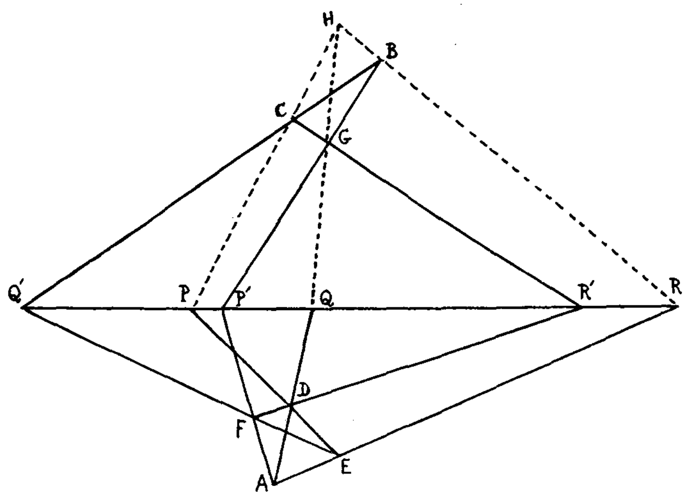
Proof. The polar of the line $(p_{23}, p_{31}, p_{12}, p_{14}, p_{24}, p_{34})$ with regard to the quadric $xz - yt = 0$ is the line whose coordinates are

$$(p_{23}, \ -p_{24}, \ -p_{12}, \ p_{14}, \ -p_{31}, \ -p_{34})$$

and this satisfies the condition (7).



The accompanying figure (*Cf.* H. F. Baker, *Principles of Geometry*, **1** (1922), 61) illustrates the construction of such a set of points by means of six given coplanar lines, as shewn, meeting another plane π in points $PP'QQ'RR'$. In the plane π three given lines through $P'Q'R'$ determine a triangle BCG, while the dotted lines complete the figure by meeting at H, the eighth of the required points.

A Series Identity

By J. C. P. Miller.

In connection with some work on the theory of probable errors, the following series identity arose:—

$$\frac{1}{2a} + \left(-\frac{1}{2}\right)^2 \frac{1}{2a+2} + \left(-\frac{1}{2}\cdot\frac{1}{4}\right)^2 \frac{1}{2a+4} + \ldots$$
$$+ \left(-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\ldots\frac{2r-3}{2r}\right)^2 \frac{1}{2a+2r} + \ldots.$$
$$= \frac{1}{2a+1} + \left(-\frac{1}{2}\right)^2 \frac{1}{2a-1} + \left(-\frac{1}{2}\cdot\frac{1}{4}\right)^2 \frac{1}{2a-3} + \ldots$$
$$+ \left(-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\ldots\frac{2r-3}{2r}\right)^2 \frac{1}{2a-2r+1} + \ldots,$$

where a is a positive integer.

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As the main interest of this identity appears to lie in itself and in the method of proof, and not in its connection with the theory of probable errors, it seems worth while to record it separately.

We note that both series are convergent, for

$$A_r \equiv \left(-\frac{1}{2} \cdot \frac{1}{4} \cdots \frac{2r-3}{3r}\right)^2 < \frac{1}{4r^2}.$$

We have to show that their difference vanishes, *i.e.* that

$$S(a, 0) \equiv \sum_{r=0}^{\infty} A_r \frac{1-4r}{(2a+2r)(2a-2r+1)} = 0.$$

Let $u(a, b; r)$, where b is a positive integer or zero, denote

$$A_r \frac{1-4r}{(2a+2r)(2a+2+2r)\ldots(2a+2b+2r).(2a+1-2r)(2a+3-2r)\ldots(2a+2b+1-2r)}.$$

Then $u(a+1, b; r) - u(a, b; r) = -2(b+1)(4a+2b+3)\, u(a, b+1; r)$;

also $\quad S(a+1, b, t) - S(a, b; t) = -2(b+1)(4a+2b+3)\, S(a, b+1; t)$,

where $$S(a, b; t) = \sum_{r=0}^{t-1} u(a, b; r).$$

Proceeding to the limit $t \to \infty$, with the notation $S(a, b) \equiv S(a, b; \infty)$ we see that the series $S(a, 0)$ can be expressed in terms of the series $S(1, b)$.

