THERMAL STABILITY OF NONSPHERICAL STARS AND THE SOLAR OBLATENESS PROBLEM

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Abstract. A brief account of the stability of distorted stars is given. From this appropriate criteria emerge, which are subsequently applied to rotational models of the sun in connection with the solar oblateness problem.

1. Stability of Distorted Stars

If we are not interested in perturbation time scales longer than the thermal diffusion time scale, we may neglect the weaker dissipative effects as viscous or material diffusion in the linear stability analysis. Then we get a polynomial of the third degree for a dimensionless growth rate:

$$\sigma^3 + \frac{\tau_h}{\tau_r}\sigma^2 + A_1\sigma + A_0\frac{\tau_h}{\tau_r} = 0. \tag{1}$$

Here $\tau_h = |r/g_{eff}|^{1/2}$ is the local hydrodynamic time scale with $\mathbf{g}_{eff}$ the acceleration of gravity modified by the distorting force and r the distance from the centre; τ_r is the local thermal relaxation time scale, which is of the order of $\tau_{KH} L^2/R^2$, where τ_{KH} is the Kelvin-Helmholtz time scale of the entire star of radius R, and L is the size of the disturbance. $1/\tau_h$ is used as scaling factor of the growth rate. The coefficients A_1 and A_0 are of the order of 1 and λ respectively, λ being the local fraction of the distortion of the star. The third degree in σ of Equation (1) is obtained from the second order time derivatives in the momentum equation and from the entropy change rate in the energy balance for nonadiabatic processes. From Equation (1), the Hurwitz criteria provide the following three conditions

$$A_1 \geqslant 0, \tag{2a}$$

$$A_1 \geqslant A_0, \tag{2b}$$

$$A_0 \geqslant 0. \tag{2c}$$

Inequalities (2) must be satisfied simultaneously in order to insure stability. They are the necessary conditions for dynamical, vibrational and secular stability respectively. This may be verified intuitively by interchanging material elements along an arbitrary path $\mathbf{l}$ in the meridional plane (axially symmetric perturbations) in an adiabatic, quasiadiabatic or diabatic manner. If the translation energy of the elements is increased by one of these processes, the system is unstable with respect to one of these three modes of instability.

In vibrationally stable layers, Equation (2b) prevents overstable convection. Equation (2) then tells us, that in vibrationally stable layers the secular stability condition

A. Slettebak (ed.), Stellar Rotation, 340–345. *All Rights Reserved*

is always stronger than the dynamic one. The explanation is simply that the stabilizing effect of buoyancy forces caused by temperature differences is wiped out if the motions are slow enough to permit an appreciable amount of heat exchange. Each of the conditions (2) has in general the form

$$a_i q^2 + b_i q + c_i \geqslant 0\,, \quad i = 1, 2, 3\,, \tag{3}$$

where $q = l_\theta / l_r$ is the ratio of the tangential and the radial component of the displacement vector of the two elements mentioned above. Equation (3) then splits into two conditions

$$c_i \geqslant 0 \quad \text{and} \quad 4a_i c_i - b_i^2 \geqslant 0\,, \quad i = 1, 2, 3\,, \tag{4}$$

because q may vary arbitrarily. The first set of criteria corresponds to radial perturbations ($q=0$) and the second one to non-radial perturbations ($q \neq 0$).

We now consider the special case of a rotating star having a spherical distribution of angular velocity $\Omega(r)$ and mean molecular weight $\mu(r)$. This case is relevant for a subsequent application to the sun. The rotation rate shall be slow, i.e. $\lambda = \Omega^2 r / |g_{\text{eff}}| \ll 1$. We obtain the following scheme for the six stability criteria valid in the equatorial plane:

mode	vibrational	dynamical	secular
radial	$\nabla_{\text{ad}} - \nabla_T \geqslant 0$	$\nabla_{\text{ad}} - \nabla_T + \nabla_\mu - 2\lambda\nabla_j \geqslant 0$	$\nabla_\mu - 2\lambda\nabla_j \geqslant 0$
non-radial	$-\lambda^2 K^2 \geqslant 0$	$\nabla_{\text{ad}} - \nabla_T + \nabla_\mu - 2\lambda\nabla_j - \lambda^2 C^2 \geqslant 0$	$\nabla_\mu - 2\lambda\nabla_j - \lambda^2 C^2 \geqslant 0$

