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The Heat Kernel and Green's Function on a Manifold with Heisenberg Group as Boundary

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Abstract. We study the Riemannian Laplace-Beltrami operator L on a Riemannian manifold with Heisenberg group H_1 as boundary. We calculate the heat kernel and Green's function for L, and give global and small time estimates of the heat kernel. A class of hypersurfaces in this manifold can be regarded as approximations of H_1. We also restrict L to each hypersurface and calculate the corresponding heat kernel and Green's function. We will see that the heat kernel and Green's function converge to the heat kernel and Green's function on the boundary.

1 Introduction

This article is a continuation of [6]. The purpose of these two articles is to study sub-Riemannian geometry on the Heisenberg group. We construct a Riemannian manifold with Heisenberg group H_1 as boundary. A class of hypersurfaces in this space can be regarded as copies of the Heisenberg group. The induced Riemannian metrics on these hypersurfaces tend to the sub-Riemannian metric of the Heisenberg group as they approach the boundary. In [6], we were basically dealing with geodesics. We explored the relations between the properties of the geodesics in the interior, on the hypersurface, and on the boundary. In this paper, we study the Riemannian Laplace-Beltrami operator L on the Riemannian manifold. We calculate the heat kernel and Green's function for L, and give global estimates and small time asymptotics of the heat kernel. In addition, we restrict L to each hypersurface and calculate the corresponding heat kernel and Green's function. When a hypersurface approaches the boundary the restriction of L to the hypersurface degenerates to the standard sub-Laplacian of the Heisenberg group H_1. Therefore the heat kernel and Green's function on the hypersurface converge to the heat kernel and Green's function for the sub-Laplacian on the boundary respectively, as the hypersurface approaches the boundary. This can be easily seen from the expressions of the heat kernel and Green's function on the hypersurface.

For convenience of the reader, we recall some basic definitions and results from [6] here. The 3-dimensional Heisenberg group H_1 can be coordinatized as $R^3 = (x_1, x_2, t) = (x, t)$, with group law

$$(x,t) \circ (x',t') = (x + x', t + t' + 2ax_2x'_1 - 2ax_1x'_2),$$

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where a is a positive real parameter. The vector fields

$$X_1 = \frac{\partial}{\partial x_1} + 2ax_2\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x_2} - 2ax_1\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

are left invariant and generate the Lie algebra of H_1. The Lie algebra relations are

$$[X_1, X_2] = -4aT, \quad [X_1, T] = [X_2, T] = 0.$$

The Heisenberg (sub-)Laplacian is the left-invariant subelliptic operator

$$\Delta_H = \frac{1}{2}(X_1^2 + X_2^2).$$

The Green's kernel G for this operator was computed by Folland [3]. With pole at the origin,

$$G(x, t; 0, 0) = -\frac{1}{2\pi\sqrt{a^2|x|^4 + t^2}}.$$

The heat kernel for Δ_H was first computed by Gaveau[4] and Hulaniki [5]:

$$(1) \qquad P_0(x, t; 0, 0; s) = \frac{1}{(2\pi s)^2}\int_{-\infty}^{+\infty} \exp\Big(-\frac{f(x, t, \tau)}{s}\Big) V(\tau)\, d\tau,$$

where $f(x, t, \tau) = a\tau\coth(2a\tau)|x|^2 - i\tau t$, $V(\tau) = 2a\tau/\sinh(2a\tau)$, and $\theta = \tau/s$ is dual to t. See [1] for another way to compute the heat kernel.

Next consider H_1 as a subset of $\mathbf{C}^2 = \{(z, w)\}$. Introduce a group operation in $\mathbf{C}^2$ by

$$(z, w) \circ (z', w') = (z + z', w + w' + 2ia\bar{z}z').$$

Use also real coordinates x_1, x_2, y_1, y_2, with

$$z = x_1 + ix_2, \quad w = y_1 + iy_2.$$

Introduce the functions

$$t = y_1, \quad u = u(z, w) = y_2 - az\bar{z}.$$

Using the coordinate $(x, t, u) = (x_1, x_2, t, u)$ the group law is

$$(x, t, u) \circ (x', t', u') = \big(x + x', t + t' + 2a(x_2x_1' - x_1x_2'), u + u'\big).$$

Since $u\colon \mathbf{C}^2 \to (\mathbf{R}, +)$ is a group homomorphism our group is isomorphic to the direct product $H_1 \times \mathbf{R}$. The corresponding Lie algebra is generated by the left-invariant vector fields

$$X_1, \quad X_2, \quad T, \quad U = \frac{\partial}{\partial u}.$$

Consider the complex vector fields

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big), \quad \frac{\partial}{\partial w} = \frac{1}{2}\Big(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}\Big),$$
$$Z = \frac{\partial}{\partial z} + 2ia\bar{z}\frac{\partial}{\partial w}, \quad W = \frac{\partial}{\partial w},$$

and their conjugates. The Siegel domain

$$\mathbf{C}^2_+ = \{\mathrm{Im}\, w > az\bar{z}\} = \{u > 0\}$$

is a sub-semigroup of $\mathbf{C}^2$ and if we identify H_1 with $\{u = 0\}$, the boundary of $\mathbf{C}^2_+$, then H_1 is a subgroup of $\mathbf{C}^2$ that acts on $\mathbf{C}^2_+$ by left and right translations. The operator

$$L = Z\bar{Z} + \bar{Z}Z + 4au(W\bar{W} + \bar{W}W) + 2aU = \frac{1}{2}(X_1^2 + X_2^2) + 2au(T^2 + U^2) + 2aU$$

is elliptic in $\mathbf{C}^2_+$, self-adjoint in $L^2(\mathbf{C}^2_+)$, and invariant with respect to the H_1 action. In fact, it is easy to see that L is symmetric on $C_c^\infty(\overline{\mathbf{C}^2_+})$, and extends to a self-adjoint operator on $L^2(\mathbf{C}^2_+)$.

For each $u > 0$, the hypersurface $H_1 \times \{u\}$ is invariant with respect to the H_1 action. The restriction of L to this hypersurface is given by

$$L_u = \frac{1}{2}(X_1^2 + X_2^2) + 2auT^2.$$

It degenerates to the Heisenberg sublaplacian Δ_H as $u \to 0$.

The paper is organized as follows. In Section 2, we calculate the heat kernel for L in the interior. The kernel with pole at $(0, 0, u_0)$ is

(2)
$$\mathbf{P}(x, t, u; 0, 0, u_0; s) = \frac{1}{(2\pi)^2 s^3}\int_{-\infty}^{+\infty} \exp\Big(-\frac{\tau\coth(2a\tau)(a|x|^2 + u + u_0) - i\tau t}{s}\Big) \cdot \frac{2a\tau^2}{\sinh^2(2a\tau)} I_0\Big(2\sqrt{uu_0}\frac{\tau}{s\sinh(2a\tau)}\Big)\, d\tau.$$

In Sections 3 and 4, we use a method similar to that used in [2] to obtain global estimates and small time asymptotics for the heat kernel in the interior. The heat kernel for the operator L_u on $H_1 \times \{u\}$ is calculated in Section 5. The heat kernel with pole at the origin is

$$P_u(x, t; 0, 0; s) = \frac{1}{(2\pi s)^2}\int_{-\infty}^{+\infty} \exp\Big(-\frac{f_u(x, t; \tau)}{s}\Big)\frac{2a\tau}{\sinh(2a\tau)}\, d\tau, \tag{3}$$

where $f_u(x, t; \tau) = a\tau\coth(2a\tau)|x|^2 - i\tau t + 2au\tau^2$. We also show that when $x \neq 0$ the critical points of $f_u(x, t; \tau)$ on the imaginary axis are one-to-one corresponding to the geodesics from $(0, 0)$ to (x, t), and the length of the geodesic corresponding to a critical point $i\theta$ is $\sqrt{2f_u(x,t;i\theta)}$. Therefore the distance comes into various estimates. When $x = 0$, the above one-to-one correspondence does not hold, and the distance from $(0,0)$ to $(0,t)$ has different forms when $t/2\pi u \le 1$ or $t/2\pi u > 1$. Nevertheless in Section 6 we show that the distance comes into small time asymptotics when $x = 0$. In the last two sections, Green's functions are calculated by integrating corresponding heat kernels.

