Can. J. Math., Vol. XLII, No. 6, 1990, pp. 1084–1097

FREE OBJECTS IN CERTAIN VARIETIES OF INVERSE SEMIGROUPS

S. W. MARGOLIS, J. C. MEAKIN AND J. B. STEPHEN

ABSTRACT. In this paper it is shown how the graphical methods developed by Stephen for analyzing inverse semigroup presentations may be used to study varieties of inverse semigroups. In particular, these methods may be used to solve the word problem for the free objects in the variety of inverse semigroups generated by the five-element combinatorial Brandt semigroup and in the variety of inverse semigroups determined by laws of the form $x^n = x^{n+1}$. Covering space methods are used to study the free objects in a variety of the form $\mathcal{V} \vee \mathcal{G}$ where $\mathcal{V}$ is a variety of inverse semigroups and $\mathcal{G}$ is the variety of groups.

1. Introduction. Inverse semigroups, considered as algebras with the two operations of multiplication and inversion, form a variety determined by associativity and the following laws:

$$(*) \qquad xx^{-1}x = x,\ (x^{-1})^{-1} = x,\ x^{-1}xy^{-1}y = y^{-1}yx^{-1}x,\ (xy)^{-1} = y^{-1}x^{-1}.$$

For an inverse semigroup S we denote the set of idempotents of S by $E(S)$. The inverse semigroup S can also be viewed as a partially ordered algebra. The *natural partial order* on S is defined by $r \geqq s$, $r, s \in S$, if there exists $e \in E(S)$ such that $er = s$. The *order filter* of $s \in S$ is $[s\uparrow] = \{r \in S : r \geqq s\}$.

We assume a familiarity with the basics of inverse semigroup theory (see Petrich [**10**]) and elementary information concerning universal algebra (see Grätzer [**2**]). We refer the reader to Clifford and Preston [**1**] for a discussion of semigroups in general; in particular, Green's lemma and the Green's relations are discussed in detail. We refer to Hopcroft and Ullman [**4**] for basic information about automata theory.

For a set Σ, and a set of formal inverses $\Sigma^{-1} = \{\sigma^{-1} | \sigma \in \Sigma\}$, a pair of elements (u, v) of $(\Sigma \cup \Sigma^{-1})^+$ will be called an *identity* and denoted $u = v$. An inverse semigroup S is said to *satisfy* the identity $u = v$ if $u\phi = v\phi$ for all homomorphisms $\phi : (\Sigma \cup \Sigma^{-1})^+ \longrightarrow S$. Given a set $\mathcal{F}$ of identities, the class of all inverse semigroups satisfying each identity in $\mathcal{F}$ is called the *variety* determined by $\mathcal{F}$. Varieties of inverse semigroups have attracted considerable attention in the literature: we refer the reader to Petrich [**10**], Chapter XII for

Received July 5, 1989.
AMS (1980) Classification: 20M05, 20M18, 03D40.
Research supported by N.S.F. grant DMS 8702019.

basic results and notation in this field. In particular, we will consistently use the following notation:

1) $\mathcal{G}$ – The variety of groups.
2) $\mathcal{I}$ – The variety of inverse semigroups.
3) $\mathcal{T}$ – The trivial variety of inverse semigroups.
4) $\mathcal{S}$ – The variety of semilattices.
5) $\mathcal{SG}$ – The variety of Clifford semigroups.
6) $\mathcal{SI}$ – The variety of strict inverse semigroups.
7) $\mathcal{B}$ – The variety of inverse semigroups determined by the five element combinatorial Brandt semigroup.
8) $[u_\alpha = v_\alpha]_{\alpha \in A}$ – The variety determined by the family of identities $\{u_\alpha = v_\alpha\}_{\alpha\in A}$.
9) $\mathcal{C}_n = [u^n = u^{n+1}]$;

In the lattice of varieties of inverse semigroups we will denote the meet of the two varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ by $\mathcal{V}_1 \cap \mathcal{V}_2$, and the join of $\mathcal{V}_1$ and $\mathcal{V}_2$ by $\mathcal{V}_1 \vee \mathcal{V}_2$.

All varieties that we will consider will be finitely based (i.e. they are defined by a finite set of identities), and we will consistently assume that the identities are given over a finite set Σ.

A variety $\mathcal{V}$ of inverse semigroups is said to be a *completely semisimple* [resp. *combinatorial*, *cryptic*] variety of inverse semigroups if every inverse semigroup in $\mathcal{V}$ is completely semisimple [resp. combinatorial, cryptic].

Our interest will be in the varieties $\mathcal{C}_n$, $\mathcal{B}$, and their joins with the variety of groups. The importance of these varieties is discussed in Petrich [**10**]. In particular we recall the following facts from Petrich [**10**].

THEOREM 1.1. (Petrich [**10**], Chapter XII)

(a) *A variety $\mathcal{V}$ of inverse semigroups is combinatorial if and only if $\mathcal{C}_n \supseteq \mathcal{V}$ for some n (and if and only if $\mathcal{V} \cap \mathcal{G} = \mathcal{T}$);*

(b) $$\bigvee_{n=1}^{\infty} \mathcal{C}_n = \mathcal{I};$$

(c) *A variety $\mathcal{V}$ of inverse semigroups is a completely semisimple cryptic variety if and only if $\mathcal{C}_n \vee \mathcal{G} \supseteq \mathcal{V}$;*

(d) $\mathcal{S} = \mathcal{C}_1 = [w^2 = w]$;

(e) $\mathcal{SG} = \mathcal{C}_1 \vee \mathcal{G} = [ww^{-1} = w^{-1}w]$

(f) $\mathcal{C}_2 \supseteq \mathcal{B} \supseteq \mathcal{C}_1$

(g) $\mathcal{B} = [(uvu^{-1})^2 = uvu^{-1}]$

(h) $\mathcal{SI} = \mathcal{B} \vee \mathcal{G}$.

We denote the free inverse semigroup on a set X by $FIS(X)$: thus $FIS(X) \cong$

$(X \cup X^{-1})^+/\rho$ where ρ is the Vagner congruence, i.e. ρ is the congruence on $(X \cup X^{-1})^+$ generated by requiring that the laws (*) should hold. A variety $\mathcal{V}$ of inverse semigroups determines the fully invariant congruence $\rho(\mathcal{V})$ on $(X \cup X^{-1})^+$ where X is a countably infinite set and $\rho(\mathcal{V}) = \{(u, v) \in (X \cup X^{-1})^+ \times (X \cup X^{-1})^+ : u = v \text{ is a law in } \mathcal{V}\}$. Clearly $\rho(\mathcal{V}) \supseteq \rho$. Conversely, every fully invariant congruence τ on $(X \cup X^{-1})^+$ such that $\tau \supseteq \rho$ determines a variety $\mathcal{V}(\tau)$ of inverse semigroups. This correspondence establishes an anti-isomorphism between the lattice of varieties of inverse semigroups and the lattice of fully invariant congruences τ on $(X \cup X^{-1})^+$ for which $\tau \supseteq \rho$ (see Petrich [**10**]). Furthermore, if X is any non-empty set and $\mathcal{V}$ is a variety of inverse semigroups, then $(X \cup X^{-1})^+/\rho(\mathcal{V})$ is the $\mathcal{V}$-free semigroup on X (i.e. $(X \cup X^{-1})^+/\rho(\mathcal{V})$ is the free semigroup on X in the variety $\mathcal{V}$).

