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AUTOMORPHISMS OF THE LIE ALGEBRAS W^* IN CHARACTERISTIC 0

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1. **Introduction.** In a recent paper [2] we defined four classes of infinite dimensional simple Lie algebras over a field of characteristic 0 which we called W^*, S^*, H^*, and K^*. As the names suggest, these classes generalize the Lie algebras of Cartan type. A second paper [3] investigates the derivations of the algebras W^* and S^*, and the possible isomorphisms between these algebras and the algebras defined by Block [1]. In the present paper we investigate the automorphisms of the algebras of type W^*. We show that the automorphism group of an algebra A of type W^* is isomorphic to a subgroup of the automorphism group of the associative algebra B of which it is a subalgebra of the derivations. In case A is all derivations of B, then $\operatorname{Aut} A \cong \operatorname{Aut} B$. We also use automorphisms to show that two algebras of type W^* can be isomorphic only if they have the same number of invertible variables and the same number of noninvertible variables.

The definition of the algebras of type W^* is given in Section 2, and much of the terminology is established there. Our basic results on the automorphisms of these algebras is found in Section 3, and some examples are given in Section 4. We also show in Section 4 that the dimension of a torus in the noninvertible part of the algebra is no more than the number of noninvertible elements. Our final result on the isomorphisms of algebras of type W^* and the machinery necessary for this result are in Section 5.

2. **Background.** We begin by giving a definition of an algebra of type W^*. (A more formal definition can be found in [2] or [3].) Let F be a field of characteristic 0, let $x_1, \ldots, x_n$ be n indeterminates or variables over F, and let the integer $k \le n$ be fixed. For each i with $1 \le i \le k$ let Δ_i denote the nonnegative integers, and for each i with $k < i \le n$ let Δ_i denote an additive subgroup of F containing the integers. Let B be the associative algebra over F spanned by all products of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where $\alpha_i \in \Delta_i$ for each i. We can write this element more succinctly as x^α where α is the n-tuple whose i-th component is α_i. Multiplication in B is then given by $x^\alpha x^\beta = x^{\alpha+\beta}$ in this notation. The variables x_i for $1 \le i \le k$ are called *noninvertible*, and those with $k < i \le n$ are called *invertible*.

For $1 \le i \le n$ let ∂_i denote the usual partial derivative with respect to x_i acting on B. We denote by A the set of all elements of the form $\sum_{1 \le i \le n} f_i \partial_i$ where each $f_i \in B$, and A

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is a Lie algebra under the product defined by letting

$$[f_i\partial_i, g_j\partial_j] = f_i\partial_i(g_j)\partial_j - g_j\partial_j(f_i)\partial_i.$$

It is obvious that B acts as on A as a left module, and that A acts on B as a ring of derivations.

The set of algebras A that we have constructed are exactly the algebras of type W^*. In the following, A will always denote a Lie algebra of type W^*, and B will always denote the associated associative algebra. The symbols n and k will always denote respectively the number of variables and the number of noninvertible variables. The symbol ϵ_i will denote the n-tuple with 1 in the i-th position and 0's elsewhere. Thus, $x^{\epsilon_i} = x_i$.

If each Δ_i for $k < i \leq n$ has rank 1, then A is easily seen to be the set of all derivations of B. Suppose, on the other hand, for some ℓ with $k < \ell \leq n$ that Δ_ℓ has rank > 1, so that we can write $\Delta_\ell = \Delta_{\ell 1} \oplus \Delta_{\ell 2}$ for two nontrivial subgroups $\Delta_{\ell 1}$ and $\Delta_{\ell 2}$. Then $x_\ell^{\alpha_\ell} = x_\ell^{\alpha_{\ell 1}} x_\ell^{\alpha_{\ell 2}}$ where $\alpha_{\ell 1} \in \Delta_{\ell 1}$ and $\alpha_{\ell 2} \in \Delta_{\ell 2}$, and $\alpha_{\ell 1} + \alpha_{\ell 2} = \alpha_\ell$. We have effectively divided x_ℓ into two new variables. The new associative algebra B' on $n + 1$ variables which we get in this way, although isomorphic to B as an algebra, gives rise to another algebra A' of type W^* which properly contains an isomorphic copy of A. In particular, A is not the set of all derivations of B.

3. Relations between the automorphisms of A and of B.

In this section we give the basic results on which everything else is grounded.

THEOREM 3.1. *Let $k \geq 1$ and let ϕ be an automorphism of A. Then there exists an automorphism σ of B such that $\phi(tw) = \sigma(t)\phi(w)$ and $\sigma(wt) = \phi(w)\sigma(t)$ for all $t \in B$ and $w \in A$.*

PROOF. Define $\Psi(t, w) = \phi^{-1}\big(t\phi(w)\big)$ for $t \in B$ and $w \in A$, and note that $\Psi(t, w)$ is linear in both t and w. Then

$$\begin{aligned}
[x_i^\ell\partial_i, \Psi(t,\partial_j)] &= \Big[x_i^\ell\partial_i, \phi^{-1}\big(t\phi(\partial_j)\big)\Big] \\
&= \phi^{-1}\Big(\big[\phi(x_i^\ell\partial_i), t\phi(\partial_j)\big]\Big) \\
&= \phi^{-1}\Big(\big(\phi(x_i^\ell\partial_i)t\big)\phi(\partial_j)\Big) + \phi^{-1}\Big(t\big[\phi(x_i^\ell\partial_i), \phi(\partial_j)\big]\Big) \\
&= \phi^{-1}\Big(\big(\phi(x_i^\ell\partial_i)t\big)\phi(\partial_j)\Big) - \ell\delta_{ij}\phi^{-1}\Big(t\phi(x_j^{\ell-1}\partial_j)\Big).
\end{aligned} \tag{3.2}$$

It is clear from (3.2) that, when we apply the ad's of the different toral elements $x_i\partial_i$ to $\Psi(t,\partial_j)$ and take linear combinations to separate the different homogeneous components of $\Psi(t,\partial_j)$, we are simultaneously separating t into its different homogeneous components with respect to the different toral elements $\phi(x_i\partial_i)$. In particular, the toral elements $\phi(x_i\partial_i)$ decompose t into elements in root spaces with exactly the same set of roots as the decomposition of $\Psi(t,\delta_j)$ induced by the elements $x_i\partial_i$.