(5)

where $j = \Omega r^2$, $\nabla_x = \mathrm{d}\ln x / \mathrm{d}\ln p$, $C = \frac{1}{2}(r/h)^{1/2}\,\nabla_J$, $J = \Omega r^4$, $\lambda\, K = |\mathbf{g}_{\text{eff}} \times \mathbf{grad}\, T|$;p, T, h, ∇_{ad} denote pressure, temperature, pressure scale height and adiabatic temperature gradient respectively.

According to the scheme (5) the transition from dynamical to secular modes shows an increase in strength of the respective stability criteria if $\nabla_T \leqslant \nabla_{\text{ad}}$. The same statement holds for the transition from radial to non-radial modes. This is not surprising as the non-radial modes represent more general disturbances than the radial ones. Such relations concerning the strength of stability criteria hold also if effects of magnetic fields are included. Incidently, for non-radial vibrational modes of rotationally distorted stars a general local instability is obtained. This result will not be discussed here. The important conclusion from the above synopsis of the stability conditions is that the 'non-radial secular' criterion provides the strongest restraint for a stellar rotation law. It can be written as

$$\tfrac{1}{4} r^2 \Omega'^2 \leqslant g_{\text{eff}} (\ln \mu)'\,, \tag{6}$$

the prime denotes $\mathrm{d}/\mathrm{d}r$.

For 'normal' composition gradients $\mu' < 0$, the r.h.s. of (6) is positive. If in addition $\Omega' < 0$, the steepest stable Ω-gradient at a certain radius is given by

$$\Omega' = -\frac{2}{r} |g_{\text{eff}}(\ln \mu)'|^{1/2}. \tag{7}$$

In a chemically homogeneous layer with no magnetic field, only uniform rotation can be stable.

2. Discussion of Dicke's Model

According to Dicke and Goldenberg (1964) (hereafter referred to as D.G.) the sun shows an oblateness of $\Delta r/r = (5 \pm 0.7) \times 10^{-5}$, a value which is larger by a factor of 5 than what one should expect if the sun were to rotate rigidly with the observed surface angular velocity $\Omega_s = 2.8 \times 10^{-6}$ rad/sec. In the present Colloquium Professor Dicke has provided evidence against a possible distortion of the solar atmosphere by stresses of velocity or magnetic surface fields and he has argued in favour of the D.G. model of the angular velocity distribution in the sun (this volume, p. 289). In the D.G. model the radiative core of the sun rotates at the high primordial rate of 4×10^{-5} rad/sec, while the outer convection zone and the atmosphere have been spun down to the present surface rotation rate by the solar wind torque. Thus, below the bottom of the convection zone at 0.8 solar radius a thin transition layer of 0.05 solar radius thickness is postulated in which Ω decreases by a factor larger than 10. This model encountered two main objections:

(1) The transition zone cannot be in hydrostatic equilibrium (Howard *et al.*, 1967),

(2) the transition zone must be thermally unstable (Goldreich and Schubert, 1967; Fricke, 1969a).

I shall not consider the onset of an Ekman flow, which transports angular momentum through the transition zone, to be imperative. As Fricke and Kippenhahn (1967) have pointed out, the stellar gas can adjust its temperature distribution to a quasi-hydrostatic state as long as the Eddington-Vogt circulations are small compared to the velocity of sound; the latter still holds in the D.G. model. I consider the instability objection to be the important one. Goldreich and Schubert used only the 'radial secular' condition of the restraints (5) to show the instability of the Ω-transition. Obviously, it is also unstable against secular non-radial motions as Equation (6) (with $\mu = \text{const.}$) shows. Dicke (1967) has proposed the following stabilizing effects: (i) magnetic fields, (ii) velocity fields.