We shall be concerned with the free semigroups in the varieties $\mathcal{B}$ and $\mathcal{C}_n$ in the present paper, as well as the free semigroups in a variety $\mathcal{V} \vee \mathcal{G}$ where $\mathcal{V}$ is a variety of inverse semigroups whose free semigroups are known and $\mathcal{G}$ is the variety of groups. In order to study these $\mathcal{V}$-free semigroups, we make use of the notion of the *Schützenberger graphs* of an inverse semigroup presentation and the iterative construction of Stephen [**14**] of these graphs. We briefly review relevant terminology and refer to the papers of Stephen [**14**], [**15**], Margolis and Meakin [**8**], Meakin [**9**] and Margolis, Meakin and Stephen [**7**] for many more details.

Let X be a non-empty set and $T = \{(u_i, v_i) : i \in I\}$ a relation on $(X \cup X^{-1})^+$ (i.e. $(X \cup X^{-1})^+ \times (X \cup X^{-1})^+ \supseteq T$). The *inverse semigroup generated by X subject to the relations* T is the inverse semigroup. $S = \mathrm{Invs}\langle X : T\rangle = (X \cup X^{-1})^+/\tau$ where τ is the congruence on $(X \cup X^{-1})^+$ generated by $T \cup \rho$. The pair (X, T) is called a *presentation* of S. If $S = \mathrm{Invs}\,\langle X : T\rangle = (X \cup X^{-1})^+/\tau$ is an inverse semigroup presentation and if u is a word in $(X \cup X^{-1})^+$ then we shall denote by $S\Gamma(X, T, u)$ (or $S\Gamma(u)$ if the presentation is understood) the *Schützenberger graph* of u relative to this presentation. Thus the set of *vertices* of $S\Gamma(u)$ is the set $R_{u\tau}$ of elements of S that are $\mathcal{R}$-related to $u\tau$ in S: there is an *edge* $\underset{v\tau}{\bigcirc}\xrightarrow{x}\underset{w\tau}{\bigcirc}$ labeled by $x \in X \cup X^{-1}$ from $v\tau$ to $w\tau$ in $S\Gamma(u)$ if $v\tau, w\tau \in R_{u\tau}$ and $w\tau = (vx)\tau$. The triple $\mathcal{A}(u) = \mathcal{A}(X, T, u) = ((uu^{-1})\tau,\ S\Gamma(u),\ u\tau)$ may be regarded as an automaton (called the *Schützenberger automaton* of u). Denote by $\mathrm{L}(\mathcal{A}(u))$ the *language* accepted by $\mathcal{A}(u)$. The importance of these automata stems from the following result of Stephen [**14**].

THEOREM 1.2. (Stephen [**14**]) For $S = \mathrm{Invs}\langle X : T\rangle = (X \cup X^{-1})^+/\tau$ *and words* $u, v \in (X \cup X^{-1})^+$ *the following statements are equivalent:*

(a) $u\tau = v\tau$;
(b) $L(\mathcal{A}(u)) = L(\mathcal{A}(v))$;

(c) *$u \in L(\mathcal{A}(v))$ and $v \in L(\mathcal{A}(u))$;*

(d) *$\mathcal{A}(u)$ and $\mathcal{A}(v)$ are isomorphic (as automata, or equivalently as birooted labeled graphs).*

Thus the automaton $\mathcal{A}(u)$ may be interpreted as a "canonical form" for the τ-equivalence class of the word u relative to the presentation. In his paper [**14**], Stephen provides an iterative procedure for constructing the automata $\mathcal{A}(u)$. Briefly, $\mathcal{A}(u)$ may be constructed from $MT(u)$ (the Munn tree of u, considered as an automaton with one initial state and one terminal state) by applying a (not necessarily finite) sequence of "expansions" and "reductions" to the intermediate automata. The automata under consideration are deterministic, injective and trim, with one initial state and one terminal state and with the property that if $\underset{\alpha}{\bigcirc} \xrightarrow{x} \underset{\beta}{\bigcirc}$ is an edge in the automaton (for $x \in X \cup X^{-1}$), then so is $\underset{\alpha}{\bigcirc} \xleftarrow{x^{-1}} \underset{\beta}{\bigcirc}$. In this paper, we call an automaton satisfying these conditions an *inverse X-automaton*. An *expansion* of such an automaton $\mathcal{A}'$ relative to a set $T = \{(u_i, v_i) : i \in I\}$ of relations $(u_i, v_i \in (X \cup X^{-1})^+)$ is an operation of the following type: if in $\mathcal{A}'$ there is a path $\underset{\alpha}{\bigcirc} \xrightarrow{u_i} \underset{\beta}{\bigcirc}$ from α to β labeled by one side (say u_i) of one of the relations in T, then we "sew on" to $\mathcal{A}'$ another path from α to β labeled by the other side (v_i) of the relation. A *reduction* of an automaton $\mathcal{A}$ is obtained by identifying two edges $\underset{\alpha}{\bigcirc} \xrightarrow{x} \underset{\beta}{\bigcirc}$ and $\underset{\alpha}{\bigcirc} \xrightarrow{x} \underset{\gamma}{\bigcirc}$ of $\mathcal{A}$ that have the same label $x \in X \cup X^{-1}$ and the same initial state (vertex) α. (A somewhat more formal definition of these operations is provided in Stephen [**14**]: see also Meakin [**9**]). We write $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$ if $\mathcal{A}_2$ is obtained from $\mathcal{A}_1$ by applying one expansion followed by a sequence of reductions so that the resulting automaton is deterministic. If $\mathcal{A}_1$ is an inverse X-automaton, then so is $\mathcal{A}_2$. We write $\mathcal{A}_1 \stackrel{*}{\Longrightarrow} \mathcal{A}$ if there is a sequence (finite or infinite) of productions $\mathcal{A}_1 \Rightarrow \mathcal{A}_2 \Rightarrow \mathcal{A}_3 \Rightarrow \cdots \Rightarrow \mathcal{A}$. We say that $\mathcal{A}_1$ is *closed* (with respect to the presentation) if no production of the form $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$ produces a new automaton $\mathcal{A}_2 \neq \mathcal{A}_1$. From the results of Stephen, we know that if $\mathcal{A}$ is a closed automaton and $MT(w) \stackrel{*}{\Longrightarrow} \mathcal{A}$, then $\mathcal{A} = \mathcal{A}(w)$. The word problem for $M = \mathrm{Invs}\langle X : T\rangle$ is decidable if this procedure for constructing the Schützenberger automata is effective in the sense that, for each $u, v \in (X \cup X^{-1})^+$, there is an algorithm for deciding whether $v \in L(\mathcal{A}(u))$ or not. We refer the reader to [**14**], [**15**], [**7**], [**8**], [**9**] for many more details and applications of this procedure. We shall apply these methods to study the free objects $X^+/\rho(\mathcal{B})$ and $X^+/\rho(\mathcal{C}_n)$ in the varieties $\mathcal{B}$ and $\mathcal{C}_n$.