We show next that $\Psi(t,\partial_j)$ is an element of B times ∂_j. It is sufficient to show this when t is in a single root space with respect to the torus consisting of the elements $\phi(x_i\partial_i)$. Say

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that, for each i, $\phi(x_i\partial_i)t = a_it$ for $a_i \in F$. If a_i is not a positive integer for some $i \neq j$, there exists $b_i \in \Delta_i$ so that $a_i + b_i$ is a positive integer. Further, we can pick b_i to be different from the exponent of x_i in every term of $\Psi(t, \delta_j)$. In this case, $\Psi(t, \partial_j)$ has the desired form if and only if

$$\begin{aligned}[x_i^{b_i}\partial_i, \Psi(t, \partial_j)] &= \phi^{-1}\Big(\big(\phi(x_i^{b_i}\partial_i)t\big)\phi(\partial_j)\Big)\\ &= \Psi\big(\phi(x_i^{b_i}\partial_i)t, \partial_j\big)\end{aligned}$$

has the desired form. This shows that we can take each a_i for $i \neq j$ to be a nonnegative integer. If any a_i is positive, we see that $[\partial_i, \Psi(t, \partial_j)]$ has the same form, and its value of a_i is one less. Thus it is sufficient to deal with the case where each a_i is 0 for $i \neq j$. But in this case,

$$[\partial_i, \Psi(t, \partial_j)] = \phi^{-1}\Big(\big[\phi(\partial_i), t\phi(\partial_j)\big]\Big) = 0,$$

for $i \neq j$. It follows that $\Psi(t, \partial_j)$ is just a multiple of ∂_j by an element of B for any t.

Then when $w = \partial_j$, we can write $\Psi(t, \partial_j) = \tau(t)\partial_j$. It is clear that τ is one-to-one, and we want to see that it is also onto. But each monomial $s \in B$ is characterized by its set of eigenvalues under the operators $x_i\partial_i$. So choosing $t \in B$ to have the same set of eigenvalues under the operators $\phi(x_i\partial_i)$ as s has under the $x_i\partial_i$'s, we see that $\tau(t)$ is a nonzero multiple of s by (3.2). Thus τ has an inverse σ, and we can write our relation in the form

$$t\partial_j = \phi^{-1}\big(\sigma(t)\phi(\partial_j)\big).$$

Applying ϕ to both sides,

(3.3) $$\phi(t\partial_j) = \sigma(t)\phi(\partial_j).$$

Let σ_j be the function σ determined by using ∂_j as above. We can change variables in B by letting $x_1' = x_1 - x_j$, and $x_i' = x_i$ for $i \neq 1$. Making the change $\partial_j' = \partial_j + \partial_1$ and $\partial_i' = \partial_i$ for $i \neq j$ in A, we arrive at new bases of B and A which act like the original bases. If σ_j' is the function σ going with ∂_j', we obtain

$$\sigma_j(t)\phi(\partial_j) + \sigma_1(t)\phi(\partial_1) = \phi\big(t(\partial_j + \partial_1)\big) = \sigma'(t)\phi(\partial_j + \partial_1),$$

and because of the independence of the B multiples of $\phi(\partial_j)$ and $\phi(\partial_1)$, we see that $\sigma_j = \sigma' = \sigma_1$. We have shown that σ is independent of the subscript j.

From the equation

$$\begin{aligned}\phi^{-1}\big(\phi(\partial_j)\sigma(t)\phi(\partial_j)\big) &= \phi^{-1}\Big(\big[\phi(\partial_j), \sigma(t)\phi(\partial_j)\big]\Big)\\ &= [\partial_j, t\partial_j]\\ &= (\partial_j t)\partial_j\\ &= \phi^{-1}\big(\sigma(\partial_j t)\phi(\partial_j)\big),\end{aligned}$$

we obtain

(3.4) $$\phi(\partial_j)\sigma(t) = \sigma(\partial_j t).$$

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We want to show next that

$$\sigma(s)\sigma(t) = \sigma(st). \tag{3.5}$$

Applying ϕ to both sides of $[s\partial_j, t\partial_j] = (s\partial_j t - t\partial_j s)\partial_j$ yields

$$[\sigma(s)\phi(\partial_j), \sigma(t)\phi(\partial_j)] = \sigma\big((s\partial_j t - t\partial_j s)\big)\phi(\partial_j).$$

The left side of this is $\{\sigma(s)\phi(\partial_j)\sigma(t) - \sigma(t)\phi(\partial_j)\sigma(s)\}\phi(\partial_j)$, so we have

$$\sigma(s)\phi(\partial_j)\sigma(t) - \sigma(t)\phi(\partial_j)\sigma(s) = \sigma(s\partial_j t - t\partial_j s).$$

Using (3.4), this becomes

$$\sigma(s)\sigma(\partial_j t) - \sigma(t)\sigma(\partial_j s) = \sigma(s\partial_j t - t\partial_j s),$$

or

$$\sigma(s)\sigma(\partial_j t) - \sigma(s\partial_j t) = \sigma(t)\sigma(\partial_j s) - \sigma(t\partial_j s). \tag{3.6}$$

If $\partial_j s = 0$, the right side of the last equation vanishes, giving

$$\sigma(s)\sigma(\partial_j t) = \sigma(s\partial_j t).$$

Thus (3.5) holds for $\deg_j(t) \neq -1$ and $\deg_j(s) = 0$. Now the right side of (3.6) also vanishes when $\deg_j(s) = 1$, showing that the left side does also. Hence, (3.5) also holds for $\deg_j(t) \neq -1$ and $\deg_j(s) = 1$. We can use this last to show that (3.5) holds when $\deg_j(t) = -1$ and $\deg_j(s) = 0$:

$$\begin{aligned}\sigma(s)\sigma(t) &= \sigma(s)\sigma(x_j)\sigma(x_j^{-1}t)\\ &= \sigma(sx_j)\sigma(x_j^{-1}t)\\ &= \sigma(sx_jx_j^{-1}t)\\ &= \sigma(st).\end{aligned}$$

