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A CUSP-LIKE FREE-SURFACE FLOW DUE TO A SUBMERGED SOURCE OR SINK

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Abstract

A solution is found for a line source or sink beneath a free surface, at a unique squared Froude number of 12.622.

1. Introduction

What is the flow induced by an isolated steady source or sink beneath a free surface? This simple question does not appear to have a simple answer. If the source is a line source of strength m, in two-dimensional irrotational flow of an incompressible inviscid fluid of infinite depth, and is situated at submergence h beneath the undisturbed level of the free surface under gravity g, then there is only one dimensionless parameter, the (squared) Froude number

$$F^2 = m^2/(gh^3), \tag{1.1}$$

and we might expect to find a solution for every value of F^2.

In fact, the problem cannot be solved without further specification of the nature of the free-surface disturbance immediately above the source. Some previous investigators ([2], [3]) have assumed a stagnation point, and have sought results for small values of F^2. Further studies of this type of flow have been made recently by the present authors, and will be reported elsewhere. A feature of these stagnation-point solutions is the presence of short waves, which steepen as F^2 increases, and these solutions seem to be confined to $F^2 < 4$.

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In the present note, we investigate a quite different type of flow, in which there is no stagnation point anywhere in the flow domain. Instead, the flow 'bifurcates' at a definite point somewhere between the source and the undisturbed free-surface level. The free surface at this point is cusp-like, the tip of the cusp pointing toward the source. Similar behaviour has been assumed in studies of stratified fluids of finite depth, surveyed by Imberger [1] and by Yih [4].

The problem is first given a mathematical formulation in which the task is reduced to that of finding a set of real coefficients b_j of an infinite series. These coefficients are required to be such that the free-surface pressure be equal to atmospheric. If the series is truncated to a finite number of terms, and the free-surface condition enforced at a finite number of points, a set of non-linear algebraic equations can be written down, that in principle enable determination of b_j for any input Froude number.

However, no such solutions appear to be obtainable for a general input Froude number. Instead, we find that the cusp-like solution can be obtained only if (in effect) the Froude number is also included as one of the unknowns of the problem. A numerical procedure with such a feature converged rapidly to the solution shown in Figure 1, whose Froude number is $F^2 = 12.622$, and whose cusp lies at 74.938% of the depth of the source.

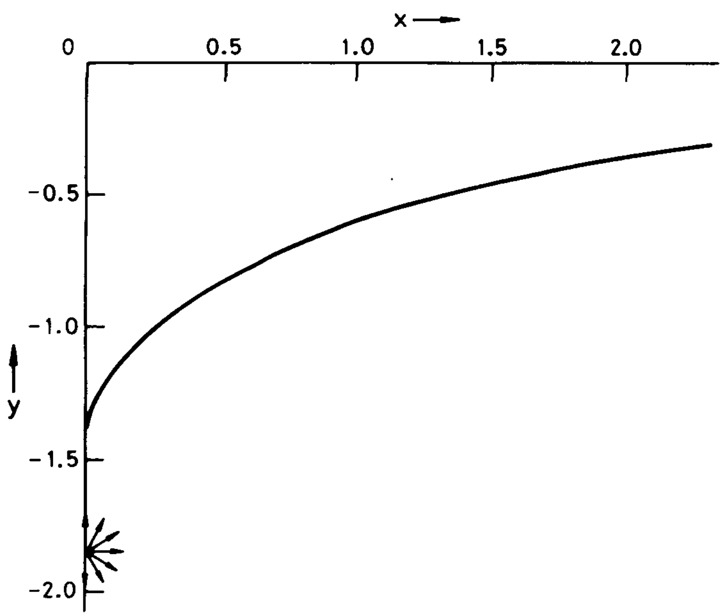

Figure 1. Free-surface shape for flow at $F^2 = 12.622$. The source is at $y = -1.84257$, and the free-surface cusp is at $y = -1.38079$, on the scale of this figure.

2. Mathematical formulation

If $f(z) = \phi(x, y) + i\psi(x, y)$ is a complex velocity potential, and a new complex variable t is defined by

$$e^f = 4t(t+1)^{-2}, \tag{2.1}$$

we represent the physical variable $z = x + iy$ as a series in powers of t, of the form

$$z(t) = -\frac{i}{t+1} \sum_{j=0}^{\infty} b_j t^j, \tag{2.2}$$

for some real coefficients b_j to be determined. The suitability of such a series follows [3] from conformal-mapping considerations. Figure 2 shows the flow region in the z, f, and t-planes.