2. The varieties $\mathcal{B}$ and $\mathcal{C}_n$ ($n \geqq 1$). We denote by $B(X)$ the free object on

X in the variety $\mathcal{B}$ and by $C_n(X)$ the free object on X in the variety C_n. From Theorem 1.1 and the discussion in the previous section it is clear that

(1) $$B(X) = \mathrm{Invs}\langle X : uvu^{-1} = (uvu^{-1})^2,\ u,\ v \in (X \cup X^{-1})^+\rangle$$

and

(2) $$C_n(X) = \mathrm{Invs}\langle X : u^n = u^{n+1},\ u \in (X \cup X^{-1})^+\rangle.$$

We indicate below how to apply the iterative procedure outlined above to construct the Schützenberger automaton $\mathcal{A}(w)$ for a word $w \in (X \cup X^{-1})^+$, relative to either the presentation (1) for $B(X)$ or the presentation (2) for $C_n(X)$. We begin by considering $C_n(X)$: the argument for $B(X)$ follows along very similar lines.

Let $\mathcal{A}_1$ be an inverse automaton and consider the effect of applying a production $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$ relative to the presentation (2). In order to apply any such production, we must be able to find states (vertices) $\alpha, \beta \in \mathcal{A}_1$ and a path $\underset{\alpha=\alpha_0}{\bigcirc} \xrightarrow{u} \underset{\alpha_1}{\bigcirc} \xrightarrow{u} \underset{\alpha_2}{\bigcirc} \xrightarrow{u} \cdots \underset{\alpha_{n-1}}{\bigcirc} \xrightarrow{u} \underset{\alpha_n=\beta}{\bigcirc}$ labeled by u^n (for some word $u \in (X \cup X^{-1})^+$) from α to β. We expand the automaton $\mathcal{A}_1$ by sewing on to $\mathcal{A}_1$ another path $\underset{\beta_0=\alpha}{\bigcirc} \xrightarrow{u} \underset{\beta_1}{\bigcirc} \xrightarrow{u} \underset{\beta_2}{\bigcirc} \xrightarrow{u} \cdots \underset{\beta_n}{\bigcirc} \xrightarrow{u} \underset{\beta_{n+1}=\beta}{\bigcirc}$ labeled by u^{n+1} from α to β. Application of a succession of reductions to the resulting automaton immediately results in identifying the path labeled by u from α_i to α_{i+1} with the path labeled by u from β_i to β_{i+1}, thus yielding a path of the form $\underset{\alpha=\alpha_0}{\bigcirc} \xrightarrow{u} \underset{\alpha_1}{\bigcirc} \xrightarrow{u} \underset{\alpha_2}{\bigcirc} \xrightarrow{u} \cdots \underset{\alpha_{n-1}}{\bigcirc} \xrightarrow{u} \underset{\alpha_n=\beta}{\bigcirc} \circlearrowright u$ with a loop labeled by u at β. Further application of reductions at β then yield $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = \alpha_n = \alpha = \beta$ and the original path labeled by u^n from α to β reduces to a loop labeled by u at $\alpha = \beta$. Thus the effect of a production $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$ is to replace a path labeled by u^n in $\mathcal{A}_1$ by a loop labeled by u. Clearly, if $\mathcal{A}_1$ is finite then $\mathcal{A}_2$ has fewer vertices than $\mathcal{A}_1$.

Now let w be any word in $(X \cup X^{-1})^+$. Since $\mathcal{A}(w)$ is obtained from the finite automaton $MT(w)$ by a sequence of productions $MT(w) \overset{*}{\Longrightarrow} \mathcal{A}(w)$, it follows that $\mathcal{A}(w)$ is finite. Furthermore, an inverse X-automaton $\mathcal{A}$ is closed with respect to the presentation (2) if and only if it satisfies the property P_n below:

P_n: if in $\mathcal{A}$ there is a path $\underset{\alpha=\alpha_0}{\bigcirc} \xrightarrow{u} \underset{\alpha_1}{\bigcirc} \xrightarrow{u} \underset{\alpha_2}{\bigcirc} \xrightarrow{u} \cdots \underset{\alpha_{n-1}}{\bigcirc} \xrightarrow{u} \underset{\alpha_n=\beta}{\bigcirc}$ from α to β labeled by u^n for some word $u \in (X \cup X^{-1})^+$, then $\alpha = \alpha_1 = \cdots = \alpha_{n-1} = \alpha_n = \beta$. Hence $\mathcal{A}(w)$ satisfies the property P_n. Note also that it is decidable whether a finite inverse X-automaton $\mathcal{A}$ satisfies property P_n or not. For if $\mathcal{A}$ is finite then its transition monoid is a finite monoid: it follows that there are finitely many

https://doi.org/10.4153/CJM-1990-058-9 Published online by Cambridge University Press

elements in this transition monoid that can be expressed as nth powers of other elements of this monoid. Hence, for all states (vertices) α, β in $\mathcal{A}$, it is decidable whether there is a path labeled by u^n from α to β for some word $u \in (X \cup X^{-1})^+$ (since such a path corresponds to an nth power in the transition monoid of $\mathcal{A}$). Hence it is decidable when a finite inverse X-automaton is closed with respect to the presentation (2) and so the word problem for $C_n(X)$ is decidable. We summarize this discussion in the following results.

THEOREM 2.1. *Let X be a non-empty set, n a fixed positive integer, w a word in $(X \cup X^{-1})^+$ and $\mathcal{A}(w)$ the Schützenberger automaton of w relative to the presentation* (2) *for $C_n(X)$. Then $\mathcal{A}(w)$ is a finite inverse X-automaton that satisfies P_n. Furthermore, the word problem for $C_n(X)$ is decidable.*

Additional information about the structure of $C_n(X)$ is obtained by classifying those automata that occur as Schützenberger automata of some word $w \in (X \cup X^{-1})^+$ relative to the presentation (2). We already know that such an automaton satisfies P_n: in fact the converse is true.

THEOREM 2.2. *Let $\mathcal{A}$ be a finite inverse X-automaton. Then $\mathcal{A}$ is the Schützenberger automaton $\mathcal{A}(w)$ for some word $w \in (X \cup X^{-1})^+$, relative to the presentation* (2) *for $C_n(X)$, if and only if $\mathcal{A}$ satisfies P_n.*