We find $\quad S(1, 0) = S(1, 0)$,

$$S(2, 0) = S(1, 0) - 14\, S(1, 1),$$

$$S(3, 0) = S(1, 0) - 36\, S(1, 1) + 792\, S(1, 2),$$

and in general that

$$S(a+1, 0) = \sum_{s=0}^{a} (-)^s \frac{a!}{(a-s)!} \frac{(a+2)!}{(2a+4)!} \frac{(2a+2s+4)!}{(a+s+2)!} S(1, s).$$

We may express $S(1, b)$ as the sum of two hypergeometric series:—

$$S(1, b) = -\{\Gamma(-\tfrac{1}{2})\}^{-2}(-4)^{-b-1}\left[\sum_{r=0}^{\infty} \frac{\Gamma(r-\frac{1}{2})\,\Gamma(r-b-\frac{3}{2})}{r!\,\Gamma(r+b+2)} + 4\sum_{r=0}^{\infty} \frac{\Gamma(r+\frac{1}{2})\,\Gamma(r-b-\frac{3}{2})}{r!\,\Gamma(r+b+2)}\right].$$

Now[1]

$$\sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)\,\Gamma(\beta+r)}{r!\,\Gamma(\gamma+r)} = \frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)},$$

whence

$$S(1, b) = -\{\Gamma(-\tfrac{1}{2})\}^{-2}(-4)^{-b-1}\left\{\frac{\Gamma(-\frac{1}{2})\,\Gamma(-b-\frac{3}{2})\,\Gamma(2b+4)}{\Gamma(b+2\frac{1}{2})\,\Gamma(2b+3\frac{1}{2})} + 4\,\frac{\Gamma(\frac{1}{2})\,\Gamma(-b-\frac{3}{2})\,\Gamma(2b+3)}{\Gamma(b+1\frac{1}{2})\,\Gamma(2b+3\frac{1}{2})}\right\} = 0.$$

There is, however, an alternative method of evaluating $S(1, b)$ which is of interest, as it is algebraic and also gives us an expression for $S(1, b; t)$, where t is finite.

Assume $S(1, b; t)$ to be of the form

$$S(1, b; t) = A_t \frac{f(t)}{(2+2t)\,..\,(2b+2t)\,.\,(3-2t)\,...\,(3+2b-2t)} + C.$$

Then $u(1, b; t) = S(1, b; t+1) - S(1, b; t)$

$$= A_t \frac{Z}{(2+2t)\,...\,(2b+2+2t)\,.\,(3-2t)\,...\,(3+2b-2t)},$$

where $Z = \left(\dfrac{2t-1}{2t+2}\right)^2 2(t+1)\,\dfrac{3+2b-2t}{1-2t}\,f(t+1) - 2(b+t+1)\,f(t).$

But $\quad Z = 1 - 4t \quad$ from the form of $u(1, b; t)$.

Thus, putting $f(t) = 2t\,\phi(t)$ we have

$$1 - 4t = -(2t-1)(3+2b-2t)\,\phi(t+1) - 4t(b+t+1)\,\phi(t),$$

which is satisfied by $\phi(t) = 1/(2b+3)$, or $f(t) = 2t/(2b+3)$. We then find that $C = 0$.

Thus

$$S(1, b; t) = A_t \frac{2t}{(2b+3).(2+2t)(4+2t)\,...\,(2b+2t)\,.\,(3-2t)\,...\,(3+2b-2t)}$$

$$= \left[\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdots\frac{2t-3}{2t-4}\cdot\frac{2t-3}{2t-2}\right]\frac{1}{2b+3}\cdot\frac{1}{(-2+2t)\,...\,(2b+2t)} \times \frac{1}{(3-2t)\,...\,(3+2b-2t)}$$

$$\sim \frac{2}{\pi}\,\frac{1/2}{2b+3}\,(-4)^{-b-1}\,t^{-2b-3} \to 0 \text{ as } t \to \infty \text{ for all } b > -3/2.$$

Hence $\quad S(1, b) = 0$ for positive integral or zero b, and so

$S(a, 0) = 0$ for all positive integral a, as required.

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[1] Whittaker and Watson. *Modern Analysis* (Ed. 4, 1927), p. 282.