We also can obtain the special case of (3.5) with $s = x_j^{\alpha}$ and $t = x_j^{\beta}$ by noting that $\deg_1(s) = 0$ and $\deg_1(t) = 0$ for $\ell \neq j$, and that we may interchange the roles of 1 and j. Finally, the general case of (3.5) follows by writing $s = x_j^{\alpha}s_0$ and $t = x_j^{\beta}t_0$ where $\deg_j(s_0) = 0$ and $\deg_j(t_0) = 0$, and by calculating that

$$\begin{aligned}\sigma(s)\sigma(t) &= \sigma(x_j^{\alpha})\sigma(x_j^{\beta})\sigma(s_0)\sigma(t_0)\\ &= \sigma(x_j^{\alpha+\beta})\sigma(s_0t_0)\\ &= \sigma(x_j^{\alpha+\beta}s_0t_0)\\ &= \sigma(st).\end{aligned}$$

As an immediate consequence of (3.5) we have

$$\phi(st\partial_j) = \sigma(st)\phi(\partial_j) = \sigma(s)\sigma(t)\phi(\partial_j) = \sigma(s)\phi(t\partial_j).$$

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We show finally that $\phi(tw) = \sigma(t)\phi(w)$ and $\phi(w)\sigma(t) = \sigma(wt)$ for all $t \in B$ and $w \in A$. Writing $w = \sum_i s_i\partial_i$, we have

$$\begin{aligned}\phi(tw) &= \phi\Big(t\sum_i s_i\partial_i\Big)\\ &= \sum_i \phi(ts_i\partial_i)\\ &= \sigma(t)\sum_i \phi(s_i\partial_i)\\ &= \sigma(t)\phi(w),\end{aligned}$$

$$\begin{aligned}\phi(w)\sigma(t) &= \phi\Big(\sum_i s_i\partial_i\Big)\sigma(t)\\ &= \sum_i \sigma(s_i)\phi(\partial_i)\sigma(t)\\ &= \sum_i \sigma(s_i)\sigma(\partial_i t)\\ &= \sum_i \sigma(s_i\partial_i t)\\ &= \sigma(wt)\end{aligned}$$

■

THEOREM 3.7. *The map χ from the group of automorphisms* Aut A *of A to the group of automorphisms* Aut B *of B defined by mapping each ϕ into its corresponding σ is an isomorphism of* Aut A *into* Aut B. *Further, χ is onto if and only if each Δ_i has rank* 1 *for $k < i \leq n$.*

PROOF. Let ϕ_1 and ϕ_2 be two automorphisms of A, and let $\sigma_1 = \chi(\phi_1)$ and $\sigma_2 = \chi(\phi_2)$. Then χ is a group homomorphism since

$$\phi_1\phi_2(tw) = \phi_1\big(\sigma_2(t)\phi_2(w)\big) = \sigma_1\big(\sigma_2(t)\big)\phi_1\phi_2(w).$$

Suppose that ϕ is in the kernel of χ, *i.e.*, that $\phi(tw) = t\phi(w)$ for all $t \in B$. If $j \neq i$, then

$$\begin{aligned}0 &= \phi([\partial_i, x_j^\ell\partial_j])\\ &= [\phi(\partial_i), \phi(x_j^\ell\partial_j)]\\ &= [\phi(\partial_i), x_j^\ell\phi(\partial_j)]\\ &= \{\phi(\partial_i)x_j^\ell\}\partial_j + x_j^\ell[\phi(\partial_i), \phi(\partial_j)]\\ &= \{\phi(\partial_i)x_j^\ell\}\partial_j + x_j^\ell\phi([\partial_i, \partial_j])\\ &= \{\phi(\partial_i)x_j^\ell\}\partial_j.\end{aligned}$$

Thus, $\phi(\partial_i)x_j^\ell = 0$ for all ℓ whenever $j \neq i$. But then $\phi(\partial_i) = f_i\partial_i$ for some $f_i \in B$. For $i \neq j$,

$$\begin{aligned}0 &= \phi([\partial_i, \partial_j])\\ &= [\phi(\partial_i), \phi(\partial_j)]\\ &= [f_i\partial_i, f_j\partial_j]\\ &= f_i(\partial_j f_i)\partial_j - f_j(\partial_i f_j)\partial_i,\end{aligned}$$

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which implies that $\partial_j f_i = 0$. Hence, f_i is just a polynomial in x_i. If f_i is not a constant polynomial, then any element on which $\phi(\partial_i)$ acts ad-nilpotently is annihilated by $\phi(\partial_i)$. But this cannot happen since $\phi(\partial_i)$ is the image of the element ∂_j which acts ad-nilpotently on some elements which it does not annihilate. It follows that $f_i \in F$. From the calculation

$$\begin{aligned} f_i\partial_i &= \phi(\partial_i) \\ &= \phi([\partial_i, x_i\partial_i]) \\ &= [\phi(\partial_i), \phi(x_i\partial_i)] \\ &= [\phi(\partial_i), x_i\phi(\partial_i)] \\ &= [f_i\partial_i, f_ix_i\partial_i] \\ &= f_i^2\partial_i \end{aligned}$$

we obtain $f_i = 1$. Thus, ϕ is just the identity automorphism, and χ is 1-1.

If Δ_ℓ has rank greater than 1 for some ℓ with $k < \ell \leq n$, then A can be embedded into a larger algebra A' of derivations of B by splitting the ℓ-th variable, as noted at the end of Section 2. The map χ' of $\operatorname{Aut} A'$ into $\operatorname{Aut} B$ is one-to-one and clearly extends $\chi\colon \operatorname{Aut} A \longrightarrow \operatorname{Aut} B$, so χ could not be onto when rank $\Delta_\ell > 1$.