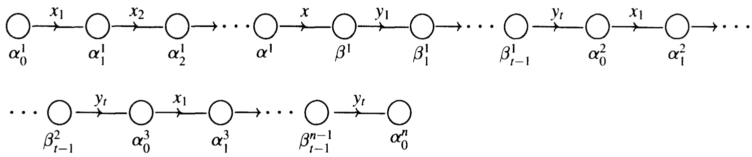
Before proceeding with the proof of this theorem we recall the notion of the *fundamental group* of a graph (or automaton) in the sense of Higgins [**3**] or Stallings [**13**]. Let $\mathcal{A}$ be an inverse X-automaton and consider $\mathcal{A}$ for the moment simply as a labeled graph (i.e. ignore the fact that it has distinguished initial and terminal states). If e is the edge $\underset{\alpha}{\bigcirc}\xrightarrow{\;x\;}\underset{\beta}{\bigcirc}$ in $\mathcal{A}$ then e^{-1} is the edge $\underset{\alpha}{\bigcirc}\xleftarrow{\;x^{-1}\;}\underset{\beta}{\bigcirc}$. For coterminal paths p, q from initial vertex α to terminal vertex β write $p \sim q$ if p can be obtained from q by a finite number of replacements of consecutive edges of the form ee^{-1} by the empty path 1 or vice-versa. (Thus, for example, if $\mathcal{A}$ is a tree and α, β are any two vertices of $\mathcal{A}$ then $p \sim q$ for any two paths p, q from initial vertex α to terminal vertex β.) The relation $\sim$ is clearly an equivalence relation (called "homotopy") on the set of paths of $\mathcal{A}$: denote the equivalence class of a path p by $[p]$. The set $\{[p] : p$ is a path in $\mathcal{A}\}$ forms a groupoid called the *fundamental groupoid* of $\mathcal{A}$ with respect to concatenation of (equivalence classes of) paths whenever this concatenation is defined. If α is a fixed vertex of $\mathcal{A}$ then the subset $\pi_1(\mathcal{A}, \alpha) = \{[p] : p$ is a path from α to α in $\mathcal{A}\}$ forms a group, called the *fundamental group of $\mathcal{A}$ based at α*, with respect to concatenation of equivalence classes of paths. It is well-known (Higgins [**3**], Stallings [**13**]) that $\pi_1(\mathcal{A}, \alpha)$ is a free group and that $\pi_1(\mathcal{A}, \alpha) \cong \pi_1(\mathcal{A}, \beta)$ if there is a path in $\mathcal{A}$ connecting α and β. Thus if $\mathcal{A}$ is connected we denote the group $\pi_1(\mathcal{A}, \alpha)$ (for any vertex α) simply by $\pi_1(\mathcal{A})$ and refer to it as the *fundamental group* of $\mathcal{A}$. If T is a spanning tree of $\mathcal{A}$ then the rank of $\pi_1(\mathcal{A})$ is $(1/2)|E(\mathcal{A} - T)|$ (the number of undirected edges in $\mathcal{A} - T$). We denote this number by $\beta(\mathcal{A})$ (and call it the first Betti number of $\mathcal{A}$).

Proof of Theorem 2.2. Let $\mathcal{A}$ be a finite inverse X-automaton. We show by induction on $\beta(\mathcal{A})$ that there is some word w such that $MT(w) \overset{*}{\Longrightarrow} \mathcal{A}$. The desired result will then follow from the fact that $\mathcal{A}$ is closed with respect to the presentation (2) if $\mathcal{A}$ satisfies condition P_n.

If $\beta(\mathcal{A}) = 0$ then $\mathcal{A}$ is a tree and so $\mathcal{A} = MT(w)$ for any word $w \in (X \cup X^{-1})^+$ whose Munn tree is $\mathcal{A}$ (such a word exists since $\mathcal{A}$ is finite, injective and trim with one initial state and one terminal state). Assume that $\beta(\mathcal{A}) > 0$ and that the result is true for all finite inverse X-automata $\mathcal{A}_1$ with $\beta(\mathcal{A}_1) < \beta(\mathcal{A})$. Let T be a fixed spanning tree of $\mathcal{A}$ and let $e = \underset{\alpha}{\bigcirc}\xrightarrow{x}\underset{\beta}{\bigcirc}$ be an edge of $\mathcal{A}$ that is not in T. (We do not require α and β to be distinct). Fix a state (vertex) α_0 of $\mathcal{A}$. Let $[\alpha_0, \alpha]$ be the geodesic in T from α_0 to α and $[\beta, \alpha_0]$ the geodesic in T from β to α_0. Thus $[\alpha_0, \alpha]$ is a path $\underset{\alpha_0}{\bigcirc}\xrightarrow{x_1}\underset{\alpha_1}{\bigcirc}\xrightarrow{x_2}\underset{\alpha_2}{\bigcirc}\xrightarrow{x_3}\cdots\underset{\alpha_{k-1}}{\bigcirc}\xrightarrow{x_k}\underset{\alpha_k=\alpha}{\bigcirc}$ with each edge in T and $[\beta, \alpha_0]$ is a path $\underset{\beta=\beta_0}{\bigcirc}\xrightarrow{y_1}\underset{\beta_1}{\bigcirc}\xrightarrow{y_2}\underset{\beta_2}{\bigcirc}\xrightarrow{y_3}\cdots\underset{\beta_{t-1}}{\bigcirc}\xrightarrow{y_t}\underset{\beta_t=\alpha_0}{\bigcirc}$ with each edge in T. Clearly $\underset{\alpha_0}{\bigcirc}\xrightarrow{x_1}\underset{\alpha_1}{\bigcirc}\xrightarrow{x_2}\underset{\alpha_2}{\bigcirc}\xrightarrow{x_3}\cdots\underset{\alpha_{k-1}}{\bigcirc}\xrightarrow{x_k}\underset{\alpha}{\bigcirc}\xrightarrow{x}\underset{\beta}{\bigcirc}\xrightarrow{y_1}\underset{\beta_1}{\bigcirc}\xrightarrow{y_2}\underset{\beta_2}{\bigcirc}\cdots\underset{\beta_{t-1}}{\bigcirc}\xrightarrow{y_t}\underset{\alpha_0}{\bigcirc}$ is a path from α_0 to α_0 in $\mathcal{A}$ and all of its edges except $e = \underset{\alpha}{\bigcirc}\xrightarrow{x}\underset{\beta}{\bigcirc}$ are in T. Without loss of generality we may assume that $x_1 \neq y_t^{-1}$ (i.e. the word $u = x_1x_2 \ldots x_kxy_1 \ldots y_t$ is cyclically reduced): for if i is the smallest integer for which $\alpha_i \neq \beta_{t-i}$ then we may choose α_i as our fixed state (and call it α_0). In fact we may also assume without loss of generality that all of the vertices $\alpha_0, \ldots, \alpha_{k-1}, \alpha, \beta, \ldots, \beta_{t-1}$ are distinct: if $\alpha = \beta$ we choose $\alpha_0 = \alpha$; if $\alpha \neq \beta$ we choose i to be the largest integer such that $\alpha_i = \beta_{t-i}$ and then choose $\alpha_0 = \alpha_i$. Thus u labels a circuit from α_0 to α_0 in $\mathcal{A}$. Let $V(e) = \{\alpha_0, \ldots, \alpha_{k-1}, \alpha, \beta, \beta_1, \ldots, \beta_{t-1}\}$ be the set of vertices in this circuit. We form a new X-automaton $\mathcal{A}_1$ as follows. The set $V(\mathcal{A}_1)$ of *vertices* of $\mathcal{A}_1$ is $V(\mathcal{A}_1) = V(\mathcal{A}) \cup V_1 \cup V_2 \cup \cdots \cup V_{n-1} \cup \{\alpha_0^n\}$ where (for $i = 1, \ldots, n-1$), $V_i = \{\gamma^i : \gamma \in V(e)\}$ is a set of $k+t+1$ elements in one-one correspondence with $V(e)$, and the sets $V(\mathcal{A})$, $V_1, \ldots, V_{n-1}$, $\{\alpha_0^n\}$ are mutually disjoint. The (labeled) *edges* of $\mathcal{A}_1$ are described in the following way. $\mathcal{A}_1$ contains all of the edges $\underset{\gamma}{\bigcirc}\xrightarrow{y}\underset{\delta}{\bigcirc}$ in $\mathcal{A}$ for $\gamma, \delta \in V(\mathcal{A})$ *except* the edge $\underset{\beta_{t-1}}{\bigcirc}\xrightarrow{y_t}\underset{\alpha_0}{\bigcirc}$, which is replaced