2 Heat Kernel for L in the Interior

We try to find the solution of the following equations:

$$\text{(4)} \qquad \begin{cases} L\mathbf{P} = \big(\frac{1}{2}(X_1^2 + X_2^2) + 2au(T^2+U^2) + 2aU\big)\,\mathbf{P} = \frac{\partial \mathbf{P}}{\partial s}, \quad s > 0 \\ \lim_{s\to 0^+} \mathbf{P} = \delta(x,t). \end{cases}$$

Since the coefficients of (4) do not depend on t, we take the Fourier transform with respect to t:

$$(L_1 + L_2)\hat{\mathbf{P}} = \frac{\partial \hat{\mathbf{P}}}{\partial s}, \quad \lim_{s\to 0} \hat{\mathbf{P}}(x,\theta,s) = \delta(x).$$

where

$$L_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - 4a^2x_2^2\theta^2 - 4a^2x_1^2\theta^2 + 4ax_2 i\theta\frac{\partial}{\partial x_1} - 4ax_1 i\theta\frac{\partial}{\partial x_2},$$

and

$$L_2 = -2au\theta^2 + 2auU^2 + 2aU.$$

Suppose that $X(x,\theta,s)$ is a solution of the following differential equation

$$\text{(5)} \qquad L_2 X(u,\theta,s) = \frac{\partial}{\partial s}X(u,\theta,s), \quad s > 0, \ \lim_{s\to 0^+} X(u,\theta,s) = \delta(u-u_0).$$

Applying the operator $L_1 + L_2$ to $\widehat{P_0}(x,\theta,s)X(u,\theta,s)$ we obtain

$$\begin{aligned}(L_1+L_2)\widehat{P_0}(x,\theta,s)X(u,\theta,s) &= \big(L_1\widehat{P_0}(x,\theta,s)\big)\,X(u,\theta,s) + \widehat{P_0}(x,\theta,s)L_2X(u,\theta,s) \\ &= \frac{\partial \widehat{P_0}}{\partial s}X + \widehat{P_0}\frac{\partial X}{\partial s} = \frac{\partial}{\partial s}\big(\widehat{P_0}(x,\theta,s)X(u,\theta,s)\big),\end{aligned}$$

where $\widehat{P_0}(x,\theta,s)$ is the Fourier transform (with respect to. t) of the heat kernel of H_1. Therefore $\widehat{P_0}(x,\theta,s)X(u,\theta,s)$ is the solution we want to find. We then only need to solve for X. Take the Laplace transform of both sides of the (5) with respect to u:

$$\text{(6)} \qquad -F(2au\theta^2X) + 2aF\Big(u\frac{\partial^2 X}{\partial u^2}\Big) + 2aF\Big(\frac{\partial X}{\partial u}\Big) = \frac{\partial F(X)}{\partial s},$$

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where $F(f)(v) = \int_0^{+\infty} f(u)\exp(-uv)\,du$ is the Laplace transform of the function f. Integration by parts gives

$$F\Big(\frac{\partial X}{\partial u}\Big) = vF(X) - X(0,\theta,s);$$

$$F\Big(u\frac{\partial^2 X}{\partial u^2}\Big) = v^2 F(uX) - 2vF(X) + X(0,\theta,s).$$

Substituting these into (6) we have

$$2a\big(v^2F(uX) - 2vF(X) + X(0,\theta,s)\big) + 2a\big(vF(X) - X(0,\theta,s)\big) - 2a\theta^2 F(uX) = \frac{\partial F(X)}{\partial s}.$$

Noticing that $F(uX) = -\frac{\partial}{\partial v}F(X)$, we may rewrite the above equation as

$$2a(\theta^2 - v^2)\frac{\partial}{\partial v}F(X) - 2avF(X) = \frac{\partial F(X)}{\partial s}. \tag{7}$$

The boundary condition $\lim_{s\to 0} X(u,\theta,s) = \delta(u-u_0)$ becomes

$$\lim_{s\to 0} F\big(X(v,\theta,s)\big) = F\big(\delta(u-u_0)\big) = \exp(-u_0 v).$$

Equation (7) is a first-order partial differential equation. We can solve it by the method of characteristic lines. The differential equations for the characteristic lines are

$$\begin{cases} \frac{dv(t)}{dt} = 2a\big(\theta^2 - v^2(t)\big), \ \frac{ds(t)}{dt} = -1, \ \frac{dz(t)}{dt} = 2av(t)z(t); \\ (v,s,z)|_{t=0} = (r, 0, e^{-u_0 r}), \end{cases}$$

which give

$$v(t) = \theta\frac{C_1 e^{4a\theta t} - 1}{C_1 e^{4a\theta t} + 1}, \quad s(t) = -t, \ z(t) = C_2 e^{-2a\theta t}(1 + C_1 e^{4a\theta t}),$$

where $C_1 = (\theta + r)/(\theta - r)$ and $C_2 = (\theta + r)e^{-u_0 r}/2\theta$. Eliminating parameters r and t we obtain

$$\begin{aligned} F(X) &= \frac{\theta}{v\sinh(2a\theta s) + \theta\cosh(2a\theta s)} \\ &\qquad \cdot \exp\Big(-u_0\theta\frac{\theta\big(1-\exp(-4a\theta s)\big) + v\big(1+\exp(-4a\theta s)\big)}{\theta\big(1+\exp(-4a\theta s)\big) + v\big(1-\exp(-4a\theta s)\big)}\Big) \\ &= \frac{\theta\exp\big(-u_0\theta\coth(2a\theta s)\big)}{v\sinh(2a\theta s) + \theta\cosh(2a\theta s)} \\ &\qquad \cdot \exp\Big(\frac{u_0\theta^2}{\sinh(2a\theta s)} \cdot \frac{1}{v\sinh(2a\theta s) + \theta\cosh(2a\theta s)}\Big) \\ &= \frac{A}{v+B}\exp\Big(\frac{C}{v+B}\Big), \end{aligned}$$

where

$$A = \frac{\theta}{\sinh(2a\theta s)} \exp(-u_0\theta \coth(2a\theta s), \ B = \theta \coth(2a\theta s), \ C = \frac{u_0\theta^2}{\sinh^2(2a\theta s)}.$$

Notice that A, B and C are all independent of v. In order to find X we need to take the inverse Laplace transform of $F(X)$:

$$X(u,\theta,s) = F^{-1}\big(F(X)\big) = \frac{A}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(iu\xi)}{i\xi + B} \exp\Big(\frac{C}{i\xi + B}\Big)\, d\xi.$$

The change of variable $\zeta = \xi - iB$ gives

$$X(u,\theta,s) = \frac{A}{2\pi i} \int_{-\infty - iB}^{+\infty - iB} \exp(iu(\zeta + iB) \exp\Big(\frac{C}{i\zeta}\Big) \frac{d\zeta}{\zeta}.$$

Let $\sigma = \sqrt{\frac{u}{C}}\zeta$. Then

$$\begin{aligned} X(u,\theta,s) &= \frac{A}{2\pi i} \exp(-uB) \int_{-\infty - iB\sqrt{\frac{C}{u}}}^{+\infty - iB\sqrt{\frac{C}{u}}} \exp\Big(i\sqrt{uC}\Big(\sigma + \frac{1}{\sigma}\Big)\Big) \frac{d\sigma}{\sigma} \\ &= A \exp(-uB)\, J_0(2i\sqrt{Cu}) \\ &= \frac{\theta}{\sinh(2a\theta s)} \exp\big(-(u_0 + u)\theta \coth(2a\theta s)\big)\, J_0\Big(2i\sqrt{uu_0}\frac{\theta}{\sinh(2a\theta s)}\Big), \end{aligned}$$

where $J_0(z)$ is Bessel's J function.

Taking the inverse Fourier transform of $\widehat{P_0}(x,\theta,s)X(u,\theta,s)$, using (1) and noticing that $\theta = \tau/s$, we have the heat kernel of the interior:

$$\begin{aligned} \mathbf{P}(x,t,u;0,0,u_0;s) &= \frac{1}{(2\pi s)^2} \int_{-\infty}^{+\infty} \exp\Big(-\frac{f}{s}\Big) V(\tau) \frac{\tau/s}{\sinh(2a\tau)} \\ &\qquad \cdot \exp(-\frac{\tau}{s}(u + u_0) \coth(2a\tau) I_0\Big(\frac{2\sqrt{uu_0}\tau}{s\sinh(2a\tau)}\Big)\, d\tau \\ &= \frac{1}{(2\pi)^2 s^3} \int_{-\infty}^{+\infty} \exp\Big(-\frac{\tau \coth(2a\tau)(a|x|^2 + u + u_0) - i\tau t}{s}\Big) \\ &\qquad \frac{2a\tau^2}{\sinh^2(2a\tau)} I_0\Big(\frac{2\tau\sqrt{uu_0}}{s\sinh(2a\tau)}\Big)\, d\tau, \end{aligned}$$

where $I_0(z)$ is Bessel's I function. Recall that $I_0(z) = J_0(iz)$.