I have investigated secular stability in the presence of toroidal or poloidal magnetic fields (Fricke, 1969a). I found toroidal fields and differential rotation to be separately unstable with respect to non-radial secular perturbations. On the other hand, a mutual stabilization is possible in principle but requires in the case of the D.G. model an unlikely physical situation. The weaker condition for radial motions already demands a field strength of the order of 10^5 Gauss with a field gradient of the order of 1 Gauss/km pointing outward. In addition, Schubert and Fricke (to be published) have shown that rotating stars with toroidal fields are secularly unstable with respect to nonaxisymmetric modes. Poloidal fields cannot suppress the instability of differential rotation against non-radial disturbances, if the rotation is steady. Bearing these results in mind the situation does not appear promising for stability even if a field of a more sophisticated structure is present, since the instabilities are of a local nature.

Dicke argues furthermore that the shear produced by meridional motions may prevent the instability motions from growing, in a similar manner as ocean currents prevent the growth of salt fingers. In stars the Eddington-Vogt circulations may take over this role, although we know very little about this mechanism. These currents have velocities of the order of $10^{-4}\ \lambda$ cm/sec in the solar core and are probably much less than the turbulent motions caused by the instability. Within the D.G. transition zone the currents are faster by a factor of 10^4 because of the Baker-Kippenhahn correction and a correction due to the steep Ω-gradient. There the velocities become of the order of λ cm/sec, where $\lambda \approx 10^{-3}$. However, the latter estimate for the velocity of the currents is not relevant for Dicke's argument. The instability according to Equation (6) operated from the beginning of the spin down process, when no Ω-transition has been present. Thus, I presume that such a transition layer could not have been formed.

3. Stable Models

The question arises whether stable rotation laws can be constructed, which can also account for the solar oblateness. On the basis of the condition (6) for non-radial secular stability I have shown that this is not possible (Fricke, 1969b). The use of the criterion for radial motions led Goldreich and Schubert (1968) to the opposite result.

The maximum possible angular momentum content which can be stably distributed in the sun can be deduced by use of Equation (7). In the absence of a μ-gradient in the sun's interior the solar oblateness would be produced by rigid body rotation with the present value of the surface velocity of 2.8×10^{-6} rad/sec. This gives $\Delta r/r = 1.0 \times 10^{-5}$. If we assume a stabilizing μ-gradient obtained from evolution theory (Schwarzschild, 1958) for the present age of the sun, 5×10^9 yr, and use the present surface velocity, Equation (7) can be integrated to give a unique Ω-distribution, which provides a first upper bound to the solar oblateness from stability arguments. The value thus obtained is 8.5×10^{-5} and is only slightly higher than the observed value of D.G. Actually, the μ-gradient has been evolved from zero to the present value by nuclear burning in the core. In order to take into account this evolution effect, two earlier Ω-distributions for the ages 3×10^7 yr and 4×10^8 yr have been integrated using the corresponding μ-distributions and surface velocities. The latter have been taken from Kraft (1967). The minimum curve resulting from the three calculated Ω-distributions is given in Figure 1. This curve represents an Ω-distribution which could have evolved from an initially constant rotation (taken as the Keplerian angular velocity at the solar equator) by three simultaneous effects: (i) evolution by nuclear burning, (ii) spin-down by the solar wind torque and (iii) angular momentum transport as an effect of the instability. This distribution produces an oblateness of 1.4×10^{-5} only, which is about a factor 4 smaller than the measured value. If the latter is correct, we must conclude that the large measured oblateness indicates an unstable Ω-distribution to be present in the solar core.

This last possibility cannot be excluded completely as long as accurate time scales for the secular instabilities are unknown. The time scale for an unstable Ω-distribution to be maintained must not be smaller than the braking time scale, which is presumably of the order of 10^8 yr. Up to now, no reliable theory is known which could support such a high value for the time scale of the instability.