https://doi.org/10.4153/CJM-1990-058-9 Published online by Cambridge University Press

by the edge $\underset{\beta_{t-1}}{\bigcirc}\xrightarrow{y_t}\underset{\alpha_0^1}{\bigcirc}$: in addition, $\mathcal{A}_1$ also contains all edges in the path from α_0^1 to α_0^n labeled by u^{n-1} as below:



The initial (terminal) state of $\mathcal{A}_1$ are the same as in $\mathcal{A}$. Essentially, the automaton (graph) of $\mathcal{A}_1$ is obtained from $\mathcal{A}$ by "unwrapping" the circuit labeled by u from α_0 to α_0 into the path $C(x)$ labeled by u^n from α_0 to α_0^n.

From the construction of the automaton $\mathcal{A}_1$ it is evident that $\mathcal{A}_1 \overset{*}{\Longrightarrow} \mathcal{A}$ relative to the presentation (2) since the path $C(x)$ is transformed into the circuit labeled by u from α_0 to α_0 and all other edges are left unchanged. Since $\mathcal{A}$ is injective and u is cyclically reduced, it follows that $\mathcal{A}_1$ is injective: it is clear that $\mathcal{A}_1$ is a finite, trim X-automaton with one initial state and one terminal state, so $\mathcal{A}_1$ is an inverse X-automaton. A spanning tree T_1 for $\mathcal{A}_1$ is obtained from the spanning tree T for $\mathcal{A}$ by taking in T_1 all the edges of the path $C(x)$ and all edges of T other than the edge $\underset{\beta_{t-1}}{\bigcirc}\xrightarrow{y_t}\underset{\alpha_0}{\bigcirc}$ (which is not in $\mathcal{A}_1$). The fact that T_1 is a tree follows since T and $C(x)$ are trees and any path in $T \cup C(x)$ from a vertex in $T - C(x)$ to a vertex in $C(x) - T$ must contain the edge $\underset{\alpha}{\bigcirc}\xrightarrow{x}\underset{\beta}{\bigcirc}$. It is clear that an edge f is in $\mathcal{A}_1 - T_1$ if and only if f is in $\mathcal{A} - T$ and $f \neq e$. Hence $\beta(\mathcal{A}_1) = \beta(\mathcal{A}) - 1$. By the induction hypothesis there is a word $w \in (X \cup X^{-1})^+$ such that $MT(w) \overset{*}{\Longrightarrow} \mathcal{A}_1$: since $\mathcal{A}_1 \overset{*}{\Longrightarrow} \mathcal{A}$ we have $MT(w) \overset{*}{\Longrightarrow} \mathcal{A}$, as required.

Remarks. (1) Some structural properties of the semigroup $C_n(X)$ follow easily from the results just established. For example $C_n(X)$ is combinatorial (well-known) and completely semisimple (with finite $\mathcal{D}$-classes) since all of its Schützenberger graphs are finite. The semigroup $C_n(X)$ is infinite if $n \geqq 2$ and $|X| \geqq 2$: this follows easily from the fact that there is an infinite square-free word on X (see, for example Lothaire [**6**]), all of whose finite prefixes must lie in different $\mathcal{D}$-classes. The Green's relations on $C_n(X)$ are easily calculated since the Schützenberger graphs are known. Corresponding structural properties for the relatively free semigroups in the variety of *semigroups* defined by the laws $x^n = x^{n+1}$ seem much more difficult to obtain, although it has long been known that these free objects on at least two generators are infinite (see, for example Lallement [**5**]): as far as we are aware it is not known how to solve the word problem for these objects.

(2) The methods developed above may also be used to solve the word problem for the relatively free objects in the variety of inverse semigroups defined by

the laws $u^n = u^{n+k}$ for $k \leqq n$. Once again, the Schützenberger automata are all finite: a path labeled by u^n for some word $u \in (X \cup X^{-1})^+$ reduces to a circuit labeled by u^k after the appropriate expansions and reductions are applied.

Similar methods may be used to study the semigroup $B(X)$ relative to the presentation (1) above. In this case a somewhat sharper condition than P_n may be obtained.

THEOREM 2.3. *Let X be a non-empty set, w a word in $(X \cup X^{-1})^+$ and $\mathcal{A}(w)$ the Schützenberger automaton of w relative to the presentation* (1) *of $B(X)$. Then $\mathcal{A} = \mathcal{A}(w)$ is a finite inverse X- automaton that satisfies the condition*

(P) *There is at most one edge in $\mathcal{A}$ labeled by each letter in $X \cup X^{-1}$.*

In fact if $\mathcal{A}$ is a finite, inverse X-automaton, then $\mathcal{A}$ is the Schützenberger automaton $\mathcal{A}(w)$ for some word $w \in (X \cup X^{-1})^+$, relative to the presentation (1) *for $B(X)$, if and only if $\mathcal{A}$ satisfies P. In particular, the word problem for $B(X)$ is decidable.*

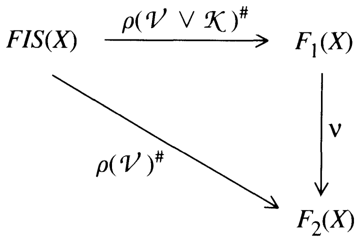
Proof. Since each relation in the presentation (1) is of the form $z^2 = z$ (for $z = uvu^{-1} \in (X \cup X^{-1})^+$) it follows as a special case of the argument used in the proof of Theorem 2.1 that each Schützenberger graph is finite. (Of course this is clear a priori if X is finite since the relatively free objects on a finite set in any variety of semigroups generated by a single finite semigroup are well-known to be finite—see, for example, Grätzer [**2**]). We next show that a finite, inverse X-automaton $\mathcal{A}$ is closed with respect to the presentation (1) if and only if $\mathcal{A}$ satisfies condition (P). Suppose first that $\mathcal{A}$ is closed. Given any two edges $\gamma_1 \xrightarrow{y} \delta_1$ and $\gamma_2 \xrightarrow{y} \delta_2$, let v label any path from δ_1 to γ_2 and then note that $y(vy)y^{-1}$ labels a path from γ_1 to γ_2. Since $\mathcal{A}$ is closed there is also a path from γ_1 to γ_2 labeled by $[y(vy)y^{-1}]^2$ and so, as in the proof of Theorem 2.1, $\gamma_1 = \gamma_2$. Since $\mathcal{A}$ is deterministic it follows that the edges $\gamma_1 \xrightarrow{y} \delta_1$ and $\gamma_2 \xrightarrow{y} \delta_2$, coincide, so $\mathcal{A}$ satisfies (P). Conversely, if $\mathcal{A}$ is not closed then there exist vertices γ, δ in $\mathcal{A}$ and words $w_1, w_2 \in (X \cup X^{-1})^+$ such that $w_1 w_2 w_1^{-1}$ labels a path from γ to δ but $(w_1 w_2 w_1^{-1})^2$ labels no such path. This implies that $\gamma \neq \delta$. Let $w_1 = yw'$ where $y \in X \cup X^{-1}$ and note that there exist vertices γ_1, δ_1 in $\mathcal{A}$ such that $\gamma \xrightarrow{y} \gamma_1$ and $\delta_1 \xrightarrow{y^{-1}} \delta$, are edges in $\mathcal{A}$. Since $\gamma \neq \delta$ we see that $\mathcal{A}$ has two distinct edges $\gamma \xrightarrow{y} \gamma_1$ and $\delta \xrightarrow{y} \delta_1$ with the same label, so $\mathcal{A}$ does not satisfy (P). It follows that the Schützenberger automata $\mathcal{A}(w)$ (for $w \in (X \cup X^{-1})^+$) satisfy (P) and that the word problem for $B(X)$ is decidable.