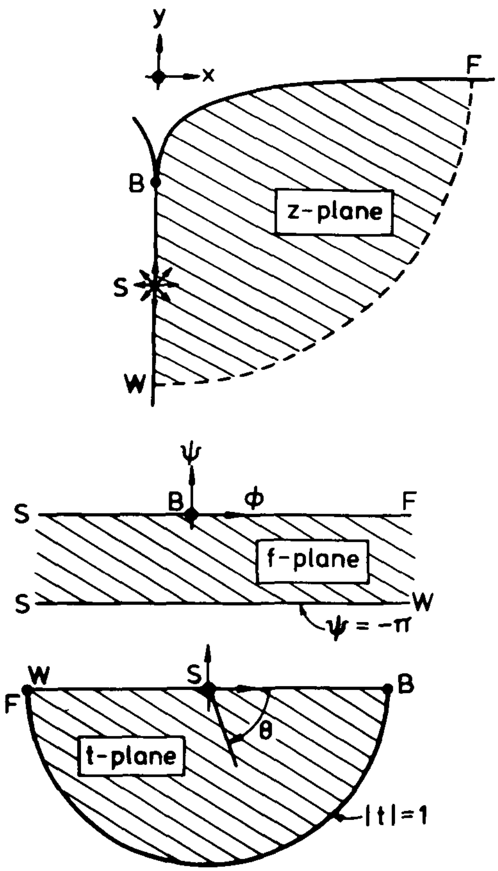

Figure 2. Flow regions in z, f and t-planes.

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The series (2.2) has been designed to satisfy automatically all requirements except for the free-surface condition. Thus, as $t \to 0$, $z \to -ib_0$ and $|f| \to \infty$, with

$$f \to \log(z + ib_0), \qquad z \to -ib_0. \tag{2.3}$$

That is, the origin $t = 0$ corresponds to a source of strength 2π located at the point $z = -ib_0$, and is labelled as the point S in Figure 2.

Similarly, as $t \to -1$, $|z|$ and $|f|$ both become infinite, with

$$f \to 2\log z, \qquad z \to \infty. \tag{2.4}$$

Thus the total flux 2π produced by the source at $z = -ib_0$ is distributed at infinity over a half plane. The flow therefore appears at infinity as if it had been generated by a source of strength 4π.

The right half $x \geqslant 0$ of the flow region maps into the t-plane as the lower half $\operatorname{Im} t \leqslant 0$ of the interior of the unit circle $|t| < 1$, and the free surface (BF in Figure 2) is the semi-circle $t = e^{-i\theta}$, $0 \leqslant \theta \leqslant \pi$. As can be shown from (2.1), ψ vanishes on $t = e^{-i\theta}$. We require that, on this free-surface, the pressure be equal to atmospheric, and this means that

$$y + |f'(z)|^2 = 0. \tag{2.5}$$

Note that the above formulation is a non-dimensional one, in which we have used $m^{2/3}(8\pi^2 g)^{-1/3}$ as the unit of length, and $(mg/\pi)^{1/3}$ as the unit of velocity, where m is the actual (dimensional) source strength, and g the acceleration of gravity. Since the source is located at $z = -ib_0$ in these dimensionless co-ordinates, if it is located at $z = -ih$ in dimensional co-ordinates, we must have

$$F^2 = 8\pi^2 b_0^{-3}. \tag{2.6}$$

Now if the free-surface condition (2.5) is transformed into the t variable, we have

$$y + \left|\frac{t-1}{t+1}\right|^2 \left|\frac{dz}{dt}\right|^{-2} = 0, \qquad t = e^{-i\theta}, \tag{2.7}$$

and, upon substitution of the series (2.2) into (2.7), we find

$$P(\theta; b_j) = 0, \qquad 0 \leqslant \theta \leqslant \pi, \tag{2.8}$$

where

$$P(\theta; b_j) = Y(\theta) + 4\sin^2\theta\left[A^2(\theta) + B^2(\theta)\right]^{-1}, \tag{2.9}$$

and

$$Y(\theta) = \sum_{j=0}^{\infty} b_j\left[-\tfrac{1}{2}\cos j\theta - \tfrac{1}{2}\tan\tfrac{1}{2}\theta \cdot \sin j\theta\right], \tag{2.10}$$

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$$\begin{Bmatrix} A(\theta) \\ B(\theta) \end{Bmatrix} = \sum_{j=0}^{\infty} b_j\left[(j-1)\begin{Bmatrix} \cos \\ \sin \end{Bmatrix} j\theta + j\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}(j-1)\theta\right]. \tag{2.11}$$

The problem has thus reduced to that of choosing a set of real coefficients b_j, such that a certain function P, physically identifiable with the free-surface pressure difference, vanishes for all θ in the range $0 \leqslant \theta \leqslant \pi$. This problem is not unlike that of finding Fourier coefficients, but of course is rendered much more difficult by the non-linear dependence of P on b_j.

Certain constraints on the coefficients b_j follow immediately from the end values $\theta = 0, \pi$. Thus, in order that $P = 0$ at $\theta = \pi$, *i.e.* that the free surface condition be satisfied at physical infinity, we must have

$$\sum_{j=0}^{\infty} (-1)^j(2j-1)b_j = 0. \tag{2.12}$$

At the other end $\theta = 0$ of the range, which corresponds to the symmetry plane $x = 0$, the requirement that $P = 0$ can be satisfied if *either*

$$\sum_{j=0}^{\infty} b_j = 0, \tag{2.13}$$

or

$$\sum_{j=0}^{\infty} (2j-1)b_j = 0. \tag{2.14}$$

Thus, as $\theta \to 0$ or $t \to 1$, (2.1) implies that

$$f \to -\tfrac{1}{4}(t-1)^2 + \tfrac{1}{4}(t-1)^3 + O(t-1)^4 \tag{2.15}$$

while

$$z \to z_1 + (t-1)z_1' + \tfrac{1}{2}(t-1)^2 z_1'' + \tfrac{1}{6}(t-1)^3 z_1''' + O(t-1)^4 \tag{2.16}$$