Suppose, on the other hand, that rank $\Delta_i = 1$ for all i with $k < i \leq n$. To see that χ is onto this time, let σ be an automorphism of B, and define the map $\phi\colon A \longrightarrow A$ by $\phi(w)t = \sigma\big(w\sigma^{-1}(t)\big)$. Then clearly $\phi(w)$ acts linearly on A, and the map ϕ is linear also. Further, $\phi(w)$ is a derivation of B since

$$\begin{aligned} \phi(w)(st) &= \sigma\big(w\sigma^{-1}(st)\big) \\ &= \sigma\Big(w\big(\sigma^{-1}(s)\sigma^{-1}(t)\big)\Big) \\ &= \sigma\Big(\big(w\sigma^{-1}(s)\big)\sigma^{-1}(t) + \sigma^{-1}(s)w\big(\sigma^{-1}(t)\big)\Big) \\ &= \sigma\big(w\sigma^{-1}(s)\big)t + s\sigma\big(w\sigma^{-1}(t)\big) \\ &= \big(\phi(w)s\big)t + s\phi(w)t, \end{aligned}$$

for all $w \in A$ and $s, t \in B$. Since A is all derivations of B in the case we are considering, it follows that $\phi(w) \in A$. Finally, $\sigma = \chi(\phi)$ since

$$\phi(sw)t = \sigma\big(sw\sigma^{-1}(t)\big) = \sigma(s)\sigma\big(w\sigma^{-1}(t)\big) = \sigma(s)\phi(w)t,$$

for all $w \in B$ and $s, t \in A$. ∎

REMARK. The following example due to Zhao Kaiming shows explicitly that the map ϕ of the last paragraph does not map A into itself when rank $\Delta_\ell > 1$ for some ℓ. Splitting x_ℓ into two variables as at the end of Section 2, we let σ be the automorphism of B which inverts one of the new variables and preserves the other new variable as well as the other old variables. Then the map ϕ of the last paragraph does not map A into itself. Equivalently, this σ has no corresponding ϕ in A.

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LEMMA 3.8. *For any automorphism ϕ of A the gradient of $\phi(\partial_m)$ is zero for each m.*

PROOF. Let $\phi(\partial_m) = \sum_i f_i^{(m)}\partial_i$, let $\sigma = \chi(\phi)$, and let $\sigma(x_j) = g_j$. Then $\sum_i f_i^{(m)}\partial_i g_j = \phi(\partial_m)\sigma(x_j) = \sigma(\partial_m x_j) = \delta_{mj}$. For fixed m, the solution to this set of equations is given by Cramer's Rule as $Cf_i^{(m)} = C_i$ where $C = \det\|\partial_i g_j\|$ and where C_i is obtained from C by replacing the i-th column by ϵ_m. To show that the gradient of $\phi(\partial_m)$ is zero, we have to establish that $\sum_i \partial_i f_i^{(m)} = 0$, or that $\sum_i \partial_i C_i = 0$. If $C_{i\ell}$ is obtained from C_i by applying ∂_i to the ℓ-th column, then $\partial_i C_i = \sum_\ell C_{i\ell}$. The assertion that the gradient of $\phi(\partial_m)$ is zero can then be written in the form $\sum_i \sum_\ell C_{i\ell} = 0$. But this follows immediately from the observation that $C_{i\ell} = -C_{\ell i}$. ∎

PROPOSITION 3.9. *The gradient of the image of an element of A under ϕ is the image of the gradient of that element under σ. In particular, the set S of elements of gradient zero is preserved under any automorphism.*

PROOF. The gradient of $x^\alpha\partial_m$ is $\partial_m x^\alpha = \alpha_m x^{\alpha-\epsilon_m}$. On the other hand, using the notation of Lemma 3.8, the gradient of

$$\phi(x^\alpha\partial_m) = \sigma(x^\alpha)\phi(\partial_m) = \sigma(x^\alpha)\sum_i f_i^{(m)}\partial_i$$

is

$$\begin{aligned}\sum_i \partial_i\Big(\sigma(x^\alpha)f_i^{(m)}\Big) &= \sum \partial_i\Big(\sigma(x^\alpha)\Big)f_i^{(m)} + \sigma(x^\alpha)\sum \partial_i f_i^{(m)}\\ &= \sum f_i^{(m)}\partial_i\Big(\sigma(x^\alpha)\Big) + 0,\end{aligned}$$

using Lemma 3.8. Then, using Theorem 3.1, this gradient is

$$\begin{aligned}\phi(\partial_m)\Big(\sigma(x^\alpha)\Big) &= \phi(\partial_m)\prod_i \sigma(x_i^{\alpha_i})\\ &= \sigma(x_1^{\alpha_1})\cdots\phi(\partial_m)\sigma(x_m^{\alpha_m})\cdots\sigma(x_n^{\alpha_n})\\ &= \alpha_m\sigma(x_1^{\alpha_1})\cdots\sigma(x_m^{\alpha_m-1})\cdots\sigma(x_n^{\alpha_n})\\ &= \alpha_m\sigma(x^{\alpha-\epsilon_m}).\end{aligned}$$

We have established the first statement of the proposition, and the second statement follows immediately from this. ∎

Hereafter let A' denote the subalgebra consisting of all elements of A of the form $\sum_{1\le i\le k} f_i\partial_i$ for $f_i \in B$. Also let $G' = A' \cap S$, *i.e.*, G' is the set of elements of A' with gradient zero. A subalgebra of A which is sent onto itself by every automorphism of A will be called *characteristic*.

PROPOSITION 3.10. *The subalgebra A' is characteristic.*

PROOF. We show first that $\phi(\partial_i) \in A'$ when $1 \le i \le k$ for any automorphism ϕ. If this is not true, then $\phi(\partial_i)x_\ell \neq 0$ for some $\ell > k$. Since x_ℓ is invertible, $\sigma^{-1}(x_\ell)$ is invertible, and so $\sigma^{-1}(x_\ell) = ax^\beta$ for some $a \in F$ and $\beta \in \Delta$ with $\beta_j = 0$ for $1 \le j \le k$. In particular, $\beta_i = 0$, and so

$$0 \neq \phi(\partial_i)x_\ell = \phi(\partial_i)\sigma(ax^\beta) = a\phi(\partial_i)\prod_j \sigma(x_j^{\beta_j}) = 0,$$

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by Theorem 3.1. This contradiction shows that $\phi(\partial_i) \in A'$. But then $\phi(A') \subset A'$ since

$$\phi\Big(\sum_{i\le k} B\partial_i\Big) \subset \sum_{i\le k} B\phi(\partial_i) \subset A'.$$

∎

Combining Propositions 3.9 and 3.10, we obtain

COROLLARY 3.11. *The subalgebra G' is characteristic.*

4. **Examples of automorphisms.** In view of Theorems 3.1 and 3.7, we can produce examples of automorphisms of A simply by picking an automorphism σ of B and finding images for ∂_i's that satisfy the second relation in Theorem 3.1.