3 A Global Estimate for the Heat Kernel of L

In this section we give a global estimate for the heat kernel of L. The method we are going to use is quite similar to that used in [2]. Taking advantage of scale invariance,

$$P(x,t,u;0,0,u_0;s) = u_0^{-3} P\Big(\frac{x}{\sqrt{u_0}}, \frac{t}{u_0}, \frac{u}{u_0}; 0, 0, 1; \frac{s}{u_0}\Big),$$

we may simplify by taking $u_0 = 1$. The heat kernel can be written as

$$\mathbf{P}(x,t,u;0,0,1;s) = \frac{1}{(2\pi)^2 s^3}\int_{-\infty}^{+\infty} \exp\Big(-\frac{\mathbf{f}}{s}\Big)\mathbf{V}(\tau,u,s)\,d\tau \tag{8}$$

where

$$\mathbf{f} = \mathbf{f}(x,t,u;0,0,1;\tau) = -it\tau + (a|x|^2+u+1)\tau\coth(2a\tau) + \frac{2\sqrt{u}\tau}{\sinh(2a\tau)},$$

is the modified complex action function;

$$\mathbf{V}(\tau,u,s) = \frac{2a\tau^2}{\sinh^2(2a\tau)} I_0(Z)\exp(-Z),$$

and

$$Z = \frac{2\sqrt{u}\tau}{s\sinh(2a\tau)}.$$

As in [6] we write $D = a|x|^2 + u + u_0 = a|x|^2 + u + 1$, and $E = -2\sqrt{uu_0} = -2\sqrt{u}$.

We have the following estimate for the heat kernel.

Theorem 1 *The heat kernel* $\mathbf{P}(x,t,u;0,0,1;s)$ *satisfies the estimate*

$$\mathbf{P}(x,t,u;0,0,1;s) \le C\frac{\exp(\frac{-d^2}{2s})}{s^3}\min\Big(1+\frac{d^{1/2}}{D^{1/4}},\frac{s^{1/2}}{D^{1/2}}\Big)\min\Big(1+\frac{d^{1/2}}{D^{1/4}},\frac{s^{1/2}}{u^{1/4}}\Big),\quad s>0, \tag{9}$$

where $d = d(x,t,u;0,0,1)$*, is the Riemannian distance between* (x,t,u) *and* $(0,0,1)$*.*

The following property of the function $I_0(z)\exp(-z)$ is easy to see.

Lemma 1

$$I_0(z)\exp(-z) \sim \frac{1}{\sqrt{2\pi z}},\quad z\to+\infty,$$

and

$$I_0(z)\exp(-z) \le C\min(1,z^{-1/2}),\quad z\in[0,+\infty),$$

where C *is a constant.*

From [6], we know that there is a unique shortest geodesic connecting two interior points (x,t,u) and $(0,0,1)$. This geodesic is given by the unique solution θ in the interval $[0.\pi/2a)$ of the equation

$$t = a\mu(2a\theta)|x|^2 + (u+1)\mu(2a\theta) - 2\sqrt{u}\Big(\frac{2a\theta\cos(2a\theta)}{\sin^2(2a\theta)} - \frac{1}{\sin(2a\theta)}\Big). \tag{10}$$

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The associated action $S(x,t,u;0,0,1;\theta)\big(= d^2(x,t,u;0,0,1)/2\big)$ is

$$S(x,t,u;0,0,1;\theta) = \frac{2a\theta^2}{\sin^2(2a\theta)}\big(a|x|^2 + u - 2\sqrt{u}\cos(2a\theta) + 1\big). \tag{11}$$

We denote by $\theta_c = \theta_c(x,t,u)$ the unique solution of (10) in the interval $[0,\pi/2a)$. Before we prove the theorem, we first consider the case when $2a\theta_c \le \pi - \epsilon_0$, where ϵ_0 is a small positive number. The contour for the integral (8) can be moved to the line $\operatorname{Im}\tau = \theta_c$:

$$\mathbf{P}(x,t,u;0,0,1;s) = \frac{1}{(2\pi)^2 s^3}\int_{\operatorname{Im}\tau=\theta_c} \exp\Big(-\frac{\mathbf{f}}{s}\Big)\mathbf{V}(\tau,u,s)\,d\tau.$$

We know from the proof of Theorem 3 in [6] that, on this line $\operatorname{Im}\tau = \theta_c$, $\operatorname{Re}\mathbf{f}$ has a strict minimum at $\tau = i\theta_c$, and $\mathbf{f}|_{\tau=i\theta_c} = d^2/2$. Therefore we have:

$$\mathbf{P}(x,t,u;0,0,1;s) \le \frac{\exp(-\frac{d^2}{2s})}{(2\pi)^2 s^3}\int_{\mathbf{R}} |\mathbf{V}(v+i\theta_c)|\,dv. \tag{12}$$

If we observe the function $\mathbf{f}$ more closely, we may get a better estimation when s/D is small.

$$\begin{aligned}\frac{\partial^2\mathbf{f}}{\partial\tau^2}\Big|_{\tau=i\theta_c} &= \frac{4aD}{\sin^2(2a\theta_c)}\big(1 - 2a\theta_c\cot(2a\theta_c)\big)\\ &\quad - E\frac{4a^2\theta_c\big(1+\cos^2(2a\theta_c)\big) - 4a\cos(2a\theta_c)\sin(2a\theta_c)}{\sin^3(2a\theta_c)}\\ &\ge \frac{4}{3}a(a|x|^2+u+1) + 2\sqrt{u}2a\frac{1}{3} = \frac{4}{3}a(a|x|^2+u+\sqrt{u}+1).\end{aligned} \tag{13}$$

If we write $\tau = v + i\theta$, over a sufficiently small interval $v \in [-\delta,\delta]$, $\delta = \delta(\epsilon_0)$, we have

$$\operatorname{Re}\mathbf{f} \ge \frac{d^2}{2} + \frac{4}{4}a(a|x|^2+u+\sqrt{u}+1)v^2 \ge \frac{d^2}{2} + aDv^2. \tag{14}$$

Outside that interval we have the following calculation:

$$\begin{aligned}&\operatorname{Re}\big(\mathbf{f}(x,t,u;v+i\theta_c) - \mathbf{f}\big(x,t,u;i\theta_c\big)\big)\\ &\quad= D\cdot\operatorname{Re}\big((v+i\theta_c)\coth(2a(v+i\theta_c)) - i\theta_c\coth(2ai\theta_c)\big)\\ &\qquad + E\cdot\operatorname{Re}\bigg(\frac{v+i\theta_c}{\sinh\big(2a(v+i\theta_c)\big)} - \frac{i\theta_c}{\sinh(2ai\theta_c)}\bigg)\end{aligned}$$

$$= \frac{\sinh^2(2av)D}{2a\big(\sinh^2(2av)+\sin^2(2a\theta_c)\big)}\big(2av\coth(2av)-2a\theta_c\coth(2a\theta_c)\big)$$
$$+\frac{E}{2a}\left(\frac{2av\sinh(2av)\cos(2a\theta_c)+2a\theta_c\cosh(2av)\sin(2a\theta_c)}{\sinh^2(2av)+\sin^2(2a\theta_c)}-\frac{2a\theta_c}{\sin(2a\theta_c)}\right)$$
$$\geq C(\epsilon_0)\frac{D}{2a},$$

where $C(\epsilon_0)$ is some positive constant depending on ϵ_0. Thus we have the following estimation:

(15)

$$\mathbf{P}(x,t,u;0,0,1;s)$$
$$\leq \frac{\exp(-\frac{d^2}{2s})}{(2\pi)^2s^3}\left(\int_{-\delta}^{\delta}\exp\Big(-\frac{aDv^2}{s}\Big)\,|\mathbf{V}(v+i\theta_c)|\,dv\right.$$
$$\left.+\int_{|v|>\delta}\exp\left(\frac{C(\epsilon_0)D}{2as}\right)|\mathbf{V}(v+i\theta_c)|\,dv\right)$$
$$\leq C\frac{\exp(-\frac{d^2}{2s})}{s^3}\left(\int_{\mathbf{R}}\exp\Big(-\frac{aDv^2}{s}\Big)\,dv+\exp\left(-\frac{C(\epsilon_0)D}{2as}\right)\right)\min\Big(1,\frac{s^{1/2}}{u^{1/4}}\Big)$$
$$\leq C'\frac{\exp(-\frac{d^2}{2s})}{s^3}\sqrt{\frac{2as}{D}}\min(1,s^{1/2}u^{-1/4}).$$

In the region under consideration $d^2=2S\sim D/a$, therefore (12) and (15) give the estimation (9) when $2a\theta_c\leq\pi-\epsilon_0$.