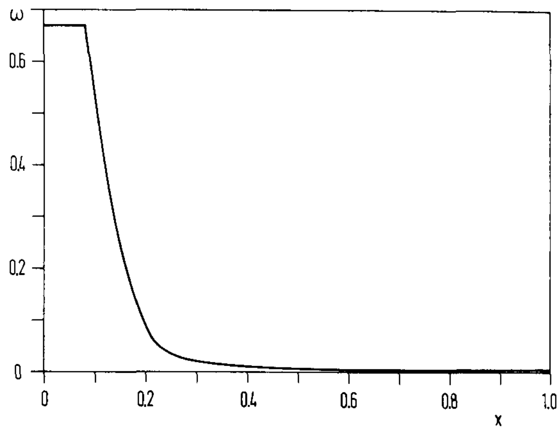

Fig. 1. The limiting angular velocity distribution of the present sun obtained from Equation (7) with regard to the history of the μ-gradient and the surface velocity. $\omega^2 = \Omega^2 R^3/GM$, where M and R are the solar mass and radius and G is the constant of gravitation; x is the relative radius r/R.

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Discussion

Roxburgh: Could you explain why your angular velocity distribution was constant in the central regions of the sun?

Fricke: I simply assumed that the sun reaches the main sequence with the Keplerian velocity and is initially in a state of rigid body rotation. During the subsequent spin down process the angular velocity is lowered in the outer parts, whereas the central distribution remains unaffected for a long time. Thus, I truncated the increasing distribution inwards at the Keplerian velocity.

Ruben: Dr. F. Krause, of the Central Institute for Astrophysics in Potsdam, asked me to make the following comment:

Using a theory of mean fields in a turbulent conducting medium one can prove the existence of an alternating field dynamo. To get such an alternating field dynamo, it is necessary that the gradient of the angular velocity at the bottom of the hydrogen convection zone be great. At the moment, it is difficult to say exactly how much the angular velocity should change. Probably for the sun the change will be of the order of the difference $\omega_{eq} - \omega_{pole}$. But it cannot be smaller than this. It is to be hoped, that in the future the analysis of the butterfly diagram and related phenomena will give some limits on the structure of the outer solar layers. The theoretical butterfly diagrams have been published in *Astronomische Nachrichten* **291** (1969), 49.

Dicke: I should like to make two comments:

(1) In connection with Dr. Fricke's paper I should like to return to my comment on Monday. A linear stability analysis cannot be carried out without first assuming a model for the stellar interior. Thus one is faced with the disagreeable task of evaluating the stability of all possible models before a general proof of instability is obtained.

(2) The Goldreich-Schubert-Fricke instability leads to the transport of angular momentum, along with the material containing the angular momentum, to the surface of the sun. Kraft's observations show that young G-type stars on the main sequence possess a large amount of angular momentum. Assuming that this angular momentum is transported to the surface implies a strong depletion of beryllium at the surface. Most of the angular momentum of a uniformly rotating solar-type star lies below $r = 0.5$, the radius at which berylium rapidly burns. Apparently depletion of berylium is not observed implying that the original angular momentum has not been transported to the surface from below $r = 0.5$ by this instability.

Fricke: First, I should like to mention that the local stability analysis does not require any specification of the equilibrium model. Conversely, the linear stability criteria prescribe the local properties of thermally stable models. Using these constraints I found (i) none of the proposed models for an oblate sun is compatible with the stability requirements, and (ii) stable models which yield a sufficiently high oblateness cannot be of a simple structure. Thus, the existence of a solar quadrupole moment of the required amount is not presently understood.

Concerning the second point, my opinion is that the depletion of lithium and the sensible deficiency of beryllium in the solar atmosphere (cf. a paper by N. Grevesse in *Solar Physics*, 1968) are favourable to the idea of mixing between core and envelope, although I have not considered the problem quantitatively.