EXAMPLE 4.1. For fixed j and for fixed nonzero $\lambda \in F$, let $\sigma(x_j) = \lambda x_j$ and $\phi(\partial_j) = \lambda^{-1}\partial_j$, and for $i \neq j$ let $\sigma(x_i) = x_i$ and $\phi(\partial_i) = \partial_i$. The automorphisms in the group generated by automorphisms of this type are called *scalar* automorphisms.

EXAMPLE 4.2. For fixed $j \le k$ and for fixed $\lambda \in F$, let $\sigma(x_j) = \lambda + x_j$ and $\sigma(x_i) = x_i$ for $i \neq j$. For all i let $\phi(\partial_i) = \partial_i$. We shall denote this automorphism by $\phi_{[i\lambda]}$. Products of automorphisms of this type are called *scalar shift* automorphisms.

EXAMPLE 4.3. For fixed $j \le k$, choose $g \in B$ with $\partial_j(g) = 0$. Define $\sigma(x_i) = x_i + \delta_{ij} g$ and $\phi(\partial_i) = \partial_i - \partial_i(g)\partial_j$. We shall call this an *elementary* automorphism and denote it by ϕ_{ig}.

EXAMPLE 4.4. For fixed $j > k$, let $\sigma(x_j) = x_j^{-1}$ and $\phi(\partial_j) = -x_j^2\partial_j$, and for $i \neq j$ let $\sigma(x_i) = x_i$ and $\phi(\partial_i) = \partial_i$. Products of automorphisms of this type are called *inverting* automorphisms.

In general there are other automorphisms besides the group of automorphisms generated by these four types. For example, if $\Delta_i = \Delta_j$ we can interchange x_i and x_j; or we can replace x_j by $x_j x_i$. We can use elementary automorphisms to establish

PROPOSITION 4.5. *Let T' be a torus in A'. Then the dimension of T' is no more than k.*

PROOF. Let $\bar{F}$ be the quotient field of the ring $F' = F[x_{k+1}, \dots, x_n, x_{k+1}^{-1}, \dots, x_n^{-1}]$, and let $\bar{A'} = \bar{F} \otimes_{F'} A'$. Then any semisimple element of A' will be a semisimple element of $\bar{A'}$. Further, any set of linearly independent elements of a torus in A' will remain linearly independent in $\bar{A'}$. Thus it is sufficient to show that the dimension of any torus T' of $\bar{A'}$ is no more than k. It will be convenient to use the standard grading of $\bar{A'}$ in which the element $x^\alpha\partial_i$ is in the component of level $\alpha_1 + \alpha_2 + \cdots + \alpha_k - 1$. As usual, we can think of the elements of level 0 as $n \times n$ matrices under Lie product, and we can think of the -1 level as an irreducible module over these matrices. The elementary automorphisms in which g is a multiple of a noninvertible variable induce elementary transformations on the matrices and its module. Thus, using products of elementary automorphisms, we can do any change of basis on the matrices and module that is convenient.

Let $t_1, \ldots, t_r$ be linearly independent elements of the torus T' of $\bar{A}$. Let d_i and e_i be respectively the -1 and 0 level components of t_i. If any set of nonzero d_i's is linearly dependent, we can subtract multiples of one t_i from another to make the nonzero d_i's linearly independent. Further, we can do a change of basis so that the nonzero d_i's are respectively $\partial_1, \partial_2, \ldots, \partial_\ell$. Then $t_{\ell+1}, \ldots, t_r$ have zero components at the -1 level. It is easy to see that $e_{\ell+1}, \ldots, e_r$ must then correspond to idempotents in the matrices. We have $e_j \neq 0$ for $j > \ell$ since otherwise t_j could not act semisimply. If there is any linear relation between the e_j's for $j > \ell$, we could subtract multiples of some t_j's from others and get a t_j without 0 component. The fact that the t_j's commute means that $e_{\ell+1}, \ldots, e_r$ correspond to a set of pairwise orthogonal idempotents.

For $1 \leq i \leq \ell < j \leq r$, the relation $[t_i, t_j] = 0$ implies that $0 = [d_i, e_j] = [\partial_i, e_j]$. Thus, $e_{\ell+1}, \ldots, e_r$ correspond to $r - \ell$ orthogonal idempotents which lie in a set of matrices with $k - \ell$ nonzero columns. It follows that $r - \ell \leq k - \ell$, or that $r \leq k$, as we wished to prove. ∎

5. **Characteristic subalgebras and autogeneration.** We say that a set C of A is *autogenerated* by an element $w \in A$ if C is contained in the set of all elements generated from w under the operations of taking images under automorphisms of A, taking Lie products, and forming linear combinations. The set of all elements autogenerated by w is clearly a subalgebra. For our final result on isomorphisms of algebras of type W^* we will need to characterize certain subalgebras intrinsically, and the concept of autogeneration will play a major role in this endeavor.

The subalgebra G' of elements of A' which have gradient zero is spanned by the elements of the form $\{x^\alpha \partial_j \mid \alpha_j = 0 \text{ and } j \leq k\}$ and of the form $\{\partial_j(f)\partial_i - \partial_i(f)\partial_j \mid i, j \leq k \text{ and } f \in B\}$. If $k = 1$, only elements of the first type will occur. Let G be the sum of G' and the span of the elements of the form $\{f_j x_j \partial_j \mid j \leq k \text{ and } \partial_i(f_j) = 0 \text{ for all } i \leq k\}$. It is clear G is a subalgebra and that $[G, G] = G'$. We will also need

$$A'' = \Big\{ \sum_{k<i\leq n} f_i \partial_i \mid f_i \in B, \text{ and } \partial_\ell(f_i) = 0 \text{ for all } \ell \leq k \Big\}.$$

We have shown in [3] that the torus $T = \sum_{i>k} F x_i \partial_i$ is the unique maximal torus of A'', and that all toral elements of A'' are contained in it.