Proof of Theorem 1 Because both $\mathbf{f}$ and $\mathbf{V}$ have a pole at $2a\theta=i\pi$, the above estimates blow up as $2a\theta_c\to\pi$. We then use another contour instead. Let $\Gamma_1=\{|\tau-\pi i/2a|=\pi/2a-\theta_c\}$, a circle around $\pi/2a$ of radius $\pi/2a-\theta_c$ and $\Gamma_2=\{\operatorname{Im}\tau=\lambda\pi/2a\}$, the line $2a\operatorname{Im}\tau=\lambda\pi$. Then

$$\mathbf{P}(x,t,u;0,0,1;s)=\frac{1}{(2\pi)^2s^3}\left(\int_{\Gamma_1}\exp\Big(-\frac{\mathbf{f}}{s}\Big)\,|\mathbf{V}(\tau)|\,d\tau+\int_{\Gamma_2}\exp\Big(-\frac{\mathbf{f}}{s}\Big)\,|\mathbf{V}(\tau)|\,d\tau\right)$$
$$\equiv\mathbf{P_0}(x,t,u;0,0,1;s)+\mathbf{P_1}(x,t,u;0,0,1;s).$$

First we consider $\mathbf{P_1}$. Choose λ in the interval $(1,3/2]$ so that Γ_1 and Γ_2 are disjoint. Without loss of generality we may assume that $t>0$. Then we have

$$\operatorname{Re}\mathbf{f}(x,t,u,v+i\lambda\pi/2a)=\frac{\lambda\pi t}{2a}+\frac{D}{4a}\frac{2av\sinh(4av)+\lambda\pi\sin(2\lambda\pi)}{\sinh^2(2av)+\sin^2(\lambda\pi)}$$
$$-\frac{\sqrt{uu^0}}{a}\frac{2av\sinh(2av)\cos(\lambda\pi)+\lambda\pi\cosh(2av)\sin(\lambda\pi)}{\sinh^2(2av)+\sin^2(\lambda\pi)}$$
$$\geq\frac{\lambda\pi t}{2a}.$$

From (10), when $\theta_c \to \pi/2a$, $t/D \to +\infty$. Using (10) and (11), we have:

$$\lim_{\theta_c \to \pi/2a} \frac{t}{2aS} = \lim_{\varphi\to\pi} \frac{\big(D + E\cos\varphi\mu(\varphi)\big) - E\sin\varphi}{\frac{\varphi^2}{\sin^2\varphi}(D + E\cos\varphi)}$$
$$= \lim_{\varphi\to\pi} \left(\frac{\mu(\varphi)}{\frac{\varphi^2}{\sin^2\varphi}} - \frac{E\sin^3\varphi}{\varphi^2(D + E\cos\varphi)} \right)$$
$$= \lim_{\varphi\to\pi} \frac{\varphi - \sin\varphi\cos\varphi}{\varphi^2} = \frac{1}{\pi},$$

where $\varphi = 2a\theta_c$. Therefore the distance d from (x, t, u) to $(0, 0, 1)$, satisfies $d^2 = 2S \to \frac{\pi t}{a}$ as $\theta_c \to \pi/2a$. Thus

$$\mathbf{P_1} \le C\frac{\exp(-\frac{d^2}{2s})}{s^3} \exp\Big(-\frac{(\lambda-1)\,d^2}{2s}\Big) \min\Big(1, \frac{s^{1/2}}{u^{1/4}}\Big).$$

Since $t/D \to +\infty$ as $2a\theta_c \to \pi$, and $\exp\big(-(\lambda-1)\,d^2/2s\big)$ is dominated by $\sqrt{s/d}$, we obtain an estimate of the form (9) for $\mathbf{P_1}$.

For $\mathbf{P_0}$, we set

$$2a\tau = i\pi - i\xi,\ F = \pi(D - E) = \pi(a|x|^2 + u + 1 + 2\sqrt{u}), \quad \varepsilon = \pi - 2a\theta_c.$$

Then the function $\mathbf{f}$ can be written as

$$\mathbf{f} = \frac{t}{2a}(\pi - \xi) + D\frac{\pi - \xi}{2a}\frac{-\cos\xi}{\sin\xi} + \frac{E}{2a}\frac{\pi - \xi}{\sin\xi}$$
$$= \frac{t}{2a}(\pi - \xi) - \frac{F}{2a\xi} + \frac{G(x, u, \xi)}{2a},$$

where $G(x, u, \xi) = O(D)$ is a holomorphic function of ξ for $|\xi| < \pi$. Therefore

$$0 = \frac{\partial\mathbf{f}}{\partial\tau}(i\theta_c) = \frac{F}{2a\varepsilon^2} + \frac{G'(\varepsilon)}{2a} - \frac{t}{2a}.$$

It follows that

$$\mathbf{f} - \mathbf{f}_c = -\frac{F}{2a\xi} + \frac{F}{2a\varepsilon} + \frac{G(\xi) - G(\varepsilon)}{2a} - \frac{t}{2a}(\xi - \varepsilon)$$
$$= \frac{F}{2a}(\frac{1}{\epsilon} - \frac{1}{\xi}) + \frac{\xi - \varepsilon}{2a}\big(G'(\varepsilon) - t\big) + O\Big(\frac{D}{2a}|\xi - \varepsilon|^2\Big)$$
$$= \frac{1}{2a}\Big(\frac{F}{\varepsilon} - \frac{F}{\xi} - \frac{F}{\varepsilon^2}(\xi - \varepsilon)\Big) + O\Big(\frac{D}{2a}|\xi - \varepsilon|^2\Big)$$
$$= \frac{F}{2a\varepsilon}\Big(1 - \frac{\xi}{\varepsilon}\Big)\Big(1 - \frac{\varepsilon}{\xi}\Big) + O\Big(\frac{D}{2a}|\xi - \varepsilon|^2\Big),$$

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uniformly for $\varepsilon \leq \pi/2$. On the circle of integration $|\xi| = \varepsilon$, we set $\xi = \varepsilon e^{i\varphi}$, so that $\mathbf{f} - \mathbf{f}_c$ can be written as

$$\mathbf{f} - \mathbf{f}_c = \frac{\pi(a|x|^2 + u + 1 + 2\sqrt{u})}{a\varepsilon}(1 - \cos\varphi) + O\Big(\frac{D}{2a}\varepsilon^2(1 - \cos\varphi)\Big). \tag{16}$$

As $\theta_c \to \pi/2a$, $\mu(2a\theta_c) \sim (\pi - 2a\theta_c)^{-2}\pi^2 = \pi^2\varepsilon^{-2}$. Using (10), we have $(a|x|^2 + u + 1 + 2\sqrt{u})\varepsilon^{-2}\pi^2 \sim t$, and therefore

$$\mathbf{f} - \mathbf{f}_c \sim \Big(\frac{\varepsilon}{a\pi} + O(\varepsilon^4)\Big)\, t(1 - \cos\varphi).$$

It follows that for some $\varepsilon_0 > 0$,

$$\operatorname{Re}\mathbf{f} \geq \operatorname{Re}\mathbf{f}_c = \mathbf{f}_c = \frac{d^2}{2}, \quad |2a\tau - i\pi| \leq \varepsilon_0.$$

From Lemma 1, $|\mathbf{V}(\tau)| \leq \min(\varepsilon^{-2}, s^{1/2}u^{-1/4}\varepsilon^{-3/2})$ on the circle $|i\pi - 2a\pi| = \varepsilon$, and the circle has length $2\pi\varepsilon$. Thus we have the estimate

$$\mathbf{P_0} \leq C\frac{\exp(-\frac{d^2}{2s})}{s^3}\min\Big(\frac{1}{\varepsilon}, \frac{s^{1/2}}{\varepsilon^{1/2}u^{1/4}}\Big). \tag{17}$$

In this range, $\varepsilon \sim \sqrt{D}/t$ and $d^2 \sim \pi t/a$, so $\varepsilon \sim \sqrt{D}/d$, and (17) becomes

$$\mathbf{P_0} \leq C\frac{\exp(-\frac{d^2}{2s})}{s^3}(1 + d^{1/2}D^{-1/4})\min(1 + d^{1/2}D^{-1/4}, s^{1/2}u^{-1/4}). \tag{18}$$

