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BOOK REVIEWS

BENSON, D. J., *Representations and cohomology*, Vol. 1: Basic representation theory of finite groups and associative algebras (Cambridge Studies in Advanced Mathematics 30, Cambridge University Press 1991), pp. xii+224, 0 521 36134 6, £25.

This is the first of two volumes on representation theory and cohomology of groups. As such, a good part of the book consists of preparatory material. In particular the first three chapters give a rapid introduction to the necessary background in rings and modules, homological algebra and modules over group algebras. It is necessary to have a reasonable background in algebra (the author suggests the three-volume work of Jacobson), as the pace is rather brisk.

The main part of this first volume is Chapter 4 which gives an introduction to the Auslander–Reiten style representation theory of finite-dimensional algebras: quivers are introduced and those of finite representation type or tame representation type are classified via Dynkin and Euclidean diagrams; almost split sequences are discussed and the important theorems of Roiter, Reidtmann and Webb are proved.

Chapter 5 introduces representation rings and various induction theorems are proved. Chapter 6 studies the Block Theory of a group algebra using the Auslander–Reiten theory from Chapter 4. The classical case of blocks with cyclic defect group is treated and also the more modern work of Erdmann analysing tame blocks is presented.

I found this to be a good book to read. Much of the material is quite technical and consequently the proofs may require careful reading. However, there is sufficient commentary around the proofs so that the reader is able to keep a clear idea of where the treatment is heading.

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BASTON, R. J. and EASTWOOD, M. G., *The Penrose transform: its interaction with representation theory* (Oxford Mathematical Monographs, Clarendon Press, Oxford 1989), pp. 232, 019 853565 1, £25.

One of the early successes of Penrose's "Twistor programme" to reformulate fundamental physics on Minkowski space in terms of the holomorphic geometry of "twistor space", $\mathbb{P}=\mathbb{CP}^3$, was the description of solutions of the massless field equations (an important series of equations on Minkowski space including the wave equation and the source-free Maxwell equations) in terms of contour integrals of homogeneous holomorphic functions on regions of $\mathbb{P}$. This process, which has since become known as the "Penrose transform", can be carried out in a far more general geometrical setting. Said very briefly, the Penrose transform as considered here is a machine, analogous to the Radon or Fourier transforms, which relates sheaf cohomology groups on one space to kernels and cokernels of differential operators on another—where the spaces involved are quotients of complex semi-simple Lie groups by parabolic subgroups. I will return to this case shortly, but it is helpful first to consider the transform in greater generality.

The basic setting in which the Penrose transform exists is that one has a "correspondence" between two complex manifolds X, Z via an intermediate complex manifold Y

$$Z \xleftarrow{\eta} Y \xrightarrow{\tau} X$$

where η, τ are surjective holomorphic maps of maximal rank. In the usual situation, X is Stein and τ has compact fibres. One is given a sheaf $\mathscr{E} \to Z$ which is the sheaf of sections of some holomorphic vector bundle.

The transform proceeds in three stages: Firstly, one computes $H^p(Z, \mathscr{E})$ in terms of $H^p(Y, \eta^{-1}\mathscr{E})$, where $\eta^{-1}\mathscr{E}$ denotes the topological inverse image sheaf. In most practical situations, the fibres of η are sufficiently topologically trivial that these groups are isomorphic—we will assume this henceforth. Secondly, one constructs on Y a resolution $0 \to \eta^{-1}\mathscr{E} \to \mathscr{R}^\bullet$ of $\eta^{-1}\mathscr{E}$ by locally free sheaves $\mathscr{R}^p$. The cohomology of $\eta^{-1}\mathscr{E}$ can then be computed in terms of the cohomology of the $\mathscr{R}^p$. Thirdly, one uses the Leray spectral sequence to compute the cohomology of the sheaves $\mathscr{R}^p$ on Y in terms of that of the direct image sheaves $\tau^q_* \mathscr{R}^p$ *on* X. When X is Stein, the last two stages collapse into a single spectral sequence which together with the first stage give the result that in the above circumstances there is a spectral sequence

$$E_1^{p,q} = \Gamma(X, \tau^q_* \mathscr{R}^p) \Rightarrow H^{p+q}(Z, \mathscr{E}).$$

This spectral sequence is the Penrose transform.

To date, the main applications of this transform have been in the setting considered by Baston and Eastwood. Here one starts with a complex semi-simple Lie group G and standard parabolic subgroups P, R of G, so that there is a double fibration:

$$G/R \overset{\eta}{\leftarrow} G/(P \cap R) \overset{\tau}{\rightarrow} G/P$$

We choose some appropriate Stein open subset $X \subset G/P$, and setting $Y = \tau^{-1}X$ and $Z = \eta Y$, we obtain a situation such as we previously considered (provided the fibres of η turn out to be sufficiently topologically trivial).

The sheaf $\mathscr{E}$ in this situation is the sheaf of sections of some homogeneous vector bundle. The direct image sheaves on X are also of this form and the transform relates the cohomology of $\mathscr{E}$ to the kernels and cokernels of G-invariant differential operators between such sheaves on X.

In this situation, the whole process becomes an extended exercise in representation theory and Baston and Eastwood essentially present an algorithm for carrying out the transform in precise detail for any correspondence of the appropriate form and any homogeneous vector bundle. The main tools are Bernstein–Gelfand–Gelfand resolutions (which provide the resolutions $\mathscr{R}^\bullet$ above) and the Bott–Borel–Weil theorem which gives the direct images. The prototype example of Penrose's twistor theory for Minkowski space is used for motivation and examples, although the reader who is not particularly interested in this should not find this a problem.

The final quarter of the book is concerned with the possible applications to representation theory—a subject of considerable interest at the moment. In particular, one can choose X, Z to be open orbits of some real form of G and use the Penrose transform to relate the different representations of the real form, firstly as cohomology groups and secondly as solutions of invariant equations.

The authors have given a good outline of the relevant representation theory and those readers wishing to have a thorough understanding of it are directed to the literature. The presentation is clear, being directed towards showing the reader how to calculate the transform and there are numerous examples. This is still a very young theory and applications in both geometry and representation theory are being actively and successfully pursued. At the moment however the pay-off has been more in the nature of substantial insights than new theorems.

In conclusion, I believe that the Penrose transform shows every sign of being a significant tool in geometry and representation theory. This book is the only published account of the transform for other than special cases, but it is in any case recommendable and I have no significant criticisms of the presentation.

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