PROPOSITION 5.1. *The subalgebra G is characteristic. Further, if f is a monomial in the invertible variables and if $i \leq k$, then $\phi(f x_j \partial_j)$ is congruent to $\sigma(f) x_j \partial_j$ modulo G' for any automorphism ϕ.*

PROOF. In view of the Corollary 3.11, it is sufficient to show that the elements $f x_j \partial_j$ for $j \leq k$ where f is a monomial in the invertible variables are congruent to $x_j \partial_j$ modulo G. Now $\phi(x_j \partial_j) \in A'$ for any automorphism ϕ by Proposition 3.10, say $\phi(x_j \partial_j) = \sum_{i\leq k} h_i \partial_i$. Using Proposition 3.9, the gradient of $\phi(x_j \partial_j)$ is $1 = \partial_j x_j = \sum_{i\leq k} \partial_i h_i$. Writing $h_i = a_i x_i + h_i'$ where h_i' involves all terms in h_i which correspond to roots of $h_i \partial_i$ other than 0, this becomes

$$1 = \sum_{i\leq k} a_i + \sum_{i\leq k} \partial_i h_i'. \tag{5.2}$$

Since the second summation involves exactly the terms which do not correspond to terms on the left side of the equation, it must vanish. Thus $\sum_{i\le k} h_i'\partial_i \in G'$, and $\phi(x_i\partial_i)$ is congruent to $\sum_{i\le k} a_i x_i\partial_i$ modulo G'. Thus (5.2) reduces to $\sum_{i\le k} a_i = 1$. Since $x_i\partial_i$ is congruent modulo G' to $x_j\partial_j$ for $i,j \le k$, in fact, $\phi(x_j\partial_j)$ is congruent to $x_j\partial_j$ modulo G'.

Considering now the general element $fx_j\partial_j$ where f is a monomial in the invertible variables, we note that $\sigma(f)$ must be an invertible element of B, so that $\sigma(f)$ is a scalar multiple of a monomial in the invertible x_i's. Thus, $\phi(fx_j\partial_j) = \sigma(f)\phi(x_j\partial_j)$ is congruent modulo G' to $\sigma(f)x_j\partial_j \in G$. ∎

LEMMA 5.3. *If $k \ge 2$ and if $i \le k$, then ∂_i autogenerates G'.*

PROOF. Let $j \le k$ and $j \ne i$. We show first that the element $f\partial_j$ with $f \in B$ where $\partial_j(f) = 0$ is autogenerated by ∂_i. Since x_i is not invertible, we can find $g \in B$ with $\partial_j(g) = 0$ and $\partial_i(g) = f$. If ϕ_{jg} is the elementary automorphism given in Example 4.3 using this g, then $\phi(\partial_i) = \partial_i + f\partial_j$. Thus $f\partial_j$ is indeed autogenerated by ∂_i. Choosing the special case when $f = 1$, we get ∂_j. Then switching the roles of i and j in the above argument, we obtain $h\partial_i$ for any $h \in B$ with $\partial_i(h) = 0$. The elements that we have shown to be autogenerated by ∂_i in fact generate G' as a subalgebra, to complete the proof. ∎

LEMMA 5.4. *(i) If $\alpha_i = 0 = \beta_i$ and $\alpha_j = 0 = \beta_j$ where $i, j \le k$, then*

$$[x_i^\ell x_j^m x^\alpha x_i\partial_i,\ \ell' x_i^{\ell'-1} x_j^{m'} x^\beta \partial_j - m' x_i^{\ell'} x_j^{m'-1} x^\beta \partial_i] \in G'$$

if and only if $0 = (\ell+1)(\ell' m - \ell m')$.

(ii) If $\alpha_i = 0 = \beta_i$ and $\alpha_j = 0 = \beta_j$ where $i, j \le k$, and if $q > k$, then

$$[x_i^\ell x_j^m x^\alpha \partial_q,\ \ell' x_i^{\ell'-1} x_j^{m'} x^\beta \partial_j - m' x_i^{\ell'} x_j^{m'-1} x^\beta x_1\partial_i] \in G'$$

if and only if $0 = (\ell' m - \ell m')$.

PROOF. Multiplying out the product in Part (i), the coefficient of the term with ∂_j will be $\ell'(\ell'-1)$, and the coefficient of the term with ∂_i will be $-m'\ell' - \ell' m + m'(\ell+1)$. Thus the condition that this lies in G' will be

$$\ell'(\ell'-1)(m+m') = (\ell+\ell')\big(-m'\ell' - \ell' m + m'(\ell+1)\big).$$

Simplifying this condition yields the one given in the statement of Part (i) of the lemma.

Computing the product in Part (ii), we see that the coefficient of the term with ∂_q will vanish if and only if $0 = (\ell' m - \ell m')$. This is of course necessary if the product is to be in $G' \subset A'$. The remaining expression is seen to be in G' if and only if $0 = (\ell' m - \ell m')$. Thus Part (ii) holds. ∎

LEMMA 5.5. *Let $k \ge 2$. Then*

(i) Any element of G' autogenerates G'.

(ii) Any element of G autogenerates a subalgebra of G containing G'.

(iii) Any element of A' not in G autogenerates A'.

(iv) Any element of A not in A' autogenerates a subalgebra containing G'.

PROOF. Suppose first that $w \in A'$. We claim first that w autogenerates an element of the form $w' = x^\alpha \partial_j$. By taking an appropriate linear combination of the images of w under different scalar automorphisms, we see that the component of w in any root space is in the subspace autogenerated by w. Thus, to see that w autogenerates all of G', it is sufficient to suppose that w is contained in a single root space. Say that w has root α. If $\alpha_i > 0$ for some $i \leq k$, apply the scalar shift automorphism $\phi_{[i\lambda]}$ to w and take the terms which have the root $\alpha - \alpha_i \epsilon_i$, which will be nonzero. Repeating this operation if necessary, we can assume that w has root α where $\alpha_i \leq 0$ for each $i \leq k$. If $\alpha_j = -1$ for some $j \leq k$ then we have arrived at the element $w' = x^{\alpha'} \partial_j$ where $\alpha' = \alpha + \epsilon_j$. If $\alpha_i = 0$ for all $i \leq k$, then $w = x^\alpha \sum_{i \leq k} c_i x_i \partial_i$ for some $c_i \in F$. Choosing j so that $c_j \neq 0$, we apply the scalar shift automorphism $\phi_{[j\lambda]}$ to w and take the component of degree -1 in j which is a nonzero multiple of $w' = x^\alpha \partial_j$.