On the other hand, notice $1 - \cos\varphi \geq \frac{2}{\pi^2}\varphi^2$ for $|\varphi| \leq \pi$, so (16) implies

$$\begin{aligned}
\mathbf{P_0} &= \frac{\exp(-\frac{d^2}{2s})}{(2\pi)^2 s^3}\int_{|\tau - i\pi/2a| = \varepsilon/2a} \exp\Big(-C\frac{D\varphi^2}{2as\varepsilon}\Big)\,|\mathbf{V}(\tau, u, s)|\, d\tau \\
&\leq C\frac{\exp(-\frac{d^2}{2s})}{s^3}\int_{-\pi}^{\pi} \exp\Big(-C\frac{D\varphi^2}{2as\varepsilon}\Big)\, d\varphi \cdot \min\Big(\frac{1}{\varepsilon}, \frac{s^{1/2}}{\varepsilon^{1/2}u^{1/4}}\Big) \\
&\leq C\frac{\exp(-\frac{d^2}{2s})}{s^3}\sqrt{\frac{as\varepsilon}{D}} \cdot \min\Big(\frac{1}{\varepsilon}, \frac{s^{1/2}}{\varepsilon^{1/2}u^{1/4}}\Big).
\end{aligned} \tag{19}$$

Again $\varepsilon \sim \sqrt{D}/d$, therefore (19) and (18) imply (9). ■

4 Small Time Behavior of the Heat Kernel of L

Theorem 2 *Given a fixed point (x, t, u) in the interior, then*

$$\mathbf{P}(x, t, u; 0, 0, 1; s) = \frac{a\exp(\frac{-d^2}{2s})}{(2\pi)^2 s^2\sqrt{u}}\big(\Theta(x, t, u) + O(\sqrt{s})\big),$$

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as $s \to 0+$, *where*

$$\Theta(x,t,u) = \sqrt{\frac{2}{\mathbf{f}''(i\theta_c)}}\left(\frac{\theta_c}{\sin(2a\theta_c)}\right)^{3/2}.$$

When t/D *is large,* $\Theta(x,t,u)$ *has the following behaviour:*

$$\Theta(x,t,u) = \frac{\pi}{4a^2}\frac{1}{\sqrt{D+2\sqrt{u}}}\big(1+O(\sqrt{D/t})\big). \tag{20}$$

Proof

$$\begin{aligned}
\mathbf{P}(x,t,u;0,0,1;s) &= \frac{1}{(2\pi)^2 s^3}\int_{-\infty}^{+\infty}\exp\Big(-\frac{\mathbf{f}}{s}\Big)\mathbf{V}(\tau,u,s)\,d\tau\\
&= \frac{\exp(-\frac{d^2}{2s})}{(2\pi)^2 s^3}\int_{-\infty}^{+\infty}\exp\left(-\frac{\Phi(v)}{s}\right)\mathbf{V}(v+i\theta_c,u,s)\,dv\\
&= \frac{\exp(-\frac{d^2}{2s})}{(2\pi)^2 s^3}\Big(\int_{-\delta}^{\delta}+\int_{|v|>\delta}\Big)\exp\left(-\frac{\Phi(v)}{s}\right)\mathbf{V}(v+i\theta_c,u,s)\,dv\\
&\equiv \frac{\exp(-\frac{d^2}{2s})}{(2\pi)^2 s^3}(I_\delta + I'_\delta).
\end{aligned}$$

Where $\Phi(v) = \mathbf{f}(x,t,u;v+i\theta_c) - \mathbf{f}(x,t,u;i\theta_c)$ and $\delta \le 1$ is to be chosen. We know that on the line $\tau = v + i\theta_c$, $v \in \mathbf{R}$, Re $\mathbf{f}$ attains its global minimum $d^2/2$ only at $i\theta_c$; it is a strictly increasing function of $|v|$. Also from (14), Re $\Phi(v) \ge aD|v|^2$, for v near 0. Therefore

$$|I'_\delta| \le \exp\left(-\frac{\Phi(\delta)}{s}\right)\int_{\mathbf{R}}|\mathbf{V}(v+i\theta_c)|\,dv \le C\exp\Big(-\frac{aD\delta^2}{s}\Big)\min\Big(1,\frac{s^{1/2}}{u^{1/4}}\Big),$$

where $C = C(x,t,u) > 0$. Now turn to the estimate of I'_δ as $s \to 0+$. $\Phi(0) = 0$, $\Phi'(0) = \mathbf{f}'(i\theta_c) = 0$ and $\Phi''(0) = \mathbf{f}''(i\theta_c) \ge 4D/3$ (see (13)). So we can write $\Phi(v)$ as

$$\Phi(v) = \Phi''(0)\frac{v^2}{2} + O(|v|^3).$$

We may choose a $\delta > 0$, $\delta \in (0, \pi/2a - \theta_c$, such that

$$\Phi(v) = \Phi''(0)\frac{z^2}{2}, \quad |v| \le \delta,$$

for some new variable $z = v + O(v^2)$. Then,

$$I_\delta = \int_{z(-\delta)}^{z(\delta)}\exp\left(-\frac{\Phi''(0)z^2}{2s}\right)\mathbf{V}\big(v(z)+i\theta_c\big)\frac{dv}{dz}\,dz.$$

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The path of the above integration may be complex. Since the integrand is holomorphic in z, by moving the path to the real axis, the error is dominated by $\exp(-c/s)$. Also if we write $z = \sigma + i\gamma$ on the path of above integration, then $|\gamma| < c\sigma^2$. Thus we have

$$I_\delta = \int_{-\delta}^{\delta} \exp\bigl(-\Phi''(0)z^2/2s\bigr)\,\mathbf{V}\bigl(v(z)+i\theta_c,s,u\bigr)\frac{dv}{dz}\,dz + O\bigl(\exp(-c/s)\bigr). \tag{21}$$

For $\mathbf{V}\bigl(v(z)+i\theta_c,s,u\bigr)$, we have the following estimation:

$$\begin{aligned}
\mathbf{V}\bigl(v(z)+i\theta_c,s,u\bigr) &= 2a\frac{\tau^2}{\sinh^2(2a\tau)}I_0(Z)\exp(-Z)\ \tau = v(z)+i\theta_c,\ Z=\frac{2\sqrt{u}\tau}{s\sinh(2a\tau)}\\
&= 2a\frac{\tau^2}{\sinh^2(2a\tau)}\frac{1}{\sqrt{2\pi Z}}\bigl(1+O(Z^{-1})\bigr)\\
&= 2a\Bigl(\frac{\tau}{\sin(2a\tau)}\Bigr)^{3/2}\sqrt{\frac{s}{4\pi\sqrt{u}}}\Bigl(1+O\Bigl(\frac{s}{u^{1/2}}\Bigr)\Bigr)\\
&= a\Bigl(\Bigl(\frac{\theta_c}{\sin(2a\theta_c)}\Bigr)^{3/2}+O(z)\Bigr)\frac{s^{1/2}}{\pi^{1/2}u^{1/4}}\Bigl(1+O\Bigl(\frac{s}{u^{1/2}}\Bigr)\Bigr).
\end{aligned}$$

Therefore (21) becomes

$$\begin{aligned}
I_\delta &= \int_{-\delta}^{\delta} a\exp\Bigl(-\frac{\Phi''(0)z^2}{2s}\Bigr)\Bigl(\Bigl(\frac{\theta_c}{\sin(2a\theta_c)}\Bigr)^{3/2}+O(z)\Bigr)\frac{s^{1/2}}{\pi^{1/2}u^{1/4}}\\
&\qquad\cdot\Bigl(1+O\Bigl(\frac{s}{u^{1/2}}\Bigr)\Bigr)\frac{dv}{dz}\,dz + O\Bigl(\exp\Bigl(-\frac{c}{s}\Bigr)\Bigr)\\
&= \int_{-\delta}^{\delta} a\exp\Bigl(-\frac{\Phi''(0)z^2}{2s}\Bigr)\Bigl(\frac{\theta_c}{\sin(2a\theta_c)}\Bigr)^{3/2}\frac{s^{1/2}}{\pi^{1/2}u^{1/4}}\Bigl(1+O\Bigl(\frac{s}{u^{1/2}}\Bigr)\Bigr)\,dz\\
&\qquad+\int_{-\delta}^{\delta} a\exp\Bigl(-\frac{\Phi''(0)z^2}{2s}\Bigr)O(z)\frac{s^{1/2}}{\pi^{1/2}u^{1/4}}\Bigl(1+O\Bigl(\frac{s}{u^{1/2}}\Bigr)\Bigr)\,dz\\
&\qquad+O\Bigl(\exp\Bigl(-\frac{c}{s}\Bigr)\Bigr)\\
&= \Bigl(\int_{\mathbf{R}}-\int_{|z|>\delta}\Bigr)a\exp\Bigl(-\frac{\Phi''(0)z^2}{2s}\Bigr)\Bigl(\frac{\theta_c}{\sin(2a\theta_c)}\Bigr)^{3/2}\frac{s^{1/2}}{\pi^{1/2}u^{1/4}}\\
&\qquad\cdot\Bigl(1+O\Bigl(\frac{s}{u^{1/2}}\Bigr)\Bigr)\,dz + O\Bigl(\frac{s^{3/2}}{u^{1/4}}\Bigr)\\
&= \frac{\sqrt{2}as}{u^{1/4}}\Bigl(\frac{\theta_c}{\sin(2a\theta_c)}\Bigr)^{3/2}\Bigl(\frac{1}{\mathbf{f}''(i\theta_c)}\Bigr)^{1/2}+O\Bigl(\frac{s^{3/2}}{u^{1/4}}\Bigr)
\end{aligned}$$

