed in one of the oscillatory dynamos. In this example the mean magnetic energy is about 2.5 times the mean kinetic energy, and the tendency for the peaks to be out of phase is consistent with Busse's quasi-equilibrium analysis at margi-

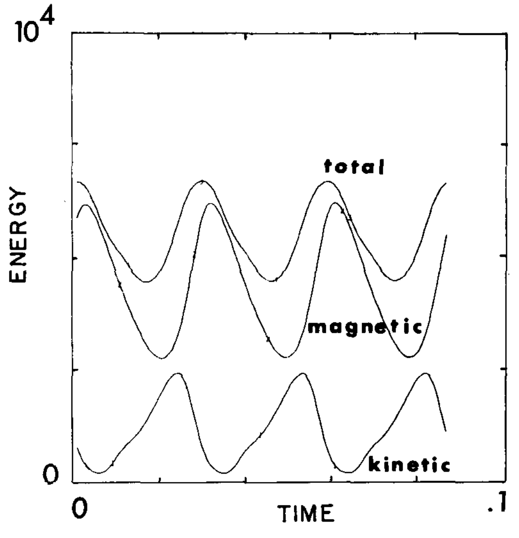

Figure 4. An oscillatory dynamo with 1 1/2 modes, for perfectly conducting rigid walls, $R = 10^5$, $P_\eta = 0.1$, $P = 1$, $a = 3.1$ (from Baker 1973)).

nal stability (Section 5). The effect of rotation was not studied in the same detail, but in some preliminary computations with 2-mode closure the rotation was found to enhance the dynamo effect. Perhaps most interesting is the fact that the calculations suggest the existence of a convective dynamo effect in a Bénard layer without rotation, for square cells whose horizontal dimension is comparable to the thickness of the layer. Unfortunately, Baker notes a rather poor convergence in going from 1-1/2 to 2 modes, so it must be regarded as possible that the dynamo effect is illusory at this low level of truncation.

Probably the simplest model of Bénard convection involves single modes for velocity, perturbed temperature, and mean temperature, and a further projection of the vertical structure onto a mode of the form $\exp(im\pi z)$. The resulting system of three first-order ordinary differential equations in time is known as the Lorenz-Howard-Malkus or "ABC" convection model (Lorenz 1963, Malkus 1972). Recently Kennett (1976) has extended this system to encompass magnetohydrodynamic convection, by the addition of terms representing the poloidal and toroidal field components. The resulting "ABCDE" model can be thought of as a projection of the vertical structure of Baker's 1-1/2 mode system, and is simpler by one equation because of the absence of one poloidal mode in the velocity. Indeed it is probably the minimal modal system for a non-rotating convective dynamo. An interesting aspect of the formulation is that it should allow systematic study of periodic and aperiodic behavior; the latter is known to occur in the ABC model (Lorenz 1963) as well as in other third-order systems (Baker, Moore, and Spiegel 1971). The "CBE" part of the system, moreover, bears a certain resemblance to the shunted disk-dynamo model studied by Robbins (1975).

Kennett shows that the system admits equilibrium solutions with non-zero magnetic field provided that

$$R > c_1 + P_\eta^2 c_2$$

where the c's are constants determined by the form of the horizontal modes. For a range of parameter values these equilibria are unstable, however, and by applying the method of averaging it is shown that there exist in that case, in the limit of large time, nearby linearly stable periodic solutions with non-zero magnetic field.

Because of their relative simplicity these systems are very useful and deserve further study. It would be interesting to know, for example, what insight could be gained concerning the role of rotation in the dynamo process, through the addition of a toroidal velocity mode. Also, it is to be hoped that as our understanding of the process deepens, a mode structure can be devised which converges rapidly with the truncation level.

8. TOWARDS SIMULATION OF THE GEODYNAMO

We have not dealt in this paper with the well known solution of the kinematic dynamo problem discovered by Braginskiĭ (see the review of this work in Roberts 1971), since this approach was not exploited in the convective dynamos discussed above. The Braginskiĭ dynamo has the advantage of making no special assumptions regarding the distribution of spatial scales. Rather, a simplification is achieved by requiring the magnetic Reynolds number of the velocity eddies to be large. This enforces a certain symmetry on the fields (near axial symmetry in a spherical core), which are then dominated by their symmetric toroidal parts.