Consider now the case when $\alpha_i \neq -1$ for all $i > k$. Choosing $\ell \leq k$ with $\ell \neq j$ and g so that $\partial_j g = x^{-\alpha}$, we apply the automorphism $\phi_{\ell g}$ which sends w' into $x^\alpha(\partial_j - x^{-\alpha}\partial_\ell)$. One component of this is ∂_ℓ, which autogenerates all of G' by Lemma 5.1. On the other hand, if $\alpha_i = -1$ for some $i > k$, we can again apply the automorphism $\phi_{\ell g}$, but this time with g chosen to be a monomial in $x_{k+1}, \ldots, x_n$ with the property that $\deg x^\alpha g \neq -1$ for $i > k$. We can then apply the first case to autogenerate G'.

We have shown that the subalgebra autogenerated by any element $w \in A'$ contains G'. If $w \in G'$, we see from Corollary 3.6 that w generates exactly G', to give Part (i). If w is in G but not in G', then the subalgebra generated by w contains G' by the above argument, and is contained in G by Proposition 5.1. Thus Part (ii) holds.

Suppose now that w is in A' but not in G. As in the first part of this argument, the component of w in any root space is autogenerated by w, so that we can assume that w is not in G but is in a single root space, say the α-root space. Modulo G' which we know is autogenerated by w, we can pick w to be a single monomial. Since $w \notin G'$, it has the form $w = x^\alpha x_j \partial_j$ for some $j \leq k$; and since $w \notin G$, $\alpha_i \neq 0$ for some $i \leq k$. If $\alpha_j \neq 0$, the automorphism ϕ_{jg} induced by $\sigma(x_j) = x_j + g$ and $\sigma(x_i) = x_i$ for $i \neq j$ applied to w will show us that $g x^\alpha \partial_j$ is autogenerated by w. Here g can be any monomial which has degree zero in x_j. If $\alpha_\ell \neq 0$ for some $\ell \neq j$ with $\ell \leq k$, then the automorphism $\phi_{\ell g}$ where $g = x_j^m$ applied to w will give that $x_j^m x_\ell^{-1} w$ is autogenerated by w. Using these two operations and the scalar shifts for lowering the powers of noninvertible variables, we can get any monomial in A' modulo G', to establish Part (iii).

Finally for Part (iv), we have to show that any element w not in A' autogenerates at least G'. As usual, we may assume that w has a single root associated with it. If any nonzero components of the root correspond to noninvertible variables, we can use scalar shift automorphisms to get rid of these variables. Thus we may assume that $w = x^\alpha \sum_{i>k} c_i x_i \partial_i$ for some $c_i \in F$ where $\alpha_i = 0$ for $i \leq k$, and where $c_j \neq 0$ for some $j > k$. As in the proof of the last part, the automorphism ϕ_{1g} where $g = x_j^m$ applied to w yields a nonzero component in A'. This component can be separated off using a scalar automorphism, and it will autogenerated all of G' by the first part of the proof. ∎

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If $k = 1$, let G'' denote the span of all elements of the form $x^\alpha \partial_1$ where $\alpha_1 = 0$ and $\alpha_i \neq 0$ for all $i \geq 2$. Then we have

PROPOSITION 5.6. *Suppose that $k = 1$. (i) Let $G^\#$ denote the intersection of all subalgebras of A which are autogenerated by elements of A which are not ad-nilpotent. Then $G'' \subset G^\# \subset G'$.*

(ii) G' is the centralizer of $G^\#$.

PROOF. The element $x_1\partial_1$ is not ad-nilpotent and $\phi(x_1\partial_1) \in Fx_1\partial_1 + G'$ for any automorphism ϕ by Proposition 5.1. Hence the subalgebra autogenerated by $x_1\partial_1$ is contained in $Fx_1\partial_1 + G'$. Similarly, the subalgebra autogenerated by $x_2x_1\partial_1$ is contained in $\sum_\sigma F\sigma(x_2)x_1\partial_1 + G'$, which does not contain $x_1\partial_1$. It follows that $G^\# \subset G'$.

If $w = \sum_i f_i\partial_i$ for $f_i \in B$ is an element of A which is not ad-nilpotent, we want to show that w autogenerates G''. Suppose first that $f_j \neq 0$ for some $j > 1$. Then w autogenerates each of its homogeneous components, so that it is sufficient to show that one component of w autogenerates G''. Select a component of w which has a nonzero coefficient for ∂_j, say, $w' = x^\alpha \sum_i c_i x_i \partial_i$ where $c_i \in F$. Applying the automorphism 4.4 to w' which sends x_j into x_j^{-1}, we obtain the element $w'' = x^{\alpha - 2\alpha_j\epsilon_j}(\sum_i x_i\partial_i - 2x_j\partial_j)$. Then the element $z = c_j^{-1}[w'', w'] = x^{2\alpha - 2\alpha_j\epsilon_j}x_j\partial_j$ is autogenerated by w. Writing $\beta = 2\alpha - 2\alpha_j\epsilon_j$, we have $z = x^\beta x_j \partial_j$. If $\beta_1 \neq 0$, we can apply a scalar shift automorphism and take the component that does not depend on x_1. Thus we may assume that $\beta_1 = 0$. Now applying the elementary automorphism ϕ_{1g} to z where $\partial_1 g = 0$, and then subtracting z, we get the element $x^\beta x_j(\partial_j g)\partial_1 \in G'$. As g ranges over all monomials in the invertible elements we will get all elements of G'' (as well as some elements not in G'').

Suppose then that the element w that is not ad-nilpotent has the form $f\partial_1$. Then some homogeneous component of it is not ad-nilpotent. We can take a scalar shift of this element and take the component which has degree zero in x_1. Thus it is sufficient to show that any element of the form $z = x^\alpha x_1 \partial_1$ with $\alpha_1 = 0$ autogenerates all of G''. Applying the automorphism ϕ_{1g} to z where $\partial_1 g = 0$, and subtracting z from this, we obtain the element $x^\alpha g \partial_1 \in G'$. As g ranges over all monomials in the invertible elements, we will get all the elements of G''. This completes the proof of part (i).