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which gives the estimation for $\mathbf{P}(x, t, u; 0, 0, 1; s)$ as $s \to 0+$:

$$\mathbf{P}(x, t, u; 0, 0, 1; s) = \frac{a \exp(-\frac{d^2}{2s})}{(2\pi)^2 s^2 u^{1/4}} \left(\sqrt{\frac{2}{\mathbf{f}''(i\theta_c)}} \left(\frac{\theta_c}{\sin(2a\theta_c)} \right)^{3/2} + O(\sqrt{s}) \right)$$

When t/D is very large, $\varepsilon = \pi - 2a\theta_c$ is very small. Using (13), the formula for $\mathbf{f}''(i\theta_c)$ we have

$$\begin{aligned}
&\Theta(x, t, u) \\
&= \sqrt{\frac{2}{\mathbf{f}''(i\theta_c)}} \left(\frac{\theta_c}{\sin(2a\theta_c)} \right)^{3/2} \\
&= \left(\frac{2\theta_c^3}{4aD\big(\sin(2a\theta_c) - 2a\theta_c \cos(2a\theta_c)\big) + 2\sqrt{u}\big(4a^2\theta_c\big(1 + \cos^2(2a\theta_c)\big) - 2a\sin(4a\theta_c)\big)} \right)^{1/2} \\
&= \frac{\pi}{4a^2} \frac{1}{\sqrt{D + 2\sqrt{u}}} \big(1 + O(\varepsilon)\big).
\end{aligned}$$

When t/D is large, $\varepsilon \sim \sqrt{(D + 2\sqrt{u})/t} \sim \sqrt{D/t}$, which yields (20). ■

5 Heat Kernel for L_u on the Hypersurface: $H_1 \times \{u\}$

In this section we calculate the heat kernel for L_u on the hypersurface $H_1 \times \{u\}$. Because of the left invariance under the H_1 action, it is enough to consider the heat kernel with pole at the origin. We need to find the solution of the following equations:

$$\begin{cases} L_u P_u = \left(\frac{1}{2}(X_1^2 + X_2^2) + 2auT^2 \right) P_u = \frac{\partial P_u}{\partial s}, & s > 0 \\ \lim_{s \to 0^+} P_u(x, t, s) = \delta(x, t). \end{cases} \tag{22}$$

Since the coefficients of (22) do not depend on t, we take the Fourier transform with respect to t:

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - 4a^2 x_2^2 \theta^2 - 4a^2 x_1^2 \theta^2 - 2au\theta^2 + 4ax_2 i\theta \frac{\partial}{\partial x_1} - 4ax_1 i\theta \frac{\partial}{\partial x_2} \right) \widehat{P}_u = \frac{\partial \widehat{P}_u}{\partial s}. \tag{23}$$

The boundary condition becomes $\lim_{s \to 0} \widehat{P}_u(x, \theta, s) = \delta(x)$. Let

$$L_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - 4a^2 x_2^2 \theta^2 - 4a^2 x_1^2 \theta^2 + 4ax_2 i\theta \frac{\partial}{\partial x_1} - 4ax_1 i\theta \frac{\partial}{\partial x_2}.$$

Then (23) can be rewritten as

$$(L_1 - 2au\theta^2)\widehat{P}_u = \frac{\partial \widehat{P}_u}{\partial s}$$

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It can be easily seen that

$$\widehat{P_u}(x,\theta,s) = \exp(-2au\theta^2 s)\widehat{P_0}(x,\theta,s),$$

where $\widehat{P_0}(x,\theta,s)$ is the Fourier transform (with respect to t) of the heat kernel of H_1. Take the inverse Fourier transform of both sides:

$$P_u = \int_{-\infty}^{+\infty} \exp(it\theta)\exp(-2au\theta^2 s)\widehat{P_0}\, d\theta$$

We plug in the formula for the heat kernel of the boundary (1) and notice that $\theta = \tau/s$, we have

$$\begin{aligned} P_u(x,t;0,0;s) &= \frac{1}{(2\pi s)^2}\int_{-\infty}^{+\infty} \exp\Big(-\frac{f(x,t,\tau)}{s}\Big) V(\tau)\exp\Big(-2au\cdot\frac{\tau^2}{s^2}\cdot s\Big)\, d\tau \\ &= \frac{1}{(2\pi s)^2}\int_{-\infty}^{+\infty} \exp\Big(-\frac{a\tau\coth(2a\tau)|x|^2 - i\tau t}{s}\Big)\frac{2a\tau}{\sinh(2a\tau)} \\ &\qquad \exp\Big(-2au\frac{\tau^2}{s}\Big)\, d\tau. \end{aligned}$$

It is obvious that $\lim_{u\to 0^+} P_u(x,t;0,0;s) = P_0(x,t;0,0;s)$, which means the heat kernel for L_u on the hypersurface $H_1\times\{u\}$ converges to the heat kernel for Δ_H on H_1. If we set $f_u(x,t;\tau) = a\tau\coth(2a\tau)|x|^2 - i\tau t + 2au\tau^2$, the heat kernel $P_u(x,t;0,0;s)$ can be written in the same form as (1):

$$P_u(x,t;0,0;s) = \frac{1}{(2\pi s)^2}\int_{-\infty}^{+\infty} \exp\Big(-\frac{f_u(x,t,\tau)}{s}\Big) V(\tau)\, d\tau.$$

We have the following proposition, which shows the connection between geodesics from $(0,0)$ to (x,t) and critical points of $f_u(x,t;\tau)$.

Proposition 1 *For any (x,t) with $x\neq 0$ and $t\geq 0$, the function $f_u(x,t;\tau)$ has finitely many critical points on the imaginary axis, which are one-to-one corresponding to the geodesics from $(0,0)$ to (x,t). The length of the geodesic corresponding to a critical point $i\theta$ is $\sqrt{2f_u(x,t;i\theta)}$.*

Proof From Theorem 5 in [6] the geodesics that join the origin to (x,t) are indexed by the solutions of

$$t = a\mu(2a\theta)|x|^2 + 4au\theta,$$

and their lengths l_θ are given by

$$l_\theta^2 = 2S(x,t,1;\theta) = \frac{(2a\theta)^2}{\sin^2(2a\theta)}|x|^2 + 4au\theta^2.$$

We set

$$F(\theta) = f_u(x,t;i\theta) = a\theta\cot(2a\theta)|x|^2 + t\theta - 2au\theta^2,$$

and note that

$$F'(\theta) = t - a\mu(2a\theta)|x|^2 - 4au\theta.$$

Then the theorem follows from an easy calculation. ∎

6 Small Time Behavior of the Heat Kernel of L_u

We see in the small time estimate of the heat kernel of L, there is a factor $\exp(-d^2/2s)$, where d is the distance from the point to the singularity. On the hypersurface $H_1 \times \{u\}$, the distance from $(0,t)$ to the singularity $(0,0)$ satisfies (see Theorem 6 in [6])

$$d\big((0,t),(0,0)\big) = \begin{cases} \frac{|t|}{\sqrt{4au}}, & |t| \leq 2\pi u, \\ \sqrt{\frac{\pi(|t|-\pi u)}{a}}, & |t| > 2\pi u. \end{cases}$$

Therefore we may expect the small time estimates of the heat kernel of L_u have different forms in these two cases. From (3), the heat kernel of L_u can be written as

$$\begin{aligned} P_u(0,t;0,0;s) &= \frac{1}{2a(2\pi s)^2} \int_{-\infty}^{+\infty} \frac{\tau}{\sinh\tau} \exp\big(-(\beta\tau^2 + \alpha\tau i)/s\big)\, d\tau \\ &= \frac{1}{2a(2\pi s)^2} \int_{-\infty}^{+\infty} \exp\Big(-\frac{\alpha^2}{4\beta s}\Big) \frac{\tau}{\sinh\tau} \exp\Big(-\frac{\beta}{s}\Big(\tau - \frac{\alpha}{2\beta}i\Big)^2\Big)\, d\tau \end{aligned}$$

where $\alpha = t/2a$, and $\beta = u/2a$. Now let us consider the behavior of $P_u(0,t;0,0;s)$ as $s \to 0$. We may assume that $t > 0$.