Finally, if $\mathcal{A}$ is any finite inverse X-automaton that satisfies (P) and if $\mathcal{A}$ is not a tree, then there exists a vertex α_0 of $\mathcal{A}$ and a cyclically reduced word

$u \in (X \cup X^{-1})^+$ such that u labels a circuit from α_0 to α_0 in $\mathcal{A}$. It follows that uuu^{-1} also labels a path from α_0 to α_0 in $\mathcal{A}$ and so by the method given in the proof of Theorem 2.2, we may "unwrap" the path labeled by uuu^{-1} from α_0 to α_0 into a tree. Notice that the word uuu^{-1} is not cyclically reduced of course, so the circuit labeled by u from α_0 to α_0 is transformed by this process into the tree $\underset{\alpha_0}{\bigcirc} \xrightarrow{u} \underset{\alpha_0^1}{\bigcirc} \xrightarrow{u} \underset{\alpha_0^2}{\bigcirc} \xrightarrow{u} \bigcirc$ in the new automaton $\mathcal{A}_1$: again, this automaton is defined as in the proof of Theorem 2.2 and induction on the first Betti number of the automaton shows that there is some word $w \in (X \cup X^{-1})^+$ such that $MT(w) \overset{*}{\Longrightarrow} \mathcal{A}$ relative to the presentation (1).

Remarks. Theorem 2.3 may be used to give very explicit information about the structure of $B(X)$, for X a finite set. In particular, it is not too difficult to deduce from this a formula for the number of elements in $B(X)$ (for X a finite set), or the number of elements in a $\mathcal{D}$-class etc. These results have also been obtained independently and using quite different methods by Reilly [**11**]: we shall not list these results here, but we refer the reader to Reilly's paper for further details.

3. Joins with groups. Let $\mathcal{V}$ be a variety of inverse semigroups and $\mathcal{K}$ a variety of groups and let $\rho(\mathcal{V})$, $\rho(\mathcal{K})$ be the corresponding fully invariant congruences on $FIS(Y)$ for Y a countably infinite set. Then $\mathcal{V} \vee \mathcal{K}$ is a variety of inverse semigroups determined by the corresponding fully invariant congruence $\rho(\mathcal{V} \vee \mathcal{K}) = \rho(\mathcal{V}) \cap \rho(\mathcal{K})$. If $F_1(X)$ denotes the free object on X in $\mathcal{V} \vee \mathcal{K}$ and $F_2(X)$ denotes the free object on X in $\mathcal{V}$ then $F_1(X) \equiv FIS(X)/\rho(\mathcal{V} \vee \mathcal{K})$ and $F_2(X) \cong FIS(X)/\rho(\mathcal{V})$. Recall (Petrich [**10**]) that the *trace* of a congruence ρ on an inverse semigroup S is defined to be the restriction of ρ to $E(S)$, the semilattice of idempotents of S. It is well-known that if ρ, σ are two congruences on an inverse semigroup S then $\mathrm{tr}(\rho \cap \sigma) = \mathrm{tr}(\rho) \cap \mathrm{tr}(\sigma)$. Hence $\mathrm{tr}(\rho(\mathcal{V} \vee \mathcal{K})) = \mathrm{tr}(\rho(\mathcal{V}) \cap \rho(\mathcal{K})) = \mathrm{tr}(\rho(\mathcal{V})) \cap \mathrm{tr}(\rho(\mathcal{K})) = \mathrm{tr}(\rho(\mathcal{V}))$ since $\rho(\mathcal{K})$ is a group congruence. If follows that there is an *idempotent-separating* homomorphism ν from $F_1(X)$ onto $F_2(X)$ such that the diagram



commutes.

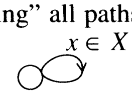
It is immediate that the word problem for $F_1(X)$ is decidable if the word problem for $F_2(X)$ is decidable and $\rho(\mathcal{K})$ is a decidable congruence. In particular if

https://doi.org/10.4153/CJM-1990-058-9 Published online by Cambridge University Press

$\mathcal{K} = \mathcal{G}$ (the variety of all groups) then $\rho(K) = \sigma$, the minimum group congruence on $FIS(X)$ and in this case $\rho(\mathcal{V} \vee \mathcal{G}) = \rho(\mathcal{V}) \cap \sigma = \rho(\mathcal{V})_{\min}$, the smallest congruence on $FIS(X)$ with the same trace as $\rho(\mathcal{V})$ (see Petrich [**10**] for further information about the congruence $\rho_{\min}$ associated with a congruence ρ on an inverse semigroup S). Clearly the word problem for the relatively free objects in $\mathcal{V} \vee \mathcal{G}$ is decidable if the corresponding word problem for the relatively free objects in $\mathcal{V}$ is decidable. In this section we examine the relationship between the Schützenberger graphs associated with $F_1(X)$ and the corresponding Schützenberger graphs associated with $F_2(X)$ and interpret the former geometrically as covering spaces of the latter.