For part (ii) we note first that G' is abelian when $k = 1$, so that G' is contained in the centralizer of $G^\#$. Conversely, let $w = \sum_i f_i\partial_i$ centralize $G^\#$. Then for any α with $\alpha_1 = 0$ and $\alpha_i \neq 0$ for all $i > 1$,

$$\begin{aligned} 0 = [w, x^\alpha \partial_1] &= \Big[\sum_i f_i\partial_i, x^\alpha\partial_1\Big] \\ &= \sum_i \alpha_i f_i x^{\alpha - \alpha_i\epsilon_i}\partial_1 - x^\alpha \sum_i (\partial_1 f_i)\partial_i. \end{aligned}$$

This implies that $f_i = 0$ for $i > 1$ and that $\partial_1 f_1 = 0$. Thus, $w \in G'$. ∎

PROPOSITION 5.7. *(i) If $k \geq 2$, then G' is the unique minimal nonzero characteristic subalgebra of A.*

(ii) G is the unique characteristic maximal ideal of the idealizer H of G' in A.

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(iii) A' *is the unique largest characteristic subalgebra which properly contains* G*, and whose intersection with* H *is* G*.*

(iv) $A'' + A'$ *is the idealizer of* A'*.*

PROOF. Part (i) follows from the fact that every nonzero element of A autogenerates at least G' by Lemma 5.5, and that elements of G' autogenerate exactly G' when $k \geq 2$. Turning to Part (ii), we know from Lemma 5.1 that G is a characteristic subalgebra, and it is clearly contained in H. To complete the proof of Part (ii) we need to show that G is the unique maximal ideal of H.

In view of Lemma 5.4(i), the only elements of A' in H are the elements of G. (Note that every element of A' that is in a single root space can be taken to have a single term modulo G'). If $w \in H$ and $w \notin A'$, then $w = w_1 + w_2$ where $w_2 \in A'$ and $w_1 = \sum_{i>k} c_i f_i \partial_i$. Each f_i must be a function only of the invertible variables, since otherwise w cannot send every element of G' into G'. Thus, $w_1 \in A''$. On the other hand, it is easy to see that each element of A'' will idealize G'. Then $w_2 = w - w_1 \in H$, and so $w_2 \in G$. We have shown that $H = A'' + G$. Since $[A'', G] \subset G$ and since A'' is a simple algebra, G is a maximal ideal of H. To establish uniqueness, let $w \in H$ and $w \notin G$, and we will show that the autoinvariant ideal of H generated by w is all of H. Now $w = w_1 + w_2$ where $w_1 \in A''$ and $w_2 \in G$, and $w_1 \neq 0$. As usual, by applying different scalar automorphisms we can assume that w is in a single root space, say, $w = x^\alpha \sum_i c_i x_i \partial_i$. Then $c_j \neq 0$ for some $j > k$, and

$$\left[x_j^2 \partial_j, [\partial_j, w]\right] = (\alpha_j^2 - \alpha_j) x^\alpha \sum_i c_i x_i \partial_i - 2c_j x_j \partial_j.$$

Subtracting $(\alpha_j^2 - \alpha_j)w$ from this we obtain that $x^\alpha x_j \partial_j$ is in the ideal. Thus, w autogenerates all of A'', since A'' is simple. Hence w autogenerates H because $G = [A'', G]$. This completes the proof of Part (ii).

To show Part (iii), we note first that A' is characteristic by Proposition 3.10, and that it properly contains G. The remaining thing to be shown for Part (iii) is that any other subalgebra with these properties which is not contained in A' must have intersection with H which is larger than G. Let w be an element in such an algebra, and we can suppose that it is not in A'. By using scalar and scalar shift automorphisms, we can assume that w is in a single root space, and that the corresponding root α has the property that $\alpha_i = 0$ for $i \leq k$. Under this reduction we retain the property that $w \notin A'$. Then $w = w_1 + w_2$ where $0 \neq w_1 = x^\alpha \sum_{i>k} x_i \partial_i \in A''$ and $w_2 = x^\alpha \sum_{i \leq k} x_i \partial_i \in G$. But this element is in H, showing that the intersection of the algebra with H is bigger than G. This completes the proof of Part (iii). When $k \geq 2$ it follows from Lemma 5.5(iii) that A' is the only characteristic subalgebra of A which properly contains G and whose intersection with H is G.

Finally, for Part (iv) we note that an element $w = \sum_i c_i f_i \partial_i$ for $c_i \in F$ and $f_i \in B$ will idealize A' if and only if, for each $i > k$, f_i is a function only of the invertible variables. But this says that w idealizes A' if and only if $w \in A'' + A'$. ∎

NOTATION. To get the analogue in an algebra A_i of the various subalgebras G', G, A', A'' of A, we shall simply add the subscript i. Similarly, the number of noninvertible variables in A_i will be denoted by k_i and the number of all variables by n_i.

THEOREM 5.8. *Let* $\theta\colon A_1 \cong A_2$ *be an isomorphism where* $A_1, A_2 \in W^*$. *Then* $A_1' \cong A_2'$ *and* $A_1'' \cong A_2''$. *In particular,* $k_1 = k_2$ *and* $n_1 = n_2$.

PROOF. If $k_1 = 0$, then $k_2 = 0$ and $n_1 = n_2$ by [3, Theorem 4.16 and Corollary 4.12]. Both isomorphisms are trivial in this case. If $k_1 = 1$, then G_1' is abelian, and so A_2 must contain an abelian characteristic subalgebra. Thus, $k_2 < 2$ by Proposition 5.7(i). The possibility that $k_1 = 0$ is ruled out by applying the first step of the proof to θ^{-1}, showing that $k_2 = 1$. Then G_1' and G_2' are invariantly characterized in A_1 and A_2 by Proposition 5.6, and so A_1' and A_2' are invariantly characterized in A_1 and A_2 by Proposition 5.7. Hence, $\theta(A_1') = A_2'$.

In order to complete the proof that $A_1' \cong A_2'$ and $k_1 = k_2$, we may assume that $k_1 \geq 2$ and $k_2 \geq 2$. By Proposition 5.7, A_1' and A_2' can be characterized in an invariant manner, so θ must map the one onto the other. Now A_1' has a torus of dimension k_1 of noninvertible variables in A_1, and hence A_2' must have a torus of dimension k_1. By Proposition 4.5, $k_1 \leq k_2$. By symmetry, $k_1 = k_2$.

Since $A_i'' + A_i'$ is the idealizer of A_i', θ must also map $A_1'' + A_1'$ onto $A_2'' + A_2'$. Then, $A_1'' \cong (A_1'' + A_1')/A_1' \cong (A_2'' + A_2')/A_2' \cong A_2''$. By [3, Corollary 4.12], A_1'' and A_2'' must have the same number of variables. ∎

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