Case I: $\frac{\alpha}{2\beta} = \frac{t}{2u} < \pi$

Since there is no pole for the integrand on the strip $\{\tau \mid 0 \leq \operatorname{Im}\tau < \pi\}$, we can shift the contour a distance $\frac{\alpha}{2\beta}$ upward:

$$P_u = \frac{\exp(-\frac{\alpha^2}{4\beta s})}{2a(2\pi s)^2} \int_{-\infty}^{+\infty} \frac{\tau + \alpha i/2\beta}{\sinh(\tau + \alpha i/2\beta)} \exp(-\beta\tau^2/s)\, d\tau.$$

Since $0 < t < 2\pi u$, the distance between $(0,t)$ and $(0,0)$ is $d = t/\sqrt{4au}$, and therefore

$$\frac{\alpha^2}{4\beta} = \frac{t^2}{8au} = \frac{d^2}{2}.$$

The heat kernel can be rewritten as

$$P_u = \frac{\exp(-\frac{d^2}{2s})}{2a(2\pi s)^2} \sqrt{\frac{s}{\beta}} \int_{-\infty}^{+\infty} \frac{\sqrt{\frac{s}{\beta}}\tau + \alpha i/2\beta}{\sinh(\sqrt{\frac{s}{\beta}}\tau + \alpha i/2\beta} \exp(-tau^2)\, d\tau.$$

As $s \to 0+$,

$$\frac{\sqrt{\frac{s}{\beta}}\tau + \alpha i/2\beta}{\sinh(\sqrt{\frac{s}{\beta}}\tau + \alpha i/2\beta)} = \frac{\alpha/2\beta}{\sin(\alpha/2\beta)}\big(1 + O(s)\big) = \frac{t/2u}{\sin(t/2u)}\big(1 + O(s)\big),$$

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which gives the following estimate:

$$P_u = \frac{t\pi^{1/2}}{(4\pi)^2 u\sqrt{au}\sin(t/2u)} \frac{\exp(-d^2/2s)}{s^{-3/2}}\big(1+O(s)\big), \quad s \to 0+.$$

Case II: $t = 2\pi u$ In this case there is a pole πi on the line $\operatorname{Im}\tau = \pi$, so we use another contour, $C(\varepsilon)$, instead. $C(\varepsilon)$ is composed of three parts, $(-\infty+\pi i, -\varepsilon+\pi i]$, $[\varepsilon+\pi i, +\infty+\pi i)$, and a semi-circle $\{\tau \mid |\tau - \pi i| = \varepsilon, \operatorname{Im}\tau < \pi\}$.

$$P_u = \frac{\exp(-\frac{\alpha^2}{4\beta s})}{2a(2\pi s)^2}\int_{C(\varepsilon)} \frac{\tau}{\sinh\tau}\exp\big(-\beta(\tau-\pi i)^2/s\big)\,d\tau.$$

Letting ε go to 0, the integral over the semi-circle goes to half of the residue, which is π^2, and P_u becomes

$$P_u = \frac{\exp(-\frac{\alpha^2}{4\beta s})}{2a(2\pi s)^2}\left(\pi^2 - \int_{-\infty}^{+\infty}\frac{\tau}{\sinh\tau}\exp\Big(-\frac{\beta}{s}\tau^2\Big)\,d\tau\right).$$

Noticing that

$$\int_{-\infty}^{+\infty}\frac{\tau}{\sinh\tau}\exp\Big(-\frac{\beta}{s}\tau^2\Big)\,d\tau = \sqrt{\frac{\pi s}{\beta}}\big(1+O(s)\big), \quad s\to 0+,$$

we have

$$P_u = \frac{\exp(-d^2/2s)}{2a(2\pi s)^2}\left(\pi^2 + \sqrt{\frac{2a\pi s}{u}}\big(1+O(s)\big)\right).$$

Case III: $2\pi u < t < 4\pi u$ As in Case I, we shift the contour $t/2u$ upward. There is only one pole of the integrand in the strip $\{\tau \mid 0 < \operatorname{Im}\tau < t/2u\}$, and the residue is $-\pi i\exp\big(-\frac{\beta}{s}(\frac{\alpha i}{2\beta}-\pi i)^2\big)$.

$$\begin{aligned}P_u &= \frac{\exp(-\frac{\alpha^2}{4\beta s})}{2a(2\pi s)^2}\left(\int_{\operatorname{Im}\tau=t/2u}\frac{\tau}{\sinh\tau}\exp\left(-\frac{\beta}{s}\Big(\tau-\frac{\alpha i}{2\beta}\Big)^2\right)d\tau\right.\\ &\qquad\left.+2\pi i\cdot(-\pi i)\exp\left(-\frac{\beta}{s}\Big(\frac{\alpha i}{2\beta}-\pi i\Big)^2\right)\right)\\ &= \frac{\exp\big(-\frac{\alpha^2}{4\beta s}+\frac{u}{2as}(\frac{t}{2u}-\pi)^2\big)}{2a(2\pi s)^2}\left(\exp\left(-\frac{\beta}{s}\Big(\frac{\alpha}{2\beta}-\pi\Big)^2\right)\right.\\ &\qquad\left.\cdot\int_{-\infty}^{+\infty}\frac{\tau+\alpha i/2\beta}{\sinh(\tau+\alpha i/2\beta)}\exp(-\beta\tau^2/s)\,d\tau + 2\pi^2\right).\end{aligned}$$

In this case, the distance between $(0,t)$ and $(0,0)$ is $d = \sqrt{\pi(t-\pi u)/a}$, therefore

$$-\frac{\alpha^2}{4\beta s} + \frac{u}{2as}\Big(\frac{t}{2u}-\pi\Big)^2 = -\frac{(t-\pi u)\pi}{2as} = -\frac{d^2}{2s}.$$

Also noticing,

$$\int_{-\infty}^{+\infty} \frac{\tau + \alpha i/2\beta}{\sinh(\tau + \alpha i/2\beta)} \exp(-\beta\tau^2/s)\, d\tau = \frac{t/2u}{\sin(t/2u)} \sqrt{\frac{2a\pi s}{u}} \big(1 + O(s)\big)\,, s \to 0+,$$

we have

$$P_u = \frac{\exp(-d^2/2s)}{4as^2} \left(1 + \frac{t/2u}{\sin(t/2u)} \sqrt{\frac{2a\pi s}{u}} \exp\left(-\frac{u}{2as}\left(\frac{t}{2u} - \pi\right)^2\right) \big(1 + O(s)\big)\right).$$

Case IV: $t \in \big(2\pi mu, 2\pi(m+1)u\big)$, where $m > 1$ is a positive integer Similar to Case III, we shift the contour $t/2u$ upward. There are m poles of the integrand in the strip $\{\tau \mid 0 < \operatorname{Im}\tau < t/2u\}$, and the residues are $(-1)^j \pi i \exp\big(-\frac{\beta}{s}(\frac{\alpha i}{2\beta} - j\pi i)^2\big)$, $j = 1, 2, \dots, m$. Therefore we have

$$\begin{aligned} P_u &= \frac{\exp(-d^2/2s)}{4as^2}(1 + \exp(-\frac{\beta}{s}(\frac{\alpha}{2\beta} - \pi)^2)(\sum_{j=2}^{m}(-1)^{j+1}\exp(\frac{\beta}{s}(\frac{\alpha}{2\beta} - j\pi)^2) \\ &\quad + \int_{-\infty}^{+\infty} \frac{\tau + \alpha i/2\beta}{\sinh(\tau + \alpha i/2\beta)} \exp(-\beta\tau^2/s)\, d\tau) \\ &= \frac{\exp(-d^2/2s)}{4as^2}\left(1 + O\Big(\exp\big((-t + 3\pi u)\pi/2as\big)\Big)\right), \quad s \to 0+. \end{aligned}$$

Case V: $t = 2\pi mu$, where $m > 1$ is a positive integer As in Case II, there is a pole $m\pi i$ on the line $\operatorname{Im}\tau = m\pi$, so we use contour $C(\varepsilon)$ instead, where $C(\varepsilon)$ is composed of three parts, $(-\infty + m\pi i, -\varepsilon + m\pi i]$, $[\varepsilon + m\pi i, +\infty + m\pi i)$, and a semi-circle $\big\{\tau \mid |\tau - m\pi i| = \varepsilon, \operatorname{Im}\tau < m\pi\big\}$. A similar calculation gives

$$P_u = \frac{\exp(-d^2/2s)}{4as^2}\left(1 + O\Big(\exp\big((-t + 3\pi u)\pi/2as\big)\Big)\right), \quad s \to 0+.$$

Remark When $t/2u$ is small($t/2u < \pi$), the heat kernel behaves like

$$Cs^{-3/2}\exp(-d^2/2s).$$

This behavior is quite similar to the Euclidean case. But when $t/2u$ is big, the heat kernel behaves like $(4as^2)^{-1}\exp(-d^2/2s)$, which is very similar to the sub-Riemannian Heisenberg case (see Theorem 2.46 in [2]).