Recently Braginskiĭ has initiated a study of the corresponding spherical convective dynamo (Braginskiĭ 1975). In this first paper the fluctuating component is assumed given, so the problem reduces to equations for the symmetric components of the fields. The questions raised by the multi-scale convective dynamos, concerning the origin of an α-effect and mean Lorentz force from the small-scale convection, are thereby avoided, and the dynamic balance for the symmetric fields can be studied at energies believed realistic for the geodynamo. Braginskii proposes a solution in which the meridional magnetic field within the core is predominantly parallel to the rotation axis. The field is matched with its mantle counterpart through a magnetic boundary layer at the core-mantle interface. Since the dynamo is of "αω" type, the azimuthal velocity which provides the "ω-effect" must be determined from the dynamics of the symmetric fields. As Roberts and Stewartson (1974) have emphasized, this is a crucial step in the construction once $M^2 \sim Ta^{1/2}$. Braginskiĭ's model, which determines the azimuthal flow by a process involving electromagnetic coupling of core and mantle, thus confronts a problem not faced in the idealized layer systems. (For a different approach to this question, devised for α^2-dynamics, see Malkus and Proctor 1975.) It is probably fair to say that the convective origin of the α-effect is only one-half, and perhaps the easier one-half, of the dynamical problem, and we await with considerable interest the further development of this approach to the spherical convective dynamo.

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We conclude with a few general observations. For the sake of argument we adopt a conservative attitude, as will be clear from the following list of postulates for the geodynamo:

(1) The field is maintained by heating at a uniform rate in a region of size L, fixed within the core relative to the rotation axis.

(2) Within this region, core motions are irregular (in particular poloidal and toroidal components are comparable) and can be characterized by a speed U and length scale L.

(3) Within this region the magnetic field is also irregular, with field strength $B > 0$ and length scale L.

(4) Within this region the Coriolis, Lorentz, and buoyancy forces acting on a fluid element are comparable.

(5) The system varies on a time scale of magnetic diffusion.

Given that heating is uniform and the system is Boussinesq, these hypotheses are close to the "worst possible" if the aim is a perturbational analysis. Indeed from (2) and the existence of a dynamo effect it follows that $R_m = UL/\eta \sim 1$, so that kinematic dynamo problem is without a small parameter. Balancing the Coriolis and Lorentz force we then have $M^2 \sim Ta^{1/2}$, an ordering already encountered in Section 2. With (5) the units are fixed and, if it is additionally postulated that viscous and inertial forces are negligible, the system reverts to the dimensionless form (13)-(15) (for example). It is plausible that (with the possible exception of Ekman layer effects and core-mantle coupling) the resulting equations contain the relevant physics and the important matter is the ordering of terms. In each perturbational model one or more of the above postulates is relaxed.

A crucial question is the appropriate magnetic Reynolds number of velocity eddies. Estimates range from 1 to 10^4 (Gubbins 1974). However, in view of the uncertainty over the possible size and location of a convecting region in the Earth's core (cf. Busse 1975) a value in the range 10-100 is not unreasonable. This would tend to favor Braginskiǐ's ordering of the kinematic dynamo, but there is a possible alternative, namely that the symmetry of the field with respect to the rotation axis is a result of the location of the convective region and the nature of the dynamo effect within it. In that case (2) and (3) might reflect irregular motion with moderate concentrations of magnetic flux (Weiss 1966).

Regarding the induction problem, it is tempting to add a postulate to our list, namely that the dynamo is of "$\alpha\omega$" type, even though if $Rm \sim 1$ the α and ω-effects are difficult to separate. We suggest that

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the α-effect could be realized as in Busse's annulus model, or as in case III of Section 4. Busse's model is especially attractive, since it also suggests how the corresponding ω-effect could be developed. Suppose we alter the direction of gravity to reflect the inclination which occurs over most of the cylindrical annulus of rolls in the marginally convective heated sphere. The convective heat transport is then oblique to gravity, so the mean temperature field is distorted, in such a way that the ω-effect arises from the "thermal wind". One can check that if the distortion is of the order of the equilibrium mean temperature profile, the magnetic Reynolds number of the thermal wind is indeed ~ 1 provided M and R_q are ordered as above. These estimates are likely to be modified somewhat if the convective zone is only a small fraction of the electrically conducting region.

The geometry of the convecting zone relevant to the α-effect may be significantly altered if, as Kennedy and Higgins (1973) suggest, convection in the Earth's core occurs only near the inner core. In that case the appropriate annulus model may involve a depth which *increases* with distance from the rotation axis, implying an α-effect from *westward*-moving waves.

Equations (13)-(15) indicate that P_η is a significant parameter in our problem, a point that has been emphasized by Roberts and Stewartson (1974) in their study of dissipative M.A.C. waves arising in rotating magnetoconvection (cf. Roberts and Soward (1972)). It is not clear whether ultimately the most profitable course will be to take $P_\eta \sim 1$, or rather to use the singular limit process $P_\eta \to \infty$ (presumably leading to localized convective heat transport and a reordering of the variables) as intrinsic to the geodynamo.

One aspect of the problem which would appear to deserve further study is the possibility of obtaining more refined estimates of solutions along the lines of the calculation of Kennett (1974), perhaps with a view to maximizing magnetic energy in a system driven by internal heating. It is likely that the geodynamo operates in a state which is "optimal" in the realized mean magnetic energy (cf. Section 2), and once the nature of this state is determined we can expect, on the basis of the considerable advances made over the last decade, that it will then not be too difficult to secure a dynamical model of the process.

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