We first recall some notions from Higgins [**3**] (see also Stallings [**13**]) which are analogues of corresponding standard ideas in algebraic topology. The *graphs* that we are interested in here are the underlying graphs of an inverse X-automaton. Thus they are graphs in the sense of Serre [**12**] with edges labeled from $X \cup X^{-1}$ and with the inverse e^{-1} of the edge $e : \underset{\alpha}{\bigcirc}\xrightarrow{\;x\;}\underset{\beta}{\bigcirc}$ being interpreted as the edge $\underset{\alpha}{\bigcirc}\xrightarrow{\;x^{-1}\;}\underset{\beta}{\bigcirc}$ for $x \in X \cup X^{-1}$. The initial [terminal] vertex of this edge e is $\alpha(e) = \alpha[\omega(e) = \beta]$. For each vertex α of such a graph Γ we denote by $\mathrm{Star}_\alpha(\Gamma)$ the set of edges of Γ with initial vertex α, i.e.

$$\mathrm{Star}_\alpha(\Gamma) = \{e \in E(\Gamma) : \alpha(e) = \alpha\}.$$

If $f : \Gamma' \longrightarrow \Gamma$ is a (labeled) graph morphism and $\alpha' \in V(\Gamma')$ then there is an associated function $f_{\alpha'} : \mathrm{Star}_{\alpha'}(\Gamma') \longrightarrow \mathrm{Star}_{\alpha' f}(\Gamma)$ that maps the edge $\underset{\alpha'}{\bigcirc}\xrightarrow{\;x\;}\underset{\beta'}{\bigcirc}$ of Γ' to the edge $\underset{\alpha' f}{\bigcirc}\xrightarrow{\;x\;}\underset{\beta' f}{\bigcirc}$ of Γ. A *covering* of the graph Γ is a pair (Γ', f) where Γ' is a graph and $f : \Gamma' \longrightarrow \Gamma$ is a graph morphism of Γ' onto Γ such that the induced maps $f_{\alpha'} : \mathrm{Star}_{\alpha'}(\Gamma') \longrightarrow \mathrm{Star}_{\alpha' f}(\Gamma)$ for $\alpha' \in V(\Gamma')$ are all one-one and onto. If $f : \Gamma' \longrightarrow \Gamma$ is a covering and $\alpha' \in V(\Gamma')$ then f induces a group embedding $f_* : \pi_1(\Gamma', \alpha') \longrightarrow \pi_1(\Gamma, \alpha' f)$ from the fundamental group of Γ' based at α' into the fundamental group of Γ based at $\alpha' f$. This sets up a correspondence between coverings of a connected graph Γ and subgroups of the fundamental group $\pi_1(\Gamma)$. It is well-known (Higgins [**3**], Stallings [**13**]) that this correspondence is one-one: thus coverings of a (connected) graph Γ are classified by subgroups of the fundamental group of Γ. In particular, if $f : \Gamma' \longrightarrow \Gamma$ is the covering of Γ associated with the *trivial* subgroup $\{1\}$ of $\pi_1(\Gamma)$, then Γ' is called the *universal covering space* of Γ. Clearly if $f : \Gamma' \longrightarrow \Gamma$ is a covering of Γ then Γ' is the universal covering space of Γ if and only if Γ' is a tree: the universal covering space of Γ is the space (graph) of paths of Γ, obtained by "unwrapping" all paths of Γ into chains. For example, if Γ is the bouquet of $|X|$ circles Γ : 

, then the universal covering space of Γ is the tree of the free group $FG(X)$ on X (i.e. the Cayley graph of $FG(X)$ relative to the usual presentation for $FG(X)$).

The notion of coverings of graphs provides a convenient geometric description of the relationship between the Schützenberger graphs of two inverse semigroups that are connected by means of an idempotent-separating morphism.

THEOREM 3.1. *Let $S_1 = \mathrm{Invs}\langle X : T_1\rangle = FIS(X)/\tau_1$ and $S_2 = \mathrm{Invs}\langle X : T_2\rangle = FIS(X)/\tau_2$ be two inverse semigroups with the same set X of generators and suppose that $\tau_2 \supseteq \tau_1$. Then the natural morphism $f : S_1 \twoheadrightarrow S_2$ of S_1 onto S_2 is idempotent-separating if and only if, for each word $w \in (X \cup X^{-1})^+$, the natural graph morphism $f_w : S\Gamma(X, T_1, w) \twoheadrightarrow S\Gamma(X, T_2, w)$ of the Schützenberger graph $S\Gamma(X, T_1, w)$ onto the corresponding Schützenberger graph $S\Gamma(X, T_2, w)$ is a covering.*

Proof. Note that f_w is the graph morphism that takes the edge $\underset{u\tau_1}{\bigcirc}\xrightarrow{x}\underset{(ux)\tau_1}{\bigcirc}$ (for $u \in FIS(X)$) of $S\Gamma(X, T_1, w)$ to the edge $\underset{u\tau_2}{\bigcirc}\xrightarrow{x}\underset{(ux)\tau_2}{\bigcirc}$ of $S\Gamma(X, T_1, w)$. (This is well-defined since $\tau_2 \supseteq \tau_1$.) Suppose first that f is idempotent-separating, so that $\mathrm{tr}(\tau_1) = \mathrm{tr}(\tau_2)$. If $\underset{u\tau_2}{\bigcirc}\xrightarrow{x}\underset{(ux)\tau_2}{\bigcirc}$ is an edge in $S\Gamma(X, T_2, w)$ for some $u \in FIS(X)$, then $(uu^{-1})\tau_2 = (uxx^{-1}u^{-1})\tau_2 = (ww^{-1})\tau_2$, so $(uu^{-1}, uxx^{-1}u^{-1}), (uu^{-1}, ww^{-1}) \in \mathrm{tr}(\tau_1) = \mathrm{tr}(\tau_2)$, so $(uu^{-1})\tau_1 = (uxx^{-1}u^{-1})\tau_1 = (ww^{-1})\tau_1$. This implies that $\underset{u\tau_1}{\bigcirc}\xrightarrow{x}\underset{(ux)\tau_1}{\bigcirc}$ is an edge in $S\Gamma(X, T_1, w)$. Since this edge maps onto $\underset{u\tau_2}{\bigcirc}\xrightarrow{x}\underset{(ux)\tau_2}{\bigcirc}$ under f_w, it follows that f_w is onto. If e_1 and e_2 are two edges of $S\Gamma(X, T_1, w)$ with $\alpha(e_1) = \alpha(e_2)$ and $e_1 f_w = e_2 f_w$ and if e_1 is the edge $\underset{u\tau_1}{\bigcirc}\xrightarrow{x}\underset{(ux)\tau_1}{\bigcirc}$ and e_2 is the edge $\underset{v\tau_1}{\bigcirc}\xrightarrow{y}\underset{(vy)\tau_1}{\bigcirc}$, then we must have $u\tau_1 = v\tau_1$ and $x = y$, so $e_1 = e_2$. Hence f_w is one-one and so f_w is a covering.

Conversely suppose that each $f_w (w \in (X \cup X^{-1})^+)$ is a covering. Let $u = x_1 \dots x_n$, $v = y_1 \dots y_m \in (X \cup X^{-1})^+$ be such that $u\tau_1$, $v\tau_1$ are idempotents of S_1 and $(u\tau_1)f = (v\tau_1)f$; i.e. $u\tau_2 = v\tau_2$. Since $u\tau_2 = v\tau_2$ is an idempotent in S_2 we have $u\tau_2 = v\tau_2 = (uv)\tau_2$. Hence $(uy_1)\tau_2 \mathcal{R}\, u\tau_2$ in S_2 since $(uy_1)\tau_2(y_2 \dots y_m)\tau_2 = u\tau_2$. It follows that $\underset{u\tau_2}{\bigcirc}\xrightarrow{y_1}\underset{(uy_1)\tau_2}{\bigcirc}$ is an edge in $S\Gamma(X, T_2, u)$. Similar arguments show that