7 The Green's Function of the Hypersurface $H_1 \times \{u\}$

Theorem 3 *The Green's function of the hypersurface* $H_1 \times \{u\}$ *is*

$$G_u(x, t; 0, 0) = -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \frac{1}{a\cosh(\tau)|x|^2 - it\sinh(\tau) + u\tau\sinh(\tau)}\, d\tau.$$

Proof Integrating the heat kernel P_u with respect to the time variable s, we get the Green's function.

$$\begin{aligned}
G_u(x,t;0,0) &= -\int_0^\infty P_u(x,t;0,0;s)\,ds \\
&= -\int_0^\infty \frac{1}{(2\pi s)^2}\int_{-\infty}^{+\infty} \exp \\
&\quad \left(-\frac{a\tau\coth(2a\tau)|x|^2 - i\tau t}{s}\right)\frac{2a\tau}{\sinh(2a\tau)}\exp\left(-2au\frac{\tau^2}{s}\right)d\tau\,ds \\
&= -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{2a\tau}{\sinh(2a\tau)} \\
&\quad \left(\int_0^\infty \frac{1}{s^2}\exp\left(\frac{1}{s}\big(-a\tau\coth(2a\tau)|x|^2 + i\tau t - 2au\tau^2\big)\right)ds\right)d\tau \\
&= -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{2a\tau}{\sinh(2a\tau)}\cdot\frac{1}{a\tau\coth(2a\tau)|x|^2 - i\tau t + 2au\tau^2)}\,d\tau \\
&= -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{2a\tau}{a\tau\cosh(2a\tau)|x|^2 - i\tau t\sinh(2a\tau) + 2au\tau^2\sinh(2a\tau)}\,d\tau
\end{aligned}$$

Changing variable $2a\tau \to \tau$, we have

$$G_u(x,t;0,0) = -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{1}{a\cosh(\tau)|x|^2 - it\sinh(\tau) + u\tau\sinh(\tau)}\,d\tau. \qquad \blacksquare$$

It can be easily seen that

$$\begin{aligned}
\lim_{u\to 0^+} G_u(x,t;0,0) &= \lim_{u\to 0^+} -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{1}{a\cosh(\tau)|x|^2 - it\sinh(\tau) + u\tau\sinh(\tau)}\,d\tau \\
&= -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{1}{a\cosh(\tau)|x|^2 - it\sinh(\tau)}\,d\tau \\
&= -\frac{1}{2\pi\sqrt{a^2|x|^4 + t^2}}.
\end{aligned}$$

Therefore Green's function for L_u on the hypersurface $H_1 \times \{u\}$ converges to Green's function for Δ_H on H_1.

8 The Green's Function of the Interior

Theorem 4 *The Green's function of L in the interior is*

$$\mathbf{G}(x,t,u;0,0,u_0) = -\frac{1}{2\pi^2}\cdot\frac{1}{(a|x|^2 + u + u_0)^2 + t^2 - 4uu_0}.$$

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Proof From (2), the expression of the heat kernel of the interior, we have

$$\begin{aligned}
&\mathbf{G}(x,t,u;0{,}0,u_0)\\
&\quad= -\int_0^\infty \mathbf{P}(x,t,u;0,0,u_0;s)\,ds\\
&\quad= -\int_0^\infty \frac{1}{(2\pi)^2 s^3}\int_{-\infty}^{+\infty} \exp\Big(-\frac{1}{s}(a|x|^2+u+u_0)\tau\coth(2a\tau)+\frac{it\tau}{s}\Big)\\
&\qquad\cdot \frac{2a\tau^2}{\sinh^2(2a\tau)} I_0\Big(2\sqrt{uu_0}\frac{\tau}{s\sinh(2a\tau)}\Big)\,d\tau\,ds.
\end{aligned}$$

Changing variable $\frac{1}{s}\to s$,
(24)

$$\begin{aligned}
&\mathbf{G}(x,t,u;0{,}0,u_0)\\
&\quad= -\frac{1}{(2\pi)^2}\int_0^\infty\int_{-\infty}^{+\infty} s\exp\big(-s(a|x|^2+u+u_0)\tau\coth(2a\tau)+it\tau s\big)\\
&\qquad\cdot\frac{2a\tau^2}{\sinh^2(2a\tau)} I_0\Big(2\sqrt{uu_0}\frac{\tau s}{\sinh(2a\tau)}\Big)\,d\tau\,ds\\
&\quad\equiv \int_{-\infty}^{+\infty}\int_0^\infty \beta\cdot\exp(\alpha\cdot s)\cdot I_0(\gamma\cdot s)s\,ds\,d\tau,
\end{aligned}$$

where

$$\alpha = -(a|x|^2+u+u_0)\tau\coth(2a\tau)+it\tau,$$

$$\beta = -\frac{1}{(2\pi)^2}\frac{2a\tau^2}{\sinh^2(2a\tau)},\qquad \gamma = 2\sqrt{uu_0}\frac{\tau}{\sinh(2a\tau)}.$$

We have the following integral formula

$$\int_0^\infty s\exp(\alpha\cdot s)\cdot I_0(\gamma\cdot s)\,ds = -\frac{\alpha}{(\alpha^2-\gamma^2)^{3/2}}.$$

Thus (24) becomes

$$\begin{aligned}
&\mathbf{G}(x,t,u;0{,}0,u_0)\\
&\quad= -\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^2}\frac{2a\tau^2}{\sinh^2(2a\tau)}\\
&\qquad\frac{(a|x|^2+u+u_0)\tau\coth(2a\tau)-it\tau}{\Big(\big((a|x|^2+u+u_0)\tau\coth(2a\tau)-it\tau\big)^2-\frac{4uu_0t^2}{\sinh^2(2a\tau)}\Big)^{3/2}}\,d\tau
\end{aligned}$$

$$= -\int_{-\infty}^{+\infty} \frac{1}{(2\pi)^2} \frac{2a}{\sinh^2(2a\tau)} \frac{(a|x|^2 + u + u_0)\coth(2a\tau) - it}{\Big(\big((a|x|^2 + u + u_0)\coth(2a\tau) - it\big)^2 - \frac{4uu_0}{\sinh^2(2a\tau)}\Big)^{3/2}}\, d\tau$$

$$= -\int_{-\infty}^{+\infty} \frac{1}{(2\pi)^2} \frac{2a}{\sinh^2(2a\tau)} \frac{(a|x|^2 + u + u_0)\coth(2a\tau) - it}{\big((D - 4uu_0)\coth^2(2a\tau) - 2itD\coth(2a\tau) + 4uu_0 - t^2\big)^{3/2}}\, d\tau.$$

where $D = a|x|^2 + u + u_0$. Changing variable $\tanh(2a\tau) \to v$, we can rewrite the above integral as

$$\int_{-1}^{1} \frac{1}{(2\pi v)^2}\Big(\frac{D}{v} - it\Big)\Big(\frac{D^2 - 4uu_0}{v^2} - 2it\frac{D}{v} + 4uu_0 - t^2\Big)^{-3/2} dv$$

$$= \int_{-1}^{1} \frac{1}{(2\pi)^2}(D - itv)\big((4uu_0 - t^2)v^2 - 2itDv + D^2 - 4uu_0\big)^{-3/2} dv$$

$$= -\frac{1}{2\pi^2} \cdot \frac{1}{(a|x|^2 + u + u_0)^2 + t^2 - 4uu_0}.$$

Thus we get the explicit form of the Green's function of L in the interior. ■

Remark We may write $\mathbf{G}$ as

$$\mathbf{G}(x, t, u; 0, 0, u_0) = -\frac{1}{2\pi^2} \cdot \frac{1}{(u - u_0)^2 + 2a(u + u_0)|x|^2 + t^2 + a^2|x|^4}.$$

Since u and u_0 are positive, around the pole $(0, 0, u_0)$, $\mathbf{G}$ behaves like the d^{-2}, where d is the distance to the singularity.

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