for some (imaginary) Taylor coefficients z_1, z_1', *etc.* Now (2.7) can be satisfied at $t = 1$ only if either z_1 or z_1' is zero. In the former case, where (2.13) holds,

$$f \to -\tfrac{1}{4}z^2/(z_1')^2 + O(z^3) \tag{2.17}$$

as $z \to 0$, so that the origin in the z-plane is a stagnation point. In the latter case, where (2.14) holds,

$$f \to -\tfrac{1}{2}(z - z_1)/z_1'' + K(z - z_1)^{3/2} + O(z - z_1)^2 \tag{2.18}$$

(for some real constant K), which consists of a (vertical) uniform stream together with a "3/2-power" velocity potential, representing bifurcating streamlines at $z = z_1$.

3. Numerical solution

If we force (2.8) to hold at some discrete set of values $\theta = \theta_k$, $k = 1, 2, \ldots, M$, with $\theta_1 > 0$ and $\theta_M < \pi$, and in addition require both (2.12) and (2.14) to hold, there results a set of $M + 2$ non-linear algebraic equations involving the $N + 1$ coefficients b_j, $j = 0, 1, 2, \ldots, N$. Various numerical methods can be used to solve this set of equations, but first we must decide just how many of the coefficients are to be considered as unknown.

If the Froude number F is prescribed, then (2.6) determines the leading coefficient b_0. In principle, it is then possible to treat $b_1, b_2, \ldots, b_N$ as a set of N unknowns. However, all attempts to solve this system with input b_0 failed. There was some indication that with $1.8 < b_0 < 1.9$, success was near, and it was then suspected that a solution might exist only for some *special* Froude number, corresponding to a b_0 value in this range. Therefore, the numerical procedures were modified to allow b_0 to be an unknown rather than an input quantity, and the result was immediate and complete success, with rapid convergence to a solution at $b_0 = 1.84257$, *i.e.* $F^2 = 12.622$.

The actual method used is a Newton iteration, in which, if b_j is an approximation to the desired solution, then a better approximation is $b_j + \delta b_j$, where δb_j is obtained by solving

$$\sum_{j=0}^{N} \delta b_j \cdot \left[\frac{\partial P(\theta_k; b_i)}{\partial b_j} \right] = -P(\theta_k; b_i). \tag{3.1}$$

If we choose $M = N - 1$, (3.1) can be solved subject to the linear constraints (2.12), (2.14) by any standard linear-equation package. It is not difficult to obtain an explicit formula for the matrix element $\partial P / \partial b_j$ by differentiation of (2.8)–(2.11). Uniformly spaced θ_k were found to be satisfactory.

The iteration process can be started with guessed values such as $b_0 = 1.8$, $b_1 = 0.5$, $b_2 = 0.6$, $b_3 = -0.1$, and all other coefficients zero. In practice it was found convenient to start with a low value (say 5) for N, and, once the iteration converged at that N, to use the resulting coefficients as a starting guess for iterations at a higher value of N. Convergence is very rapid at any fixed N, no more than 5 iterations being ever needed to reduce the maximum value of $P(\theta_k; b_j)$ below 10^{-5}.

TABLE 1

N	b_0
5	1.86935
10	1.84223
15	1.84260
20	1.84256
25	1.84257

TABLE 2

j	b_j
0	1.84257
1	0.41325
2	0.55982
3	−0.06731
4	0.01766
5	−0.00613
6	0.00248
7	−0.00111
8	0.00053
9	−0.00027
10	0.00014

Table 1 shows values of b_0 from a run in which N was successively increased in steps of 5. The final value $b_0 = 1.84257$ at $N = 25$ is accurate to at least 5 figures. Table 2 shows the coefficients b_j, $j = 0, 1, 2, \ldots, 10$. The free surface is shown in Figure 1. All computations were carried out on a TRS-80 microcomputer.

4. Conclusion

We have provided here both negative (failure to achieve solutions at general input b_0 values) and positive (success to high accuracy with b_0 as an unknown) numerical evidence that a cusp-like flow exists only for a unique Froude number, close to $F^2 = 12.622$.

Several questions are raised by this conclusion. If this cusped solution exists only at $F^2 = 12.622\ldots$, what happens at $F^2 = 12$ or $F^2 = 13$? The present conclusion relates only to existence of a steady flow, and an obvious but hardly satisfying answer to the above question is that, if the source-like flow is started from rest, a steady state cannot be achieved if $F^2 \neq 12.622\ldots$. But then, what happens *instead* of a steady state?

The present results may be compared with some empirical estimates of Craya, *e.g.* as discussed in [4], pages 192–196. Craya used an exact solution for a submerged 120° sea-mount, assuming that results for infinite water depth can be obtained simply by rescaling this exact solution. The resulting approximation to F^2 is about 9.3, rather than the present exact value of 12.622.

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