$$\underset{u\tau_2}{\bigcirc}\xrightarrow{y_1}\underset{(uy_1)\tau_2}{\bigcirc}\xrightarrow{y_2}\underset{(uy_1y_2)\tau_2}{\bigcirc}\xrightarrow{y_3}\cdots\ \underset{(uy_1\dots y_{m-1})\tau_2}{\bigcirc}\xrightarrow{y_m}\underset{(uv)\tau_2=u\tau_2}{\bigcirc}$$

is in fact a path in $S\Gamma(X, T_2, u)$. Since $f_u : S\Gamma(X, T_1, u) \longrightarrow S\Gamma(X, T_2, u)$ is a covering it follows that this path lifts to a path

$$\underset{u\tau_1}{\bigcirc}\xrightarrow{y_1}\underset{(uy_1)\tau_1}{\bigcirc}\xrightarrow{y_2}\underset{uy_1y_2)\tau_1}{\bigcirc}\xrightarrow{y_3}\cdots\underset{(uy_1\dots y_{m-1})\tau_1}{\bigcirc}\xrightarrow{y_m}\underset{(uv)\tau_1}{\bigcirc}$$

in $S\Gamma(X, T_1, u)$, and hence $(u\tau_1)(v\tau_1)\mathcal{R}\, u\tau_1$ in S_1. Hence $(u\tau_1)(v\tau_1) = u\tau_1$: similarly $(v\tau_1)(u\tau_1) = v\tau_1$, so $u\tau_1 = (u\tau_1)(v\tau_1) = (v\tau_1)(u\tau_1) = v\tau_1$, whence f is idempotent-separating.

COROLLARY 3.2. *Let $\mathcal{V}$ be a variety of inverse semigroups and $F_{\mathcal{V}}(X)$ [resp. $F_{\mathcal{V}\vee\mathcal{G}}(X)$] the relatively free (inverse) semigroup on the set X in $\mathcal{V}$ [resp. $\mathcal{V} \vee \mathcal{G}$]. For each word $w \in (X \cup X^{-1})^+$ let $S\Gamma(X, \rho(\mathcal{V}), w)$ and $S\Gamma(X, \rho(\mathcal{V} \vee \mathcal{G}), w)$ denote the Schützenberger graphs of w relative to the presentation $(X, \rho(\mathcal{V}))$ [resp. $(X, \rho(\mathcal{V} \vee \mathcal{G}))$] of $F_{\mathcal{V}}(X)$ [resp. $F_{\mathcal{V}\vee\mathcal{G}}(X)$]. Then $S\Gamma(X, \rho(\mathcal{V} \vee \mathcal{G}), w)$ is the universal covering space of $S\Gamma(X, \rho(\mathcal{V}), w)$ for each $w \in (X \cup X^{-1})^+$.*

Proof. Since the natural map f from $F_{\mathcal{V}\vee\mathcal{G}}(X)$] onto $F_{\mathcal{V}}(X)$ is idempotent-separating, it follows that each graph morphism $f_w : S\Gamma(X, \rho(\mathcal{V} \vee \mathcal{G}), w) \twoheadrightarrow S\Gamma(X, \rho(\mathcal{V}), w)$ is a covering, by Theorem 3.1. Now $\rho(\mathcal{V} \vee \mathcal{G}) = \rho(\mathcal{V})_{\min} = \rho(\mathcal{V}) \cap \sigma$ and so the natural morphism from $FIS(X)$ onto $F_{\mathcal{V}\vee\mathcal{G}}(X) = FIS(X)/\rho(\mathcal{V})_{\min}$ is idempotent-pure since $FIS(X)$ is E-unitary (see Petrich [**10**]). It follows from Lemma 1.4 of Margolis and Meakin [**8**] that each Schützenberger graph $S\Gamma(X, \rho(\mathcal{V} \vee \mathcal{G}), w)$ is a tree and so $S\Gamma(X, \rho(\mathcal{V} \vee \mathcal{G}), w)$ is the universal covering space of $S\Gamma(X, \rho(\mathcal{V}), w)$.

Remarks. (1) Since the universal covering space of a graph Γ is the space of paths of Γ, it may be effectively constructed from Γ if there is an effective procedure for deciding, given a vertex $\alpha \in V(\Gamma)$ and a word $u \in (X \cup X^{-1})^*$, whether there is a path in Γ labelled by u and starting at α. It follows that Corollary 3.2 and Theorems 2.1 and 2.3 immediately yield solutions to the word problem for the relatively free objects in the varieties $\mathcal{B} \vee \mathcal{G}$ and $\mathcal{C}_n \vee \mathcal{G}$. In fact, if we interpret the Schützenberger automaton of a word u relative to a presentation of an inverse monoid M as a "canonical form" for u in M, then these results provide us with "canonical forms" for words in these cases.

(2) The results of Theorems 2.1, 2.2 and 2.3 were announced briefly by one of the authors at a semigroups conference in Szeged, Hungary (August 1987) and in more detail at a special session on semigroups at the October, 1987 meeting of the American Mathematical Society in Lincoln, Nebraska, U.S.A. The results of section 3 of the paper were announced at a special session on semigroups at the December, 1988 meeting of the Canadian Mathematical Society in Toronto, Canada.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math Surveys No. 7, Amer. Math. Soc. Providence, Vol. I (1961), Vol. II (1967).

2. G. Grätzer, *Universal algebra* 2nd ed., Springer (1979).

3. P. Higgins, *Categories and groupoids*, Van Nostrand-Reinhold, New York, 1971.

4. J. E. Hoproft and J. D. Ullman, *Formal languages and their relation to automata*, Addison-Wesley, 1969.

5. G. Lallement, *Semigroups and combinatorial applications*, Wiley, 1979.

6. M. Lothaire, *Combinatorics on words*, Encyclopedia of mathematics and its applications, Addison-Wesley, 1983.

7. S. Margolis, J. Meakin and J. Stephen, *Some decision problems for inverse monoid presentations*, in "Semigroups and their applications", Goberstein, Higgins, (Ed). D. Reidel (1987), 99–110.

8. S. Margolis and J. Meakin, *Inverse monoids, trees and context-free languages*, Trans. Amer. Math. Soc. (to appear).

9. J. Meakin, *Automata and the word problem*, in "Formal Properties of Finite Automata and Applications", Lecture Notes in Computer Science 386, J. E. Pin, (Ed). Springer-Verlag (1989), 89–103.

10. M. Petrich, *Inverse semigroups*, Wiley Interscience, 1984.

11. N. R. Reilly, *Free combinatorial strict inverse semigroups*, J. Lond. Math. Soc. (2) *39* (1989), 102–120.

12. J. P. Serre, *Trees*, Springer-Verlag, 1980.

13. J. Stallings, *Topology of finite graphs*, Inv. Math. *71* (1983), 551–565.

14. J. B. Stephen, *Presentations of inverse monoids*, J. Pure Appl. Algebra *63* (1990) no. 1, 81–112.

15. J. B. Stephen, *Contractive presentations of inverse monoids*, preprint.

Department of Computer Science
Department of Mathematics and Statistics
University of Nebraska-Lincoln
Lincoln, Nebraska 68588, U.S.A.

Department of Mathematics
Northern Illinois University
DeKalb, Illinois 60615